A note on gradient solitons on para-Sasakian manifolds

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Abstract. The purpose of the offering exposition is to characterize gradient Yamabe, gradient Einstein and gradient m-quasi Einstein solitons within the framework of three-dimensional para-Sasakian manifolds. Finally, we consider a non-trivial example to validate a result of our paper.

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1 Introduction

The Yamabe flow lies in the fact that it is an inherent geometric deformation to metrics of constant scalar curvature which is very significant in coeval mathematics. It is pointed that Yamabe flow fits to the fast diffusion case of the plasma equation in mathematical physics. The Yamabe flow is identical to the Ricci flow (defined by $\frac{\partial}{\partial_t} g(t) = -2S(t)$, where S stands for the Ricci tensor) for dimension $n = 2$. However, in higher dimension they do not assent, since the first one preserves the conformal class of the metric but the Ricci flow does not in general.

In [15] several years ago, Hamilton presented the concept of Yamabe soliton. According to Hamilton, a semi-Riemannian metric g of a semi-Riemannian manifold (M^n, g) is said to be a Yamabe soliton if it obeys

$$\frac{1}{2}\pounds_{\bar{W}} g = (r - \lambda)g, \tag{1.1}$$

for a smooth vector field $\bar{W}$ and a real number λ, where r is the scalar curvature of g and $\pounds$ denotes the Lie-derivative operator. The vector field $\bar{W}$ is termed as the soliton field of the Yamabe soliton. When the vector field $\bar{W}$ is gradient of a smooth function $f : M^n \to \mathbb{R}$, then the manifold will be called **gradient Yamabe soliton**. In this case the antecedent equation reveals the form

$$\nabla^2 f = (r - \lambda)g, \tag{1.2}$$

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where $\nabla^2 f$ indicates the Hessian of f.
Yamabe solitons have been pointed out by numerous researchers in different context (see, [2], [3], [7], [8], [9], [10], [21], [22]).

The concept of Gradient Einstein solitons was introduced by Catino and Mazzieri [6]. Gradient Einstein solitons are semi-Riemannian manifolds satisfying

$$S - \frac{1}{2} r g + \nabla^2 f = \lambda g, \tag{1.3}$$

for some smooth function f and some constant $\lambda \in \mathbb{R}$.

A semi-Riemannian metric g on a semi-Riemannian manifold M is called a **gradient generalized m-quasi Einstein metric** [1] if there exists a smooth function $f : M^n \to \mathbb{R}$ and obeys

$$S + \nabla^2 f = \frac{1}{m} df \otimes df + \lambda g, \tag{1.4}$$

where$0 < m \leq \infty$ is an integer, ∇^2 and $\otimes$ indicate the Hessian of g and tensor product, respectively. Here f denotes the m-quasi Einstein potential function of the m-quasi Einstein soliton [1]. The expression $S + \nabla^2 f - \frac{1}{m} df \otimes df$ is the Bakry-Emery Ricci tensor, which is proportional to the metric g [23]. The trace of (1.4) is given by

$$r + \nabla f - \frac{1}{m} \mid \nabla f \mid^2 = n\lambda.$$

If λ is constant and $m = \infty$, then (1.4) reduces to a gradient Ricci soliton. Also, the metric becomes almost gradient Ricci soliton if λ is a smooth function and $m = \infty$. A gradient generalized m-quasi Einstein metric in a semi-Riemannian manifold becomes gradient m-quasi Einstein if m is a positive integer and λ is constant. He et al. [16] presented some classifications of m-quasi Einstein metrics on Einstein product manifold with a non-empty base. In this connection, the properties of m-quasi Einstein solitons in different geometrical structures have been studied (in details) by ([13], [14], [17]) and others.

