Some results of conformal Ricci solitons on $N(\kappa)$-paracontact manifolds

Tarak Mandal

Abstract. In the present paper, we have deduced conformal Ricci solitons on $N(\kappa)$-paracontact metric manifolds and obtained a relation between $\lambda$ and $p$. We have also studied second order parallel tensor, projective curvature tensor, concircular curvature tensor on an $N(\kappa)$-paracontact metric manifold admitting conformal Ricci solitons. Also we have proved that, there does not exist conformal Ricci solitons on $N(\kappa)$-paracontact metric manifolds.

Key words: $N(\kappa)$-paracontact manifolds; Ricci soliton; conformal Ricci soliton; second order parallel tensor; projective curvature tensor; concircular curvature tensor.

1 Introduction

In 1985, Paracontact geometry was introduced by Kaneyuki and Williams in the paper [11]. The dimension of a paracontact metric manifold is any positive integer whereas the dimension of contact metric manifold is always odd. In 2010, Montano, Erken and Murathan were introduced a class of paracontact metric manifolds for which the characteristic vector field $\xi$ belongs to the $(\kappa, \mu)$-nullity distribution, where $\kappa, \mu$ are real constants. This type of new manifolds are known as $(\kappa, \mu)$-paracontact metric manifolds. If $\mu = 0$, then we call the manifolds as $N(\kappa)$-paracontact metric manifolds. The paracontact metric manifold has also been studied by several authors such as De and Mondal [5], Mandal and Mandal [14], Zamkovoy [18], Zamkovoy and Tzanov [19]. In 1926, Levy introduced the notion of second order parallel tensors. Later many authors such as Chandra, Hui and Shaikh [3], Mondal and De [13], Sharma {[16], [17]} have studied second order parallel tensors on several manifolds.

The notion of Ricci soliton was introduced by Hamilton [10] which is the generalization of Einstein metric and is defined by

$$ (L_X g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0, $$
where \( L_X \) denotes the Lie-derivatives of Riemannian metric \( g \) along the vector field \( X \), \( \lambda \) is a constant, \( S \) the Ricci tensor of type \((0,2)\) and \( Y, Z \) are arbitrary vector fields on the manifold. Here \( X \) is called the potential vector field. A Ricci soliton is called shrinking or steady or expanding according as \( \lambda \) is negative or zero or positive. A Ricci soliton is the limit of the solutions of Ricci flow equation given by

\[
\frac{\partial g}{\partial t} = -2S.
\]

Ricci soliton on different kind of manifolds has been studied in the papers [2], [4], [5], [15] by several authors.

Conformal Ricci flow equation was introduced by A. E. Fisher [9] in the year 2005 which is a variation of the classical Ricci flow equation and the equation is given by

\[
\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg
\]

and \( r = -1 \), where \( p \) is a time dependent non-dynamical scalar field, \( r \) is the scalar curvature of the manifold and \( n \) is the dimension of the manifold.

In 2015, the notion of conformal Ricci soliton was introduced by N. Basu and A. Bhattacharyya [1] which is the generalization of the Ricci soliton and the equation is given by

\[
L_X g + 2S = [2\lambda - (p + \frac{2}{n})]g.
\]

The above equation also satisfies the conformal Ricci flow equation. Conformal Ricci soliton has been studied in the paper [7], [8].

In this paper we would like to study some properties of conformal Ricci solitons on an \( N(\kappa) \)-paracontact metric manifold.

The paper is organized as follows: After introduction, we give some preliminaries in the Section 2. Also we give an example of \( N(\kappa) \)-paracontact metric manifold in the Section 3. In Section 4, we have studied the conformal Ricci solitons on \( N(\kappa) \)-paracontact metric manifolds. In Section 5, we have derived second order parallel tensor and conformal Ricci tensor. In Section 6, we devoted to study projectively semi-symmetric \( N(\kappa) \)-paracontact metric manifolds admitting conformal Ricci solitons. In Section 7, we deduced some results of \( N(\kappa) \)-paracontact metric manifolds admitting conformal Ricci solitons and satisfy \( C(\xi,X).S(Y,Z) = 0 \). In the last Section, we have proved that, there does not exist a conformal Ricci soliton in an \( N(\kappa) \)-paracontact manifold.

