Some results of conformal Ricci solitons on $N(\kappa)$-paracontact manifolds

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Abstract. In the present paper, we have deduced conformal Ricci solitons on $N(\kappa)$-paracontact metric manifolds and obtained a relation between λ and p. We have also studied second order parallel tensor, projective curvature tensor, concircular curvature tensor on an $N(\kappa)$-paracontact metric manifold admitting conformal Ricci solitons. Also we have proved that, there does not exist conformal Ricci solitons on $N(\kappa)$-paracontact metric manifolds.

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1 Introduction

In 1985, Paracontact geometry was introduced by Kaneyuki and Williams in the paper [11]. The dimension of a paracontact metric manifold is any positive integer whereas the dimension of contact metric manifold is always odd. In 2010, Montano, Erken and Murathan were introduced a class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, where κ, μ are real constants. This type of new manifolds are known as (κ, μ)-paracontact metric manifolds. If $\mu = 0$, then we call the manifolds as $N(\kappa)$-paracontact metric manifolds. The paracontact metric manifold has also been studied by several authors such as De and Mondal [5], Mandal and Mandal [14], Zamkovoy [18], Zamkovoy and Tzanov [19]. In 1926, Levy introduced the notion of second order parallel tensors. Later many authors such as Chandra, Hui and Shaikh [3], Mondal and De [13], Sharma {[16], [17]} have studied second order parallel tensors on several manifolds.

The notion of Ricci soliton was introduced by Hamilton [10] which is the generalization of Einstein metric and is defined by

$$(L_X g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0,$$

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where L_X denotes the Lie-derivatives of Riemannian metric g along the vector field X, λ is a constant, S the Ricci tensor of type $(0, 2)$ and Y, Z are arbitrary vector fields on the manifold. Here X is called the potential vector field. A Ricci soliton is called shrinking or steady or expanding according as λ is negative or zero or positive. A Ricci soliton is the limit of the solutions of Ricci flow equation given by

$$\frac{\partial g}{\partial t} = -2S.$$

Ricci soliton on different kind of manifolds has been studied in the papers [2], [4], [5], [15] by several authors.

Conformal Ricci flow equation was introduced by A. E. Fisher [9] in the year 2005 which is a variation of the classical Ricci flow equation and the equation is given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg$$

and $r = -1$, where p is a time dependent non-dynamical scalar field, r is the scalar curvature of the manifold and n is the dimension of the manifold.

In 2015, the notion of conformal Ricci soliton was introduced by N. Basu and A. Bhattacharyya [1] which is the generalization of the Ricci soliton and the equation is given by

$$L_X g + 2S = [2\lambda - (p + \frac{2}{n})]g.$$

The above equation also satisfies the conformal Ricci flow equation. Conformal Ricci soliton has been studied in the paper [7], [8].

In this paper we would like to study some properties of conformal Ricci solitons on an $N(\kappa)$-paracontact metric manifold.

The paper is organized as follows: After introduction, we give some preliminaries in the Section 2. Also we give an example of $N(\kappa)$-paracontact metric manifold in the Section 3. In Section 4, we have studied the conformal Ricci solitons on $N(\kappa)$-paracontact metric manifolds. In Section 5, we heve derived second order parallel tensor and conformal Ricci tensor. Section 6 is devoted to study projectively semi-symmetric $N(\kappa)$-paracontact metric manifolds admitting conformal Ricci solitons. In Section 7, we deduced some results of $N(\kappa)$-paracontact metric manifolds admitting conformal Ricci solitons and satisfy $C(\xi, X).S(Y, Z) = 0$. In the last Section, we heve proved that, there does not exist a conformal Ricci soliton in an $N(\kappa)$-paracontact manifold.