Yamabe solitons on three-dimensional Sasakian manifolds and Kenmotsu manifolds were investigated respectively by Sharma [20] and Wang [22]. Also, in a recent paper Erken [12] studied Yamabe solitons in normal almost paracontact metric manifolds. Motivated from the above studies, we make the contribution to investigate gradient Yamabe, gradient Einstein and gradient m-quasi Einstein solitons, respectively in a 3-dimensional para-Sasakian manifold. Precisely, we establish the accompanying outcomes:

Theorem 1.1. *Let the semi-Riemannian metric of a three-dimensional para-Sasakian manifold M^3 be the gradient Yamabe soliton. Then either the manifold is locally isometric to the hyperbolic space $H^3(-1)$ or the gradient Yamabe soliton is trivial.*

Theorem 1.2. *Let the semi-Riemannian metric of a three-dimensional para-Sasakian manifold M^3 be the gradient Einstein metric. Then either M^3 is η-Einstein or a manifold of constant sectional curvature.*

Theorem 1.3. *Let the semi-Riemannian metric of a three-dimensional para-Sasakian manifold M^3 be the closed gradient m-quasi Einstein metric. Then the manifold is locally isometric to the hyperbolic space $H^3(-1)$. Moreover, either the m-quasi Einstein potential vector field is pointwise collinear with ξ or $\lambda = m - 2$.*

2 Para-Sasakian manifolds

In this section, we gather the formulas and results of the para-Sasakian manifold, which will be required in later chapters. To know more facts about paracontact metric geometry, we may refer to ([4], [18]) and references therein. Several years ago, the notion of paracontact metric structures was presented in [18]. Since the publication of [4], paracontact metric manifolds have been pointed out by numerous authors. The significance of paracontact geometry has been indicated minutely in the previous years by various papers illuminating the exchanges with the theory of para-Kähler manifolds and its appearance in pseudo-Riemannian geometry([11], [12]).

Let M be a $(2n+1)$-dimensional smooth differentiable manifold endowed with a vector field ξ, a $(1,1)$-type tensor field ϕ, and a 1-form η such that

$$\phi^2\bar{U} = \bar{U} - \eta(\bar{U})\xi, \;\; \eta(\phi\bar{U}) = 0, \;\; \phi\xi = 0, \;\; \eta(\xi) = 1 \tag{2.1}$$

hold for all vector field $\bar{U}$ on M, and the almost paracomplex structure on each fibre of $\mathcal{D} = ker\,\eta$ is induced by the tensor field ϕ. In other words, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have the equal dimension. Then the triplet (ϕ, ξ, η) satisfying (2.1) is named as an almost paracontact structure and the manifold M is an almost paracontact manifold. In addition, if a semi-Riemannian metric g of M satisfies

$$g(\xi, \bar{U}) = \eta(\bar{U}), \quad g(\bar{U}, \bar{V}) + g(\phi\bar{U}, \phi\bar{V}) = \eta(\bar{U})\eta(\bar{V}) \tag{2.2}$$

for all vector fields $\bar{U}$ and $\bar{V}$ on M, then the quadruple (ϕ, ξ, η, g) is claimed to be an *apm*-structure and M an *apm*-manifold [19].

The Nijenhuis torsion is defined by

$$[\phi, \phi](\bar{U}, \bar{V}) = [\phi\bar{U}, \phi\bar{V}] + \phi^2[\bar{U}, \bar{V}] - \phi[\bar{U}, \phi\bar{V}] - \phi[\phi\bar{U}, \bar{V}]$$

for all $\bar{U},\ \bar{V} \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields of M. The almost paracontact manifold is called normal if $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes. The fundamental 2-form of the *apm*-manifold is defined by $\Phi(\bar{U}, \bar{V}) = g(\bar{U}, \phi\bar{V})$. If $d\eta(\bar{U}, \bar{V}) = g(\bar{U}, \phi\bar{V})$, then the manifold M endowed with structure (ϕ, ξ, η, g) is known as a paracontact metric manifold.

A symmetric trace-free operator $h = \frac{1}{2}\pounds_\xi\phi$ in a paracontact manifold satisfies $h\xi = 0$ and

$$\nabla_{\bar{U}}\xi = -\phi\bar{U} + \phi h\bar{U}, \;\; \forall\ \bar{U} \in \mathfrak{X}(M).$$

It is to be noted that ξ being Killing is equivalent to the condition $h = 0$ and (ϕ, ξ, η, g) is called K-paracontact structure. If the normality condition is satisfied in a paracontact metric manifold, then it is called a para-Sasakian manifold. It is well circulated that every para-Sasakian manifold is necessarily K-paracontact. The converse is not true, in general, but it holds when the manifold is of dimension three [5].