2 Preliminaries

A smooth \((2n+1), (n>1)\), dimensional manifold \( M \) is said to be an almost paracontact manifold if it admits a \((1,1)\)-tensor field \( \phi \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying the conditions [5]

\[
\phi^2 X = X - \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1
\]

and on each fibre of \( D = ker(\eta) \), the tensor field \( \phi \) induces an almost paracomplex structure, i.e., the eigen distribution \( D^\phi_+ \) and \( D^\phi_- \) of \( \phi \) corresponding to the respective eigenvalue 1 and \(-1\) have the same dimension \( n \).
Some results of conformal Ricci solitons on $N(\kappa)$-paracontact manifolds

An almost paracontact manifold $M$ is said to be an almost paraconcact metric manifold if there is a pseudo-Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \chi(M)$ and $(\phi, \xi, \eta, g)$ is said to be an almost paracontact metric structure. Here the signature of $g$ is necessarily $(n+1, n)$.

An almost paracontact structure is said to be a paracontact structure if $\Phi(X, Y) = d\eta(X, Y)$, the fundamental 2-form is defined by $\Phi(X, Y) = g(X, \phi Y)$. An almost paracontact structure is called normal if the $(1, 2)$-torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$.

For a paracontact metric manifold, we define a symmetric $(1, 1)$-tensor field $h = \frac{1}{2} L_\xi \phi$, where $L_\xi$ stands for the Lie derivative in the direction $\xi$, satisfying the following conditions:

$$\phi h + h \phi = 0, \quad h \xi = 0, \quad tr(h) = tr(\phi h) = 0,$$

(2.1)

$$\nabla_X \xi = -\phi X + \phi h X,$$

for all $X \in \chi(M)$, where $\nabla$ is the Levi-Civita connection of the pseudo-Riemannian manifold.

A paracontact metric manifold is said to be a paracontact $(\kappa, \mu)$-manifold if the curvature tensor $R$ satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for all $X, Y \in \chi(M)$ and $\kappa, \mu$ are real constants. If $\mu = 0$, then the paracontact $(\kappa, \mu)$-manifold reduces to an $N(\kappa)$-paracontact manifold. Thus for an $N(\kappa)$-paracontact manifold, we get

(2.2)$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y),$$

for all vector fields $X, Y \in \chi(M)$ and $\kappa$ is a real constant.

In an $N(\kappa)$-paracontact manifold of dimension $(2n + 1)(n>1)$, the following relations hold for $\kappa \neq -1$ [5]:

$$h^2 = (\kappa + 1)\phi^2,$$

(2.3)$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X),$$

$$S(X, Y) = 2(1 - n)g(X, Y) + 2(n - 1)g(hX, Y) + \{2(n - 1) + 2n\kappa\}\eta(X)\eta(Y),$$

(2.4)$$QX = 2(1 - n)X + 2(n - 1)hX + \{2(n - 1) + 2n\kappa\}\eta(X)\xi,$$

$$S(X, \xi) = 2n\kappa\eta(X),$$

(2.5)$$QX = 2(1 - n)X + 2(n - 1)hX + \{2(n - 1) + 2n\kappa\}\eta(X)\xi,$$

$$S(X, \xi) = 2n\kappa\eta(X),$$

(2.6)$$\nabla_X \phi Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX),$$

for all $X, Y \in \chi(M)$.
structure on $M$ for any vector fields $g$.

Let us define the metric tensor $M$ which are linearly independent at each point of $M$.