2 Preliminaries

A smooth $(2n+1), (n>1)$, dimensional manifold M is said to be an almost paracontact manifold if it admits a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying the conditions [5]

$$\phi^2 X = X - \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

and on each fibre of $D = ker(\eta)$, the tensor field ϕ induces an almost paracomplex structure, i.e., the eigen distribution D_ϕ^+ and D_ϕ^- of ϕ corresponding to the respective eigenvalue 1 and -1 have the same dimension n.

An almost paracontact manifold M is said to be an almost paraconcact metric manifold if there is a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all X, $Y \in \chi(M)$ and (ϕ, ξ, η, g) is said to be an almost paracontact metric structure. Here the signature of g is necessarily (n+1,n).

An almost paracontact structure is said to be a paracontact structure if $\Phi(X, Y) = d\eta(X, Y)$, the fundamental 2-form is defined by $\Phi(X, Y) = g(X, \phi Y)$. An almost paracontact structure is called normal if the $(1, 2)$-torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$.

For a paracontact metric manifold, we define a symmetric $(1, 1)$-tensor field $h = \frac{1}{2} L_\xi \phi$, where L_ξ stands for the Lie derivative in the direction ξ, satisfying the following conditions:

$$\phi h + h\phi = 0, \quad h\xi = 0, \quad tr(h) = tr(\phi h) = 0,$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.1}$$

for all $X \in \chi(M)$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold.

A paracontact metric manifold is said to be a paracontact (κ, μ)-manifold if the curvature tensor R satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for all X, $Y \in \chi(M)$ and κ, μ are real constants. If $\mu = 0$, then the paracontact (κ, μ)-manifold reduces to an $N(\kappa)$-paracontact manifold. Thus for an $N(\kappa)$-paracontact manifold, we get

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y), \tag{2.2}$$

for all vector fields X, $Y \in \chi(M)$ and κ is a real constant.

In an $N(\kappa)$-paracontact manifold of dimension $(2n + 1)(n>1)$, the following relations hold for $\kappa \neq -1$ [5]:

$$h^2 = (\kappa + 1)\phi^2,$$

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X), \tag{2.3}$$

$$\begin{aligned} S(X, Y) = &2(1 - n)g(X, Y) + 2(n - 1)g(hX, Y) \\ &+ \{2(n - 1) + 2n\kappa\}\eta(X)\eta(Y), \end{aligned} \tag{2.4}$$

$$QX = 2(1 - n)X + 2(n - 1)hX + \{2(n - 1) + 2n\kappa\}\eta(X)\xi, \tag{2.5}$$

$$S(X, \xi) = 2n\kappa\eta(X),$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{2.6}$$

$$(\nabla_X h)Y = -\{(1+\kappa)g(X,\phi Y) + g(X,\phi hY)\}\xi + \eta(Y)\phi h(hX - X), \tag{2.7}$$

$$(\nabla_X \eta)Y = g(X,\phi Y) + g(\phi hX, Y), \tag{2.8}$$

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X = &-(1+\kappa)\{2g(X,\phi Y)\xi + \eta(X)\phi Y \\ &-\eta(Y)\phi X\} + \eta(X)\phi hY - \eta(Y)\phi hX,\end{aligned} \tag{2.9}$$

for all vector fields X, $Y \in \chi(M)$ and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Before ending the present section, we recall a result

Lemma 2.1. *[19] Let M be a paracontact metric manifold of dimension $(2n+1), (n>1)$ which satisfies $R(X,Y)\xi = 0$ for all X, $Y \in \chi(M)$, then M is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

3 Example of $N(\kappa)$-paracontact metric manifold

Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where $\{x, y, z\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M, we get

$$[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

Let us define the metric tensor g as $g(e_1, e_1) = 1$, $g(e_2, e_2) = -1$, $g(e_3, e_3) = 1$ and $g(e_i, e_j) = 0$ for $i \neq j$. The 1-form η is defined by $\eta(X) = g(X, e_3)$, for any X on M. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then we find that

$$\eta(e_3) = 1, \quad \phi^2 X = X - \eta(X)e_3, \quad d\eta(X,Y) = g(X,\phi Y),$$

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Hence (ϕ, e_3, η, g) defines a paracontact metric structure on M.