In a para-Sasakian manifold the subsequent relations hold :

$$R(\bar{U}, \bar{V})\xi = \eta(\bar{U})\bar{V} - \eta(\bar{V})\bar{U}, \tag{2.3}$$

$$(\nabla_{\bar{U}}\phi)\bar{V} = -g(\bar{U}, \bar{V})\xi + \eta(\bar{V})\bar{U}, \tag{2.4}$$

$$\nabla_{\bar{U}}\xi = -\phi\bar{U}, \tag{2.5}$$

$$R(\bar{U},\xi)\bar{V} = g(\bar{U},\bar{V})\xi - \eta(\bar{V})\bar{U}, \tag{2.6}$$

$$S(\bar{U},\xi) = -(n-1)\eta(\bar{U}),\ Q\xi = -(n-1)\xi, \tag{2.7}$$

for any vector fields $\bar{U},\bar{V}$ where Q is the Ricci operator, i.e., $g(Q\bar{U},\bar{V}) = S(\bar{U},\bar{V})$ of the manifold.

The Riemannain curvature tensor of a semi-Riemannian manifold M^3 is written by

$$\begin{aligned} R(\bar{U},\bar{V})\bar{Z} &= g(\bar{V},\bar{Z})Q\bar{U} - g(\bar{U},\bar{Z})Q\bar{V} + S(\bar{V},\bar{Z})\bar{U} - S(\bar{U},\bar{Z})\bar{V} \\ &\quad -\frac{r}{2}[g(\bar{V},\bar{Z})\bar{U} - g(\bar{U},\bar{Z})\bar{V}], \end{aligned} \tag{2.8}$$

for any vector fields $\bar{U},\bar{V},\bar{Z}$. Replacing $\bar{V}=\bar{Z}=\xi$ in the foregoing equation and utilizing (2.3) and (2.7) we obtain (see [12])

$$Q\bar{U} = \frac{1}{2}[(r+2)\bar{U} - (r+6)\eta(\bar{U})\xi]. \tag{2.9}$$

In view of (2.9) the Ricci tensor is written as

$$S(\bar{U},\bar{V}) = \frac{1}{2}[(r+2)g(\bar{U},\bar{V}) - (r+6)\eta(\bar{U})\eta(\bar{V})]. \tag{2.10}$$

Now before introducing the detailed proof of our main theorems, we first state the following result [12]:

Lemma 2.1. *In a para-Kenmotsu manifold* (M^3,ϕ,ξ,η,g)*, we have*

$$\xi r = 0. \tag{2.11}$$

3 Proof of the main Theorems

Proof of Theorem 1.1. Let us consider a gradient Yamabe soliton on a 3-dimensional para-Sasakian manifold. Then from (1.2) we obtain

$$\nabla_{\bar{U}}Df = (r-\lambda)\bar{U}, \tag{3.1}$$

from which we get

$$R(\bar{U},\bar{V})Df = dr(\bar{V})\bar{U} - dr(\bar{U})\bar{V}. \tag{3.2}$$

The contraction of above equation along $\bar{U}$ gives

$$S(\bar{V},Df) = 2dr(\bar{V}). \tag{3.3}$$

Now, from (2.10) we infer that

$$S(\bar{U}, Df) = (\frac{r}{2} + 1)(\bar{U}f) - (\frac{r}{2} + 3)\eta(\bar{U})(\xi f). \tag{3.4}$$

Equation (3.3) and (3.4) together reveal that

$$2dr(\bar{U}) = (\frac{r}{2} + 1)(\bar{U}f) - (\frac{r}{2} + 3)\eta(\bar{U})(\xi f). \tag{3.5}$$

Putting $\bar{U} = \xi$ and using (2.11), we have

$$(\xi f) = 0. \tag{3.6}$$

Hence, using (3.6) in (3.5), we get

$$2dr(\bar{U}) = (\frac{r+2}{2})(\bar{U}f). \tag{3.7}$$

Now, from (3.2) we infer that

$$g(R(\bar{U}, \bar{V})\xi, Df) = dr(\bar{U})\eta(\bar{V}) - dr(\bar{V})\eta(\bar{U}). \tag{3.8}$$

Again (2.3) implies that

$$g(R(\bar{U}, \bar{V})\xi, Df) = \eta(\bar{U})(\bar{V}f) - \eta(\bar{V})(\bar{U}f). \tag{3.9}$$

Combining equation (3.8) and (3.9) reveal that

$$dr(\bar{U})\eta(\bar{V}) - dr(\bar{V})\eta(\bar{U}) = \eta(\bar{U})(\bar{V}f) - \eta(\bar{V})(\bar{U}f). \tag{3.10}$$

Setting $\bar{V} = \xi$ in the foregoing equation gives

$$\bar{U}r = -(\bar{U}f). \tag{3.11}$$

Utilizing (3.11) in (3.7) we infer that

$$(r + 6)(\bar{U}f) = 0. \tag{3.12}$$

This shows that either $r = -6$ or $f = constant$. Next, we consider the above two cases as follows.