(2.7) \[(\nabla_X h)Y = -(1 + \kappa)g(X, \phi Y) + g(X, \phi h Y)\xi + \eta(Y)\phi h(hX - X),\]

(2.8) \[(\nabla_X \eta)Y = g(X, \phi Y) + g(\phi hX, Y),\]

\[(\nabla_X h)Y - (\nabla_Y h)X = -(1 + \kappa)\{2g(X, \phi Y)\xi + \eta(X)\phi Y\}
- \eta(Y)\phi X + \eta(X)\phi h Y - \eta(Y)\phi h X,\]

for all vector fields $X, Y \in \chi(M)$ and $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

Before ending the present section, we recall a result

Lemma 2.1. [19] Let $M$ be a paracontact metric manifold of dimension $(2n + 1)$, $(n > 1)$ which satisfies $R(X, Y)\xi = 0$ for all $X, Y \in \chi(M)$, then $M$ is locally isometric to a product of a flat $(n + 1)$-dimensional manifold and an $n$-dimensional manifold of negative constant curvature equal to $-4$.

3 Example of $N(\kappa)$-paracontact metric manifold

Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where $\{x, y, z\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

\[e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},\]

which are linearly independent at each point of $M$, we get

\[[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.\]

Let us define the metric tensor $g$ as $g(e_1, e_1) = 1$, $g(e_2, e_2) = -1$, $g(e_3, e_3) = 1$ and $g(e_i, e_j) = 0$ for $i \neq j$. The 1-form $\eta$ is defined by $\eta(X) = g(X, e_3)$, for any $X$ on $M$.

Let $\phi$ be the $(1, 1)$-tensor field defined by

\[\phi(e_1) = e_2, \quad \phi(e_2) = e_3, \quad \phi(e_3) = 0.\]

Then we find that

\[\eta(e_3) = 1, \quad \phi^2 X = X - \eta(X)e_3, \quad d\eta(X, Y) = g(X, \phi Y),\]

\[g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),\]

for any vector fields $X, Y$ on $M$. Hence $(\phi, e_3, \eta, g)$ defines a paracontact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection on $M$, using Koszul’s formula, we obtain

\[\nabla_{e_i} e_2 = -e_3, \quad \nabla_{e_2} e_1 = e_3, \quad \nabla_{e_1} e_3 = -e_2,\]

\[\nabla_{e_i} e_3 = -e_i, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_3} e_1 = -e_2\]

and the remaining $\nabla_{e_i} e_j = 0$ for $i, j = 1, 2, 3$.

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

Tarak Mandal
$R(e_1, e_2)e_1 = -3e_2, \quad R(e_1, e_2)e_2 = -3e_1, \quad R(e_1, e_2)e_3 = 0,$
$R(e_3, e_1)e_1 = -e_3, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = -e_1,$
$R(e_2, e_3)e_1 = 0, \quad R(e_3, e_2)e_2 = e_3, \quad R(e_2, e_3)e_3 = -e_2.$

From the above expressions of the curvature tensor, we conclude that $M$ is an $N(\kappa)$-paracontact metric manifold with $\kappa = -1$.

4 Conformal Ricci solitons on $N(\kappa)$-paracontact manifolds

Let $M$ be an $N(\kappa)$-paracontact metric manifold of dimension $(2n + 1), (n > 1)$. Then the conformal Ricci soliton is given by

\[(4.1) \quad L_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g.\]

Let $V$ be the Reeb vector field $\xi$, then with the help of (2.1), we get

\[(4.2) \quad (L_\xi g)(X,Y) = 2g(\phi hX,Y).\]

Therefore, from (4.1) and (4.2), we get

\[(4.3) \quad S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]g(X,Y) - g(\phi hX,Y).\]

Since $S(X,Y) = g(QX,Y)$, where $Q$ is the Ricci operator, we get

\[(4.4) \quad QX = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]X - \phi hX.\]

Also, we get from (4.3) and (4.4)

\[(4.5) \quad S(X,\xi) = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]\eta(X),\]
\[(4.6) \quad S(\xi,\xi) = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})],\]
\[(4.7) \quad Q\xi = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]\xi.\]

Again, putting $X = Y = e_i$ in (4.3), where $\{e_i\}$ is the orthogonal basis of the tangent space of the manifold and summing over $i$, we get

\[(4.8) \quad r = S(e_i, e_i) = \frac{2n+1}{2}[2\lambda - (p + \frac{2}{2n+1})] - tr(\phi h).\]

Since for conformal Ricci soliton, $r = -1$ and $tr(\phi h) = 0$, we get from (4.8)

$$\lambda = \frac{p}{2}.$$

Thus we can state
Theorem 4.1. If an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$, $(n > 1)$ admits conformal Ricci soliton, then the value of the scalar $\lambda$ is $\frac{p^2}{2}$.