Let ∇ be the Levi-Civita connection on M, using Koszul's formula, we obtain

$$\nabla_{e_1} e_2 = -e_3, \quad \nabla_{e_2} e_1 = e_3, \quad \nabla_{e_1} e_3 = -e_2,$$

$$\nabla_{e_2} e_3 = -e_1, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_3} e_1 = -e_2$$

and the remaining $\nabla_{e_i} e_j = 0$ for $i, j = 1, 2, 3$.
Using the formula $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

$$R(e_1,e_2)e_1=-3e_2,\quad R(e_1,e_2)e_2=-3e_1,\quad R(e_1,e_2)e_3=0,$$
$$R(e_3,e_1)e_1=-e_3,\quad R(e_1,e_3)e_2=0,\quad R(e_1,e_3)e_3=-e_1,$$
$$R(e_2,e_3)e_1=0,\quad R(e_3,e_2)e_2=e_3,\quad R(e_2,e_3)e_3=-e_2.$$

From the above expressions of the curvature tensor, we conclude that M is an $N(\kappa)$-paracontact metric manifold with $\kappa=-1$.

4 Conformal Ricci solitons on $N(\kappa)$-paracontact manifolds

Let M be an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1),(n>1)$. Then the conformal Ricci soliton is given by

$$L_V g+2S=[2\lambda-(p+\frac{2}{2n+1})]g. \tag{4.1}$$

Let V be the Reeb vector field ξ, then with the help of (2.1), we get

$$(L_\xi g)(X,Y)=2g(\phi hX,Y). \tag{4.2}$$

Therefore, from (4.1) and (4.2), we get

$$S(X,Y)=\frac{1}{2}[2\lambda-(p+\frac{2}{2n+1})]g(X,Y)-g(\phi hX,Y). \tag{4.3}$$

Since $S(X,Y)=g(QX,Y)$, where Q is the Ricci operator, we get

$$QX=\frac{1}{2}[2\lambda-(p+\frac{2}{2n+1})]X-\phi hX. \tag{4.4}$$

Also, we get from (4.3) and (4.4)

$$S(X,\xi)=\frac{1}{2}[2\lambda-(p+\frac{2}{2n+1})]\eta(X), \tag{4.5}$$

$$S(\xi,\xi)=\frac{1}{2}[2\lambda-(p+\frac{2}{2n+1})], \tag{4.6}$$

$$Q\xi=\frac{1}{2}[2\lambda-(p+\frac{2}{2n+1})]\xi. \tag{4.7}$$

Again, putting $X=Y=e_i$ in (4.3), where $\{e_i\}$ is the orthogonal basis of the tangent space of the manifold and summing over i, we get

$$r=S(e_i,e_i)=\frac{2n+1}{2}[2\lambda-(p+\frac{2}{2n+1})]-tr(\phi h). \tag{4.8}$$

Since for conformal Ricci soliton, $r=-1$ and $tr(\phi h)=0$, we get from (4.8)

$$\lambda=\frac{p}{2}.$$

Thus we can state

Theorem 4.1. *If an $N(\kappa)$-paracontact manifold of dimension $(2n+1), (n>1)$ admits conformal Ricci soliton, then the value of the scalar λ is $\frac{p}{2}$.*

Proposition 4.2. *For an $N(\kappa)$-paracontact manifold of dimension $(2n+1), (n>1)$, the conformal Ricci soliton is given by*

$$L_V g + 2S + \frac{2}{2n+1} g = 0.$$

5 Second order parallel tensor and conformal Ricci solitons

Definition 5.1. [13] Let M be an $N(\kappa)$-paracontact metric manifold of dimension n with metric g. A tensor field γ of type $(0,2)$ is called parallel tensor if $\nabla\gamma = 0$, where ∇ is the operator of covariant differentiation with respect to the metric tensor g.