Case i: If r=-6 but $f \neq constant$, then from (2.10) we get $S = -2g$, that is the manifold is an Einstein manifold and hence from (2.8) it follows that the manifold is of constant sectional curvature -1. Hence the manifold is locally isometric to the hyperbolic space $H^3(-1)$.

Case ii: If $f = constant$ but $r \neq -6$, then the gradient Yamabe soliton is trivial. This completes the proof. □

Proof of Theorem 1.2. Let us suppose that the semi-Riemannian metric of a 3-dimensional para-Sasakian manifold is a gradient Einstein metric. Then from (1.3) we obtain

$$\nabla_{\bar{U}} Df = (\lambda + \frac{r}{2})\bar{U} - Q\bar{U}, \tag{3.13}$$

from which using (2.9), we get

$$\nabla_{\bar{U}} Df = (\lambda + 1)\bar{U} + (\frac{r}{2} + 3)\eta(\bar{U})\xi, \tag{3.14}$$

Now, from (3.14) we infer that

$$R(\bar{V}, \bar{U})Df = \frac{dr(\bar{V})}{2}\eta(\bar{U})\xi - \frac{dr(\bar{U})}{2}\eta(\bar{V})\xi + (\frac{r}{2} + 3)[-\eta(\bar{U})\phi\bar{V} + \eta(\bar{V})\phi\bar{U}]. \tag{3.15}$$

The contraction of above equation along $\bar{V}$ and using (2.11), gives

$$S(\bar{U}, Df) = -\frac{dr(\bar{U})}{2}. \tag{3.16}$$

Equation (3.4) and (3.16) together reveal that

$$-\frac{dr(\bar{U})}{2} = (\frac{r}{2} + 1)(\bar{U}f) - (\frac{r}{2} + 3)\eta(\bar{U})(\xi f). \tag{3.17}$$

Putting $\bar{U} = \xi$ and using (2.11), we have

$$(\xi f) = 0. \tag{3.18}$$

Hence, using (3.18) in (3.17), we get

$$-\frac{dr(\bar{U})}{2} = (\frac{r}{2} + 1)(\bar{U}f). \tag{3.19}$$

Now, from (3.15) we infer that

$$g(R(\bar{V}, \bar{U})Df, \xi) = \frac{dr(\bar{V})}{2}\eta(\bar{U}) - \frac{dr(\bar{U})}{2}\eta(\bar{V}). \tag{3.20}$$

Combining equation (3.9) and (3.20) reveal that

$$\frac{dr(\bar{U})}{2}\eta(\bar{V}) - \frac{dr(\bar{V})}{2}\eta(\bar{U}) = \eta(\bar{U})(\bar{V}f) - \eta(\bar{V})(\bar{U}f). \tag{3.21}$$

Putting $\bar{U} = \xi$ in the above equation gives

$$-\frac{dr(\bar{V})}{2} = (\bar{V}f). \tag{3.22}$$

Utilizing the foregoing equation in (3.19), yields that

$$\bar{U}f = (\frac{r}{2} + 1)(\bar{U}f). \tag{3.23}$$

This shows that either $r = 0$ or $f = constant$. Next, we consider the above two cases as follows.

Case i: If r=0 but $f \neq constant$, then we get from (2.10) that the manifold is η-Einstein.

Case ii: If $f = constant$ but $r \neq 0$, then we get from (3.13) that the manifold is an Einstein manifold. Since the manifold is under consideration of dimension 3, hence the manifold is of constant sectional curvature.