Proposition 4.2. For an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$, $(n > 1)$, the conformal Ricci soliton is given by

$$LVg + 2S + \frac{2}{2n+1}g = 0.$$ 

5 Second order parallel tensor and conformal Ricci solitons

Definition 5.1. [13] Let $M$ be an $N(\kappa)$-paracontact metric manifold of dimension $n$ with metric $g$. A tensor field $\gamma$ of type $(0,2)$ is called parallel tensor if $\nabla \gamma = 0$, where $\nabla$ is the operator of covariant differentiation with respect to the metric tensor $g$.

Let $\gamma$ be a second order symmetric tensor field on an $N(\kappa)$-paracontact manifold $M$ of dimension $(2n+1)$, $n > 1$, that is, $\gamma(X,Y) = \gamma(Y,X)$, for all vector fields $X$, $Y$ on $M$ such that $\nabla \gamma = 0$. Then, from the Ricci identity, we have

$$\nabla^2 \gamma(X,Y;Z,W) = \nabla^2 \gamma(X,Y;W,Z).$$

From above, we obtain

$$\gamma(R(X,Y)Z,W) + \gamma(R(X,Y)W,Z) = 0,$$

for all vector fields $X$, $Y$, $Z$ and $W$ on $M$.

Substituting $X = Z = W = \xi$ in (5.1), we get

$$\gamma(R(\xi,Y)\xi,\xi) = 0.$$

From (2.3), we get

$$R(\xi,Y)\xi = \kappa(\eta(Y)\xi - Y).$$

From (5.2) and (5.3), we obtain

$$\kappa(\eta(Y)\gamma(\xi,\xi) - \gamma(Y,\xi)) = 0.$$

Let $\kappa \neq 0$, then from (5.4), we get

$$\gamma(Y,\xi) = g(Y,\xi)\gamma(\xi,\xi).$$

Taking differentiation of (5.5) covariantly, we get

$$\gamma(\nabla_X Y,\xi) + \gamma(Y,\nabla_X \xi) = g(\nabla_X Y,\xi)\gamma(\xi,\xi) + g(Y,\nabla_X \xi)\gamma(\xi,\xi) + 2g(Y,\xi)\gamma(\nabla_X \xi,\xi).$$

Again, from (5.5), we get

$$\gamma(\nabla_X Y,\xi) = g(\nabla_X Y,\xi)\gamma(\xi,\xi).$$
Using (5.7) in (5.6), we obtain
\[-\gamma(Y,\phi X) + \gamma(Y,\phi h X) = -g(Y,\phi X)\gamma(\xi,\xi) + g(Y,\phi h X)\gamma(\xi,\xi) \]
\[-2g(Y,\xi)\gamma(\phi X,\xi) + 2g(Y,\xi)\gamma(\phi h X,\xi).\]

From (5.5), we get
\[\gamma(\phi X,\xi) = \gamma(\phi h X,\xi) = 0.\]
Therefore, from (5.8), we obtain
\[\gamma(Y,\phi X) = -\gamma(\phi X,\xi) = \gamma(\phi h X,\xi) = 0.\]

Interchanging $X$ and $Y$ in (5.9), we get
\[\gamma(X,\phi Y) = \gamma(X,\phi h Y) = 0.\]
Subtracting (5.10) from (5.9), we get
\[\gamma(X,\phi Y) = \gamma(X,\phi h Y) = 0.\]

From (2.4) and (4.2), we obtain
\[\gamma(X,\phi Y) = \gamma(X,\phi h Y) = 0.\]
Comparing (4.1) and (5.17), we get
\[\kappa = \frac{\lambda}{2n} - \frac{p}{4n} - \frac{1}{2n(2n+1)}.\]
Thus we can state the following