Let γ be a second order symmetric tensor field on an $N(\kappa)$-paracontact manifold M of dimension $(2n+1)$, $n>1$, that is, $\gamma(X,Y) = \gamma(Y,X)$, for all vector fields X, Y on M such that $\nabla\gamma = 0$. Then, from the Ricci identity, we have

$$\nabla^2\gamma(X,Y;Z,W) = \nabla^2\gamma(X,Y;W,Z).$$

From above, we obtain

$$\gamma(R(X,Y)Z,W) + \gamma(R(X,Y)W,Z) = 0, \tag{5.1}$$

for all vector fields X, Y, Z and W on M.
Substituting $X = Z = W = \xi$ in (5.1), we get

$$\gamma(R(\xi,Y)\xi,\xi) = 0. \tag{5.2}$$

From (2.3), we get

$$R(\xi,Y)\xi = \kappa(\eta(Y)\xi - Y). \tag{5.3}$$

From (5.2) and (5.3), we obtain

$$\kappa(\eta(Y)\gamma(\xi,\xi) - \gamma(Y,\xi)) = 0. \tag{5.4}$$

Let $\kappa \neq 0$, then from (5.4), we get

$$\gamma(Y,\xi) = g(Y,\xi)\gamma(\xi,\xi). \tag{5.5}$$

Taking differentiation of (5.5) covariantly, we get

$$\begin{aligned}\gamma(\nabla_X Y,\xi) + \gamma(Y,\nabla_X\xi) =& g(\nabla_X Y,\xi)\gamma(\xi,\xi) + g(Y,\nabla_X\xi)\gamma(\xi,\xi)\\ &+2g(Y,\xi)\gamma(\nabla_X\xi,\xi).\end{aligned} \tag{5.6}$$

Again, from (5.5), we get

$$\gamma(\nabla_X Y,\xi) = g(\nabla_X Y,\xi)\gamma(\xi,\xi). \tag{5.7}$$

Using (5.7) in (5.6), we obtain

$$\begin{aligned}-\gamma(Y,\phi X)+\gamma(Y,\phi hX) = &- g(Y,\phi X)\gamma(\xi,\xi)+g(Y,\phi hX)\gamma(\xi,\xi)\\ &-2g(Y,\xi)\gamma(\phi X,\xi)+2g(Y,\xi)\gamma(\phi hX,\xi).\end{aligned} \tag{5.8}$$

From (5.5), we get

$$\gamma(\phi X,\xi)=\gamma(\phi hX,\xi)=0.$$

Therefore, from (5.8), we obtain

$$-\gamma(Y,\phi X)+\gamma(Y,\phi hX) = -g(Y,\phi X)\gamma(\xi,\xi)+g(Y,\phi hX)\gamma(\xi,\xi). \tag{5.9}$$

Interchanging X and Y in (5.9), we get

$$-\gamma(X,\phi Y)+\gamma(X,\phi hY) = -g(X,\phi Y)\gamma(\xi,\xi)+g(X,\phi hY)\gamma(\xi,\xi). \tag{5.10}$$

Subtracting (5.10) from (5.9), we get

$$\gamma(X,\phi Y)=g(X,\phi Y)\gamma(\xi,\xi). \tag{5.11}$$

Putting $Y=\phi Y$ in (5.11), we get

$$\gamma(X,Y)-\eta(Y)\gamma(X,\xi)=g(X,Y)\gamma(\xi,\xi)-\eta(Y)g(X,\xi)\gamma(\xi,\xi). \tag{5.12}$$

From(5.5), we get

$$\eta(Y)\gamma(X,\xi)=\eta(Y)g(X,\xi)\gamma(\xi,\xi). \tag{5.13}$$

Therefore, from (5.12) and (5.13), we obtain

$$\gamma(X,Y)=g(X,Y)\gamma(\xi,\xi). \tag{5.14}$$

Thus we can state

Theorem 5.1. *If γ is a second order parallel tensor on an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, $n>1$, then γ is given by*

$$\gamma(X,Y)=g(X,Y)\gamma(\xi,\xi),$$

for all vector fields X, Y on M.