This finishes the proof. □

Proof of Theorem 1.3. Let us assume that the semi-Riemannian metric of a 3-dimensional para-Sasakian manifold is a gradient m-quasi Einstein metric. Then generalized the equation (1.4) by considering a 1-form $\bar{W}^b(\bar{U})$ instead of df, we get

$$S(\bar{U},\bar{V}) + \frac{1}{2}(\pounds_{\bar{W}} g)(\bar{U},\bar{V}) - \frac{1}{m}\bar{W}^b(\bar{U})\bar{W}^b(\bar{V}) = \lambda g(\bar{U},\bar{V}), \tag{3.24}$$

where $\bar{W}$ is the m-quasi Einstein potential function. If $\bar{W}^b$ is closed, then

$$g(\nabla_{\bar{U}}\bar{W},\bar{V}) = g(\nabla_{\bar{V}}\bar{W},\bar{U}).$$

Then equation (3.24) transforms to

$$Q\bar{U} + \nabla_{\bar{U}}\bar{W} - \frac{1}{m}\bar{W}^b(\bar{U})\bar{W} = \lambda\bar{U}. \tag{3.25}$$

Now, using (2.9) the above equation reduces to

$$\nabla_{\bar{U}}\bar{W} = (\lambda - \frac{r}{2} - 1)\bar{U} + (\frac{r}{2} + 3)\eta(\bar{U})\xi + \frac{1}{m}\bar{W}^b(\bar{U})\bar{W}. \tag{3.26}$$

Taking covariant derivative of (3.26) along the vector field $\bar{V}$, we get

$$\begin{aligned}\nabla_{\bar{V}}\nabla_{\bar{U}}\bar{W} &= (\lambda - \frac{r}{2} - 1)\nabla_{\bar{V}}\bar{U} + \frac{dr(\bar{V})}{2}\{\eta(\bar{U})\xi - \bar{U}\}\\ &+(\frac{r}{2} + 3)\{\nabla_{\bar{V}}\eta(\bar{U})\xi + \eta(\bar{U})\nabla_{\bar{V}}\xi\}\\ &+\frac{1}{m}\nabla_{\bar{V}}\bar{W}^b(\bar{U})\bar{W} + +\frac{1}{m}\bar{W}^b(\bar{U})\nabla_{\bar{V}}\bar{W}.\end{aligned} \tag{3.27}$$

Interchanging $\bar{U}$ and $\bar{V}$ in (3.27), we lead

$$\begin{aligned}\nabla_{\bar{U}}\nabla_{\bar{V}}\bar{W} &= (\lambda - \frac{r}{2} - 1)\nabla_{\bar{U}}\bar{V} + \frac{dr(\bar{U})}{2}\{\eta(\bar{V})\xi - \bar{V}\}\\ &+(\frac{r}{2} + 3)\{\nabla_{\bar{U}}\eta(\bar{V})\xi + \eta(\bar{V})\nabla_{\bar{U}}\xi\}\\ &+\frac{1}{m}\nabla_{\bar{U}}\bar{W}^b(\bar{V})\bar{W} + +\frac{1}{m}\bar{W}^b(\bar{V})\nabla_{\bar{U}}\bar{W}.\end{aligned} \tag{3.28}$$

Equations (3.26)-(3.28) and the symmetric property of Levi-Civita connection together with $R(\bar{U},\bar{V})\bar{W} = \nabla_{\bar{U}}\nabla_{\bar{V}}\bar{W} - \nabla_{\bar{V}}\nabla_{\bar{U}}\bar{W} - \nabla_{[\bar{U},\bar{V}]}\bar{W}$ we lead

$$\begin{aligned}R(\bar{U},\bar{V})\bar{W} &= \frac{dr(\bar{U})}{2}\{\bar{V} - \eta(\bar{V})\xi\} - \frac{dr(\bar{V})}{2}\{\bar{U} - \eta(\bar{U})\xi\}\\ &+(\frac{r}{2} + 3)\{\eta(\bar{V})\phi\bar{U} - \eta(\bar{U})\phi\bar{V} - 2g(\phi\bar{U},\bar{V})\}\\ &+\frac{1}{m}[(\lambda - \frac{r}{2} - 1)\{\bar{W}^b(\bar{V})\bar{U} - \bar{W}^b(\bar{U})\bar{V}\}\\ &+(\frac{r}{2} + 3)\{\bar{W}^b(\bar{V})\eta(\bar{U})\xi - \bar{W}^b(\bar{U})\eta(\bar{V})\xi\}].\end{aligned} \tag{3.29}$$