**Theorem 5.1.** If $\gamma$ is a second order parallel tensor on an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, $n > 1$, then $\gamma$ is given by
\[\gamma(X,\phi Y) = \gamma(X,\phi h Y) = 0.\]

for all vector fields $X, Y$ on $M$. Since $\nabla\{2\lambda - (p + \frac{2}{2n+1})\}g(X,Y) = 0$ for all vector fields $X, Y$ on $M$, we can say that $\{L_{\xi}g(X,Y) + 2S(X,Y)\}$ is a second order parallel tensor. So,
\[\kappa = \frac{\lambda}{2n} - \frac{p}{4n} - \frac{1}{2n(2n+1)}.\]
Thus we can state the following

**Proposition 5.2.** If an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, $n > 1$, admits conformal Ricci soliton, then the value of $\kappa$ is
\[\kappa = \frac{\lambda}{2n} - \frac{p}{4n} - \frac{1}{2n(2n+1)}.\]
6 Projectively semi-symmetric $N(\kappa)$-paracontact manifolds admitting conformal Ricci solitons

**Definition 6.1.** [6] The Weyl projective curvature tensor in an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$ is defined by
\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],
\]
for all $X, Y, Z \in \chi(M)$.

**Definition 6.2.** [7] An $N(\kappa)$-paracontact manifold of dimension $(2n+1)$ is said to be projectively semi-symmetric if
\[
R(\xi,X)(P(Y,Z)W) - P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W - P(Y,X)R(\xi,Z)W = 0,
\]
for all $X, Y, Z, W \in \chi(M)$.

Putting $Z = \xi$ in (6.2), we get with the help of (2.3)
\[
\kappa[g(X,Y P(\xi,W)\xi) - \eta(P(\xi,W)X - g(X,Y)P(\xi,W)
+ \eta(Y)P(X,\xi)W - \eta(X)P(Y,\xi)W + P(Y,X)W
- g(X,W)P(Y,\xi)\xi + \eta(W)P(Y,\xi)X] = 0.
\]
Using (4.3), (4.5) in (6.1), we get
\[
P(X,Y)\xi = (\kappa - \frac{A}{2n})[\eta(Y)X - \eta(X)Y],
\]
\[
P(X,\xi)Z = (\frac{A}{2n} - \kappa)[g(X,Z)\xi - \eta(Z)X] + \frac{1}{2n}g(\phi h X, Z)\xi,
\]
\[
P(X,\xi)\xi = (\frac{A}{2n} - \kappa)[\eta(X)\xi - X],
\]
\[
P(\xi,\xi)Z = 0,
\]
where $A = \frac{1}{4}[2\lambda - (p + \frac{2}{n+1})]$.

Putting $W = \xi$ in (6.3) and using (6.4), (6.5), (6.6) and (6.7), we get
\[
\frac{\kappa}{2n}g(X,\phi h Y)\xi = 0,
\]
which gives $\kappa = 0$.

Thus, with the help of lemma 2.1, we can state the following theorem

**Theorem 6.1.** If an $N(\kappa)$-paracontact manifold of dimension $(2n+1)$, $(n > 1)$ admits conformal Ricci soliton and is projectively semi-symmetric, then the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of negative constant curvature equal to $-4$. 
7 \(N(\kappa)\)-paracontact metric manifolds with conformal Ricci solitons satisfying \(C'(\xi, X).S(Y, Z) = 0\)

**Definition 7.1.** [6] The concircular curvature tensor of type (1, 3) on an \((2n + 1)\)-dimensional \(N(\kappa)\)-paracontact metric manifold \(M\) is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y],
\]

for all vector fields \(X, Y, Z \in \chi(M)\) and \(r\) is the scalar curvature of the manifold.