Since $\nabla\{[2\lambda-(p+\frac{2}{2n+1})]g(X,Y)\}=0$ for all vector fields X, Y on M, we can say that $\{L_\xi g(X,Y)+2S(X,Y)\}$ is a second order parallel tensor. So,

$$(L_\xi g)(X,Y)+2S(X,Y)=\{(L_\xi g)(\xi,\xi)+2S(\xi,\xi)\}g(X,Y). \tag{5.15}$$

From (2.4) and (4.2), we obtain

$$(L_\xi g)(\xi,\xi)+2S(\xi,\xi)=4n\kappa. \tag{5.16}$$

From (5.15) and (5.16), we get

$$(L_\xi g)(X,Y)+2S(X,Y)=4n\kappa g(X,Y). \tag{5.17}$$

Comparing (4.1) and (5.17), we get

$$\kappa=\frac{\lambda}{2n}-\frac{p}{4n}-\frac{1}{2n(2n+1)}.$$

Thus we can state the following

Proposition 5.2. *If an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, $n>1$, admits conformal Ricci soliton, then the value of κ is $\left(\frac{\lambda}{2n}-\frac{p}{4n}-\frac{1}{2n(2n+1)}\right)$.*

6 Projectively semi-symmetric $N(\kappa)$-paracontact manifolds admitting conformal Ricci solitons

Definition 6.1. [6] The Weyl projective curvature tensor in an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$ is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{6.1}$$

for all X, Y, $Z \in \chi(M)$.

Definition 6.2. [7] An $N(\kappa)$-paracontact manifold of dimension $(2n+1)$ is said to be projectively semi-symmetric if $R(\xi, X).P(Y,Z)W = 0$, which gives

$$\begin{aligned} R(\xi,X)(P(Y,Z)W) - &P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W \\ &-P(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{6.2}$$

for all X, Y, Z, $W \in \chi(M)$

Putting $Z = \xi$ in (6.2), we get with the help of (2.3)

$$\begin{aligned} &\kappa[g(X,P(Y,\xi)W)\xi - \eta(P(Y,\xi)W)X - g(X,Y)P(\xi,\xi)W \\ &+ \eta(Y)P(X,\xi)W - \eta(X)P(Y,\xi)W + P(Y,X)W \\ &- g(X,W)P(Y,\xi)\xi + \eta(W)P(Y,\xi)X] = 0. \end{aligned} \tag{6.3}$$

Using (4.3), (4.5) in (6.1), we get

$$P(X,Y)\xi = (\kappa - \frac{A}{2n})[\eta(Y)X - \eta(X)Y], \tag{6.4}$$

$$P(X,\xi)Z = (\frac{A}{2n} - \kappa)[g(X,Z)\xi - \eta(Z)X] + \frac{1}{2n}g(\phi hX,Z)\xi, \tag{6.5}$$

$$P(X,\xi)\xi = (\frac{A}{2n} - \kappa)[\eta(X)\xi - X], \tag{6.6}$$

$$P(\xi,\xi)Z = 0, \tag{6.7}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]$.
Putting $W = \xi$ in (6.3) and using (6.4), (6.5), (6.6) and (6.7), we get

$$\frac{\kappa}{2n}g(X,\phi hY)\xi = 0,$$

which gives $\kappa = 0$.

Thus, with the help of lemma 2.1, we can state the following theorem

Theorem 6.1. *If an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$, $(n > 1)$ admits conformal Ricci soliton and is projectively semi-symmetric, then the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

7 $N(\kappa)$-paracontact metric manifolds with conformal Ricci solitons satisfying $C(\xi, X).S(Y, Z) = 0$

Definition 7.1. [6] The concircular curvature tensor of type $(1, 3)$ on an $(2n + 1)$-dimensional $N(\kappa)$-paracontact metric manifold M is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y], \tag{7.1}$$

for all vector fields X, Y, $Z \in \chi(M)$ and r is the scalar curvature of the manifold.