Taking inner product of (3.29) with ξ, we have

$$\begin{aligned}g(R(\bar{U},\bar{V})\bar{W},\xi) &= (r+6)g(\bar{U},\phi\bar{V})\\ &+\frac{\lambda+2}{m}\{\bar{W}^b(\bar{V})\eta(\bar{U}) - \bar{W}^b(\bar{U})\eta(\bar{V})\}.\end{aligned} \tag{3.30}$$

Now, from (2.3) we infer that

$$g(R(\bar{U},\bar{V})\xi,\bar{W}) = \eta(\bar{U})g(\bar{V},\bar{W}) - \eta(\bar{V})g(\bar{U},\bar{W}). \tag{3.31}$$

Combining equation (3.30) and (3.31) reveal that

$$\frac{m-\lambda-2}{m}\{\bar{W}^b(\bar{V})\eta(\bar{U}) - \bar{W}^b(\bar{U})\eta(\bar{V})\} = (r+6)g(\phi\bar{U},\bar{V}). \tag{3.32}$$

Replacing $\bar{U}=$ by $\phi\bar{U}$, $\bar{V}$ by $\phi\bar{V}$ in the onward equation gives

$$(r+6)d\eta(\bar{U},\bar{V}) = 0. \tag{3.33}$$

Since $d\eta$ is non-vanishing in para-Sasakian manifolds, then $r=-6$. Then from (2.10) we get $S=-2g$, that is the manifold is an Einstein manifold and hence from (2.8) it follows that the manifold is of constant sectional curvature -1. Hence the manifold is locally isometric to the hyperbolic space $H^3(-1)$.

Putting the value of r in (3.32), we have

$$\frac{m-\lambda-2}{m}\{\bar{W}^b(\bar{V}) - \eta(\bar{W})\eta(\bar{V})\} = 0. \tag{3.34}$$

This shows that either $\bar{W}=\eta(\bar{W})\xi$ or $\lambda = m-2$.
This put an end on the Proof.

□

We know that when λ is constant and $m=\infty$, then gradient m-quasi Einstein metric reduces to a gradient Ricci soliton. Therefore, from (3.32) we get $r=-6$ which implies that the manifold is of constant sectional curvature -1. Hence the manifold is locally isometric to the hyperbolic space $H^3(-1)$. Thus, we can state:

Corollary 3.1. *Let the semi-Riemannian metric of a three-dimensional para-Sasakian manifold M^3 be the gradient Ricci soliton. Then the manifold is locally isometric to the hyperbolic space $H^3(-1)$.*

4 Example

Here we consider an example of the paper [12]. In this paper the author considers the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$ and the vector fields

$$\phi\bar{e_2} = \bar{e_1} = 2y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad \phi\bar{e_1} = \bar{e_2} = \frac{\partial}{\partial y}, \quad \xi = \bar{e_3} = \frac{\partial}{\partial x}$$

and shows that the manifold is a para-Sasakian manifold. Also the author has obtained the expressions of the curvature tensor and the Ricci tensor respectively as follows:

$$R(\bar{e_1},\bar{e_2})\xi = 0, \quad R(\bar{e_2},\xi)\xi = -\bar{e_2}, \quad R(\bar{e_1},\xi)\xi = -\bar{e_1},$$

$$R(\bar{e_1},\bar{e_2})\bar{e_2} = -3\bar{e_1}, \quad R(\bar{e_2},\xi)\bar{e_2} = -\xi, \quad R(\bar{e_1},\xi)\bar{e_2} = 0,$$

$$R(\bar{e_1},\bar{e_2})\bar{e_1} = -3\bar{e_2}, \quad R(\bar{e_2},\xi)\bar{e_1} = 0, \quad R(\bar{e_1},\xi)\bar{e_1} = \xi$$

and

$$\begin{aligned} S(\bar{e_1},\bar{e_1}) &= -g(R(\bar{e_1},\bar{e_2})\bar{e_2},\bar{e_1}) + g(R(\bar{e_1},\bar{e_3})\bar{e_3},\bar{e_1}) \\ &= 2 \\ &= 2g(\bar{e_1},\bar{e_1}). \end{aligned}$$

Similarly, we have

$$S(\bar{e_2},\bar{e_2}) = 2g(\bar{e_2},\bar{e_2}) \quad and \quad S(\bar{e_3},\bar{e_3}) = 2g(\bar{e_3},\bar{e_3}).$$

Therefore,

$$r = S(\bar{e_1},\bar{e_1}) - S(\bar{e_2},\bar{e_2}) + S(\xi,\xi) = 2.$$

Further $r = 2 \neq -6$. Then for $\lambda = -2$, we get from (3.1) that the function f is constant. Then the gradient Yamabe soliton is trivial. Thus the **Theorem 1.1.** is verified.

References

[1] A. Barros and J. N. Gomes, *A compact gradient generalized quasi-Einstein metric with constant scalar curvature*, J. Math. Analysis and Applications, **401** (2013), 702-705.

[2] A. M. Blaga, *A note on warped product almost quasi-Yamabe solitons*, Filomat, **33** (2019), 2009-2016.

[3] A. M. Blaga, *Some Geometrical Aspects of Einstein, Ricci and Yamabe solitons*, J. Geom. symmetry Phys. **52** (2019), 17-26.

[4] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1-13.

[5] Calvaruso, G., *Homogeneous paracontact metric three-manifolds*, III. J. Math. **55** (2011), 697-718.

[6] Catino, G. and Mazzieri, L., *Gradient Einstein-solitons*, arXiv:1201.6620.

[7] K. De and U.C. De, *A note on gradient Solitons on para-Kenmotsu manifolds*, Int. J.Geom. Methods Mod. Phys. **18**, no. 01, (2021),2150007 (11 pages).

[8] K. De and U.C. De, *Almost quasi-Yamabe solitons and gradient almost quasi-Yamabe solitons in paracontact geometry*, quaestiones mathematicae, DOI: 10.2989/16073606.2020.1799882.

[9] K. De and U.C. De, *k-Almost Yamabe Solitons On Kenmotsu Manifolds* , Honam Math. J. **43** (2021), 115-122.

[10] S. Deshmukh and B.Y. Chen, *A note on Yamabe solitons*, Balkan. J. Geo. Appl. **23**(2018), 37-43.

[11] I.K. Erken and C. Murathan, *A complete study of three-dimensional paracontact (κ,μ,ν)-spaces*, Int. J.Geom. Methods Mod. Phys. (2017). https://doi.org/10.1142/S0219887817501067.

[12] I.K. Erken, *Yamabe solitons on three-dimensional normal almost paracontact metric manifolds*, Periodica Mathematica Hungarica https://doi.org/10.1007/s10998-019-00303-3.

[13] A. Ghosh, *Generalized m-quasi-Einstein metric within the framework of Sasakian and Kcontact manifolds*, Ann. Polonici Math. **115** (2015), 33-41.

[14] A. Ghosh, *m-quasi-Einstein metric and contact geometry*, RACSAM **113** (2019), 2587–2600. https://doi.org/10.1007/s13398-019-00642-3

[15] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity* (Santa Cruz, CA, 1986), 237–262, Contemp. Math. **71**, American Math. Soc., 1988.

[16] C. He, P. Petersen and W. Wylie, *On the classification of warped product Einstein metrics*, Comm. Anal. Geom. **20** (2012) 271-312.

[17] Z. Hu, D. Li and J. Xu, *On generalized m-quasi-Einstein manifolds with constant scalar curvature*, J. Math. Anal. Appl. 432 (2015), 733-743.

[18] S. Kaneyuki and F.L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math.J. **99**(1985), 173-187 .

[19] I. Satō, *On a structure similar to the almost contact structure*, Tensor (N.S.) **30**(1976), 219-224.

[20] R. Sharma, *A 3-dimensional Sasakian metric as a Yamabe soliton.* Int. J. Geom. Methods Mod. Phys. 9, 1220003 (2012).

[21] Y. J. Suh and U.C. De, *Yamabe solitons and Ricci solitons on almost co-Kähler Manifolds*, Canad. Math. Bull., **62** (2019), 653-661.

[22] Y. Wang, *Yamabe solitons on three dimensional Kenmotsu manifolds*, Bull. Belg. Math. Soc. **23** (2016), 345-355.

[23] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, Journal of Differential Geometry, **83**(2009), 337-405. doi:10.4310/jdg/1261495336.

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