From (7.1), we get

\[
C(\xi, X)Y = R(\xi, X)Y - \frac{r}{2n(2n + 1)}[g(X, Y)\xi - \eta(Y)X].
\]

Using (2.3) in the above equation, we obtain

\[
(7.2) \quad C(\xi, X)Y = [\kappa - \frac{r}{2n(2n + 1)}][g(X, Y)\xi - \eta(Y)X].
\]

Let us assume that \(C(\xi, X).S(Y, Z) = 0\) holds. Then we have

\[
(7.3) \quad S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0.
\]

From (4.3) and (7.3), we get

\[
Ag(C(\xi, X)Y, Z) - g(\phi hC(\xi, X)Y, Z) + Ag(Y, C(\xi, X)Z)
\]

\[
- g(\phi hY, C(\xi, X)Z) = 0,
\]

where \(A = \frac{1}{2}[2\lambda - (p + \frac{2}{2n+1})]\).

Using (7.2) in (7.4), we obtain

\[
B[g(\phi hX, Z)\eta(Y) + g(X, \phi hY)\eta(Z)] = 0,
\]

where \(B = \kappa - \frac{r}{2n(2n + 1)}\), which implies \(B = 0\), i.e., \(\kappa = \frac{r}{2n(2n + 1)}\). But for a conformal Ricci soliton, we have \(r = -1\). So we get \(\kappa = -\frac{1}{2n(2n + 1)}\).

Thus we can state the following

**Theorem 7.1.** An \(N(\kappa)\)-paracontact metric manifold of dimension \((2n + 1)\), \(n > 1\) admitting conformal Ricci soliton and satisfy \(C(\xi, X).S(Y, Z) = 0\), then the value of \(\kappa\) is \(-\frac{1}{2n(2n + 1)}\).

8 Non-existence of conformal Ricci solitons in \(N(\kappa)\)-paracontact metric manifolds

Let \(M\) be an \(N(\kappa)\)-paracontact metric manifold of dimension \((2n + 1)\), \(n > 1\) admitting conformal Ricci soliton with potential vector as the Reeb vector field. Then from (4.3), we get

\[
S(X, Y) = \frac{1}{2} \left[2\lambda - \left(p + \frac{2}{2n + 1}\right)\right] g(X, Y) - g(\phi hX, Y).
\]
Using (2.4) in (8.1), we get
\[
\begin{align*}
[2(1-n) & - \frac{1}{2}(2\lambda - (p + \frac{2}{2n+1}))]g(X, Y) + 2(n-1)g(hX, Y) \\
+ 2(n-1) + 2\kappa \eta(X)\eta(Y) + g(\phi hX, Y) = 0.
\end{align*}
\]
(8.2)

Replacing $X$ by $\phi X$ in (8.2), we get
\[
\begin{align*}
[2(1-n) & - \frac{1}{2}(2\lambda - (p + \frac{2}{2n+1}))]g(\phi X, Y) + 2(n-1)g(h\phi X, Y) + g(hX, Y) = 0.
\end{align*}
\]
(8.3)

Interchanging $X$ and $Y$, we get
\[
\begin{align*}
[2(1-n) & - \frac{1}{2}(2\lambda - (p + \frac{2}{2n+1}))]g(\phi Y, X) + 2(n-1)g(h\phi Y, X) + g(hY, X) = 0.
\end{align*}
\]
(8.4)

Subtracting (8.3) from (8.4), we obtain
\[
[2(1-n) - \frac{1}{2}(2\lambda - (p + \frac{2}{2n+1}))]g(\phi X, Y) = 0.
\]

This implies
\[
2(1-n) - \frac{1}{2}(2\lambda - (p + \frac{2}{2n+1})) = 0.
\]
(8.5)

Using $\lambda = \frac{p}{2}$ in (8.5) and simplifying, we get
\[
4n^2 - 2n - 3 = 0,
\]
which has no integer root. Thus our assumption is wrong.

Hence we can state the following

**Theorem 8.1.** There does not exist conformal Ricci soliton in an $N(\kappa)$-paracontact metric manifold $M$ of dimension $(2n+1)$, $n > 1$, with potential vector field as the Reeb vector field.

**References**


Some results of conformal Ricci solitons on $N(\kappa)$-paracontact manifolds


Author’s address:

Tarak Mandal
Department of Mathematics, Jangipur College,
Murshidabad 742213, West Bengal, India.
E-mail: mathtarak@gmail.com