From (7.1), we get

$$C(\xi, X)Y = R(\xi, X)Y - \frac{r}{2n(2n + 1)}[g(X, Y)\xi - \eta(Y)X].$$

Using (2.3) in the above equation, we obtain

$$C(\xi, X)Y = [\kappa - \frac{r}{2n(2n + 1)}][g(X, Y)\xi - \eta(Y)X]. \tag{7.2}$$

Let us assume that $C(\xi, X).S(Y, Z) = 0$ holds. Then we have

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0. \tag{7.3}$$

From (4.3) and (7.3), we get

$$\begin{aligned} Ag(C(\xi, X)Y, Z) - g(\phi hC(\xi, X)Y, Z) + Ag(Y, C(\xi, X)Z) \\ -g(\phi hY, C(\xi, X)Z) = 0, \end{aligned} \tag{7.4}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]$.
Using (7.2) in (7.4) , we obtain

$$B[g(\phi hX, Z)\eta(Y) + g(X, \phi hY)\eta(Z)] = 0,$$

where $B = \kappa - \frac{r}{2n(2n+1)}$, which implies $B = 0$, i.e., $\kappa = \frac{r}{2n(2n+1)}$. But for a conformal Ricci solition, we have $r = -1$. So we get $\kappa = -\frac{1}{2n(2n+1)}$.

Thus we can state the following

Theorem 7.1. *An $N(\kappa)$-paracontact metric manifold of dimension $(2n + 1)$, $n > 1$ admitting conformal Ricci soliton and satisfy $C(\xi, X).S(Y, Z) = 0$, then the value of κ is $-\frac{1}{2n(2n+1)}$.*

8 Non-existence of conformal Ricci solitons in $N(\kappa)$-paracontact metric manifolds

Let M be an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, $n > 1$ admitting conformal Ricci soliton with potential vector as the Reeb vector field. Then from (4.3), we get

$$S(X, Y) = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{2n + 1}\right)\right] g(X, Y) - g(\phi hX, Y). \tag{8.1}$$

Using (2.4) in (8.1), we get

$$\begin{aligned}&[2(1-n)-\frac{1}{2}\{2\lambda-(p+\frac{2}{2n+1})\}]g(X,Y)+2(n-1)g(hX,Y)\\ &+\{2(n-1)+2n\kappa\}\eta(X)\eta(Y)+g(\phi hX,Y)=0. \qquad (8.2)\end{aligned}$$

Replacing X by ϕX in (8.2), we get

$$[2(1-n)-\frac{1}{2}\{2\lambda-(p+\frac{2}{2n+1})\}]g(\phi X,Y)+2(n-1)g(h\phi X,Y)+g(hX,Y)=0. \qquad (8.3)$$

Interchanging X and Y, we get

$$[2(1-n)-\frac{1}{2}\{2\lambda-(p+\frac{2}{2n+1})\}]g(\phi Y,X)+2(n-1)g(h\phi Y,X)+g(hY,X)=0. \qquad (8.4)$$

Subtracting (8.3) from (8.4), we obtain

$$[2(1-n)-\frac{1}{2}\{2\lambda-(p+\frac{2}{2n+1})\}]g(\phi X,Y)=0.$$

This implies

$$2(1-n)-\frac{1}{2}\{2\lambda-(p+\frac{2}{2n+1})\}=0. \qquad (8.5)$$

Using $\lambda=\frac{p}{2}$ in (8.5) and simplifying, we get

$$4n^2-2n-3=0,$$

which has no integer root. Thus our assumption is wrong.

Hence we can state the following

Theorem 8.1. *There does not exist conformal Ricci soliton in an $N(\kappa)$-paracontact metric manifold M of dimension $(2n+1)$, $n>1$, with potential vector field as the Reeb vector field.*

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