On special weakly symmetric Riemannian manifolds

M. Ali, Q. Khan and M. Vasiulla

Abstract. In the present paper, we have investigated some geometric properties of special weakly symmetric Riemannian manifold. Besides these, we have obtained some interesting and fruitful results on it. Finally, the existence of special weakly symmetric Riemannian manifolds have been shown by a non-trivial example.

M.S.C. 2010: 53C15, 53C21, 53C25.
Key words: symmetric Riemannian manifold; Einstein manifold; conharmonic curvature tensor, W_2-curvature tensor; Ricci tensor; Codazzi type Ricci tensor.

1 Introduction

The notion of a weakly symmetric Riemannian manifold have been introduced and studied by L. Tamassy and T. Q. Binh ([1], [2]). Recently, Singh and Khan [3] introduced the notion of special weakly symmetric Riemannian manifolds. Let (M^n, g) be an n-dimensional Riemannian manifold and $\chi(M)$ denotes the set of differentiable vector fields on (M^n, g). Let K(X,Y,Z) be the Riemannian curvature tensor of type (1,3) for X,Y,Z $\in \chi(M)$. A non-flat Riemannian manifold (M^n, g), (n≥2) is called a special weakly symmetric Riemannian manifold [3] if the curvature tensor K of type (1,3) satisfies the condition:

$$\begin{aligned}(\nabla_U K)(X,Y,Z) =& 2A(U)K(X,Y,Z) + A(X)K(U,Y,Z) + A(Y)K(X,U,Z)\\ &+ A(Z)K(X,Y,U),\end{aligned} \tag{1.1}$$

where A is non-zero 1-form, P is associated vector vector field such that

$$A(X) = g(X,P), \tag{1.2}$$

for every vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g.
The conharmonic curvature tensor N is given by [4]

$$\begin{aligned}N(X,Y,Z) =& K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y\\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)],\end{aligned} \tag{1.3}$$

BSG Proceedings, vol. 28 (2021), pp. 1-11.

here R is the Ricci tensor of type (1,1), defined by

$$Ric(X,Y) = g(R(X),Y). \tag{1.4}$$

In 1970, Pokhariyal and Mishra [11] introduced and studied a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ denoted by W_2 and defined by

$$W_2(X,Y,Z) = K(X,Y,Z) + \frac{1}{n-1}[g(X,Z)R(Y) - g(Y,Z)R(X)]. \tag{1.5}$$

Such a tensor is known as W_2- curvature tensor. The W_2- curvature tensor have also been studied by various authors such as De and Sarkar [5], Ianus and Mihai [6], Pokhariyal ([7], [8]), Shaikh, Jana and Eyasmin [10], yildiz and De [9] and many others.
From (1.5), we can define a (0,4) type W_2- curvature tensor as follows:

$$\begin{aligned}\tilde{W_2}(X,Y,Z,V) =& \tilde{K}(X,Y,Z,V) + \frac{1}{n-1}[g(X,Z)Ric(Y,V) \\ & - g(Y,Z)Ric(X,V)],\end{aligned} \tag{1.6}$$

where $\tilde{K}$ denotes the Riemannian curvature tensor of type (0,4) defined by

$$\tilde{K}(X,Y,Z,V) = g(K(X,Y,Z),V)$$

and

$$\tilde{W_2}(X,Y,Z,V) = g(W_2(X,Y,Z),V).$$

Let

$$P(Y,Z) = \sum_{i=1}^{n} \tilde{W_2}(e_i, Y, Z, e_i), \tag{1.7}$$

then from (1.6), we get

$$P(Y,Z) = \frac{n}{n-1}Ric(Y,Z) - \frac{r}{n-1}g(Y,Z), \tag{1.8}$$

where r is the scalar curvature.
If the conharmonic curvature tensor N and W_2-curvature tensor W_2 satisfy the conditions:

$$\begin{aligned}(\nabla_U N)(X,Y,Z) =& 2A(U)N(X,Y,Z) + A(X)N(U,Y,Z) \\ & + A(Y)N(X,U,Z) + A(Z)N(X,Y,U)\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}(\nabla_U W_2)(X,Y,Z) =& 2A(U)W_2(X,Y,Z) + A(X)W_2(U,Y,Z) \\ & + A(Y)W_2(X,U,Z) + A(Z)W_2(X,Y,U),\end{aligned} \tag{1.10}$$

respectively. Such types of manifolds are known as special weakly conharmonically symmetric Riemannian manifold and special weakly W_2- symmetric Riemannian manifold, respectively.

A Riemannian manifold (M^n, g) is called special weakly Ricci symmetric Riemannian manifold if its Ricci tensor Ric of type (0,2) is not identically zero and satisfies the condition [3]

$$(\nabla_U Ric)(X,Y) = 2A(U)Ric(X,Y) + A(X)Ric(U,Y) + A(Y)Ric(X,U), \tag{1.11}$$

where A is non-zero 1-form which is stated earlier.
A Riemannian manifold is said to be an Einstein manifold [8] if

$$Ric(Y,Z) = \frac{r}{n}g(Y,Z). \tag{1.12}$$

From (1.12), we have

$$dr(X) = 0 \quad and \quad (\nabla_X Ric)(Y,Z) = 0, \tag{1.13}$$

also

$$R(X) = \frac{r}{n}X. \tag{1.14}$$

In section 2 and 3, we have obtained some interesting and fruitful results on special weakly symmetric Riemannian manifold and special weakly Ricci symmetric Riemannian manifold, respectively. In the last section, we have proved the existence of special weakly symmetric Riemannian manifold by a non-trivial example.

2 Special weakly symmetric Riemannian manifolds

Taking covariant derivative of (1.3) with respect to U and then using (1.9), we get

$$\begin{aligned}
&(\nabla_U K)(X,Y,Z) - \frac{1}{n-2}[(\nabla_U Ric)(Y,Z)X - (\nabla_U Ric)(X,Z)Y \\
&+ g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y)] \\
&= 2A(U)N(X,Y,Z) + A(X)N(U,Y,Z) \\
&+ A(Y)N(X,U,Z) + A(Z)N(X,Y,U).
\end{aligned} \tag{2.1}$$

By virtue of (1.3), the relation (2.1) reduces to

$$\begin{aligned}
&(\nabla_U K)(X,Y,Z) - 2A(U)K(X,Y,Z) - A(X)K(U,Y,Z) \\
&- A(Y)K(X,U,Z) - A(Z)K(X,Y,U) \\
&= \frac{1}{n-2}[(\nabla_U Ric)(Y,Z)X - (\nabla_U Ric)(X,Z)Y \\
&+ g(Y,Z)(\nabla_U R)(X) - g(X,Z)(\nabla_U R)(Y) \\
&- 2A(U)\{Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)R(X) - g(X,Z)R(Y)\} \\
&- A(X)\{Ric(Y,Z)U - Ric(U,Z)Y + g(Y,Z)R(X) - g(U,Z)R(Y)\} \\
&- A(Y)\{Ric(U,Z)X - Ric(X,Z)U + g(U,Z)R(X) - g(X,Z)R(U)\} \\
&- A(Z)\{Ric(Y,U)X - Ric(X,U)Y + g(Y,U)R(X) - g(X,U)R(Y)\}].
\end{aligned} \tag{2.2}$$

Permuting (2.2) twice with respect to X,Y,U; adding the three obtained equations and using Bianchi's first and second identities, we have

$$
\begin{aligned}
&2A(U)K(X,Y,Z)+2A(X)K(Y,U,Z)+2A(Y)K(U,X,Z)\\
&+A(X)K(U,Y,Z)+A(U)K(Y,X,Z)+A(Y)K(X,U,Z)\\
&+A(Y)K(X,U,Z)+A(X)K(U,Y,Z)+A(U)K(Y,X,Z)\\
&+\frac{1}{n-2}[(\nabla_U Ric)(Y,Z)X+(\nabla_Y Ric)(X,Z)U+(\nabla_X Ric)(U,Z)Y\\
&-(\nabla_U Ric)(X,Z)Y-(\nabla_X Ric)(Y,Z)U-(\nabla_Y Ric)(U,Z)X\\
&+g(Y,Z)(\nabla_U R)(X)+g(U,Z)(\nabla_X R)(Y)+g(X,Z)(\nabla_Y R)(U)\\
&-g(X,Z)(\nabla_U R)(Y)-g(U,Z)(\nabla_Y R)(X)-g(Y,Z)(\nabla_X R)(U)\\
&-2A(U)\{Ric(Y,Z)X-Ric(X,Z)Y+g(Y,Z)R(X)-g(X,Z)R(Y)\}\\
&-2A(X)\{Ric(U,Z)Y-Ric(Y,Z)U+g(U,Z)R(Y)-g(Y,Z)R(U)\}\\
&-2A(Y)\{Ric(X,Z)U-Ric(U,Z)X+g(X,Z)R(U)-g(U,Z)R(X)\}\\
&-A(X)\{Ric(Y,Z)U-Ric(U,Z)Y+g(Y,Z)R(U)-g(U,Z)R(Y)\}\\
&-A(Y)\{Ric(U,Z)X-Ric(X,Z)U+g(U,Z)R(X)-g(X,Z)R(U)\}\\
&-A(U)\{Ric(X,Z)Y-Ric(Y,Z)X+g(X,Z)R(Y)-g(Y,Z)R(X)\}\\
&-A(Y)\{Ric(U,Z)X-Ric(X,Z)U+g(U,Z)R(X)-g(X,Z)R(U)\}\\
&-A(U)\{Ric(X,Z)Y-Ric(Y,Z)X+g(X,Z)R(Y)-g(Y,Z)R(X)\}\\
&-A(X)\{Ric(Y,Z)U-Ric(U,Z)Y+g(Y,Z)R(U)-g(U,Z)R(Y)\}\\
&-A(Z)\{Ric(Y,U)X-Ric(X,U)Y+g(Y,U)R(X)-g(X,U)R(Y)\\
&+Ric(U,X)Y-Ric(U,Y)X+g(U,X)R(Y)-g(U,Y)R(X)\\
&+Ric(X,Y)U-Ric(Y,X)U+g(X,Y)R(U)-g(Y,X)R(U)\}]=0.
\end{aligned}
\tag{2.3}
$$

Using symmetric properties of Ricci tensor and the skew- symmetric properties of curvature tensor in (2.3), we get

$$
\begin{aligned}
&(\nabla_U Ric)(Y,Z)X+(\nabla_Y Ric)(X,Z)U+(\nabla_X Ric)(U,Z)Y\\
&-(\nabla_U Ric)(X,Z)Y-(\nabla_X Ric)(Y,Z)U-(\nabla_Y Ric)(U,Z)X\\
&+g(Y,Z)(\nabla_U R)(X)+g(U,Z)(\nabla_X R)(Y)+g(X,Z)(\nabla_Y R)(U)\\
&-g(X,Z)(\nabla_U R)(Y)-g(U,Z)(\nabla_Y R)(X)-g(Y,Z)(\nabla_X R)(U)=0.
\end{aligned}
\tag{2.4}
$$

Contracting (2.4) with respect to U, we get

$$
\begin{aligned}
&(\nabla_X Ric)(Y,Z)+n(\nabla_Y Ric)(X,Z)+(\nabla_X Ric)(Y,Z)\\
&-(\nabla_Y Ric)(X,Z)-n(\nabla_X Ric)(Y,Z)-(\nabla_Y Ric)(X,Z)\\
&+\frac{1}{2}dr(X)g(Y,Z)+g((\nabla_X R)(Y),Z)+dr(Y)g(X,Z)\\
&-\frac{1}{2}dr(Y)g(X,Z)-g((\nabla_Y R)(X),Z)-dr(X)g(Y,Z)=0.
\end{aligned}
\tag{2.5}
$$

By virtue of (1.4), the relation (2.5), becomes

$$
\begin{aligned}
&(n-3)[(\nabla_Y Ric)(X,Z)-(\nabla_X Ric)(Y,Z)]\\
&\qquad+\frac{1}{2}\{dr(Y)g(X,Z)-dr(X)g(Y,z)\}=0.
\end{aligned}
\tag{2.6}
$$

If the scalar curvature r is constant, then

$$dr(X) = 0. \tag{2.7}$$

By virtue of (2.7), (2.6) yields

$$(\nabla_Y Ric)(X, Z) - (\nabla_X Ric)(Y, Z) = 0.$$

This shows that the Ricci tensor of this manifold is of Codazzi type [13]. Conversely, if the Ricci tensor of this manifold is Codazzi type then from (2.6), the scalar curvature of this manifold is constant. Thus, we are in the position to state the following theorem:

Theorem 2.1. *The scalar curvature of a special weakly conharmonically symmetric Riemannian manifold is non-zero constant if and only if the Ricci tensor of this manifold is of Codazzi type.*

Let m be an arbitrary point on an n-dimensional Riemannian manifold (M^n, g) and $e_i \in \chi(M)$ (i=1,2,...,n) an orthonormal and parallel vector system around m. We consider at m the following relation

$$\begin{aligned}\sum_{i=1}^{n}(\nabla_U \tilde{K})(e_i, Y, Z, e_i) = &\sum_{i=1}^{n} \nabla_U \tilde{K}(e_i, Y, Z, e_i) - \sum_{i=1}^{n} \tilde{K}(\nabla_X e_i, Y, Z, e_i) \\ &- \sum_{i=1}^{n} \tilde{K}(e_i, \nabla_U Y, Z, e_i) - \sum_{i=1}^{n} \tilde{K}(e_i, Y, \nabla_U Z, e_i).\end{aligned} \tag{2.8}$$

Since we have

$$\sum_{i=1}^{n} \tilde{K}(e_i, Y, Z, e_i) = Ric(Y, Z) = Ric(Z, Y) \tag{2.9}$$

and $\nabla_U e_i|m = 0$, then from (2.8), we obtain

$$\begin{aligned}\sum_{i=1}^{n}(\nabla_U \tilde{K})(e_i, Y, Z, e_i) =&\nabla_U Ric(Y, Z) - Ric(\nabla_U Y, Z) - Ric(Y, \nabla_U Z) \\ =& (\nabla_U Ric)(Y, Z).\end{aligned} \tag{2.10}$$

On the other hand we have from (1.1)

$$\begin{aligned}\sum_{i=1}^{n}(\nabla_U \tilde{K})(e_i, Y, Z, e_i) = &\sum_{i=1}^{n}[2A(U)\tilde{K}(e_i, Y, Z, e_i) + A(e_i)\tilde{K}(U, Y, Z, e_i) \\ &+ A(Y)\tilde{K}(e_i, U, Z, e_i) + A(Z)\tilde{K}(e_i, Y, U, e_i)].\end{aligned} \tag{2.11}$$

From (2.9), (2.10) and (2.11), we obtain

$$\begin{aligned}(\nabla_U Ric)(Y, Z) =&2A(U)Ric(Y, Z) + A(K(U, Y, Z)) \\ &+ A(Y)Ric(U, Z) + A(Z)Ric(Y, U).\end{aligned} \tag{2.12}$$

Contracting (2.12) over Y and Z, we get

$$dr(U) = 2A(U)r + Ric(U, \rho) + 2A(R(U)), \tag{2.13}$$

where R is the Ricci tensor of type (1,1) stated in (1.4) and ρ is a vector field defined by

$$A(X) = g(X, \rho). \tag{2.14}$$

Now we suppose that the scalar curvature r of a special weakly symmetric Riemannian manifold to be non-zero constant. Then (2.13) reduces to

$$2A(U)r + Ric(U, \rho) + 2A(R(U)) = 0. \tag{2.15}$$

Using (1.4) and (2.14) in (2.15), we get

$$Ric(U, \rho) = \frac{2r}{3} g(U, \rho). \tag{2.16}$$

This leads to the following:

Theorem 2.2. *In a special weakly symmetric Riemannian manifold, if the scalar curvature is non-zero constant then* $\frac{2r}{3}$*, is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector* ρ *defind in* (2.14).

By virtue of (1.14), the equation (1.5) reduces to

$$W_2(X, Y, Z) = K(X, Y, Z) + \frac{r}{n(n-1)}[g(X, Z)Y - g(Y, Z)X]. \tag{2.17}$$

Taking covariant derivative of (2.17) with respect to U and using (1.13), we get

$$(\nabla_U W_2)(X, Y, Z) = (\nabla_U K)(X, Y, Z). \tag{2.18}$$

By virtue of (2.17) and (2.18), the equation (1.10) reduces to the form

$$\begin{aligned}(\nabla_U K)(X, Y, Z) = 2A(U)[K(X, Y, Z) &+ \frac{r}{n(n-1)}\{g(X, Z)Y - g(Y, Z)X\}] \\ +A(X)[K(U, Y, Z) &+ \frac{r}{n(n-1)}\{g(U, Z)Y - g(Y, Z)U\}] \\ +A(Y)[K(X, U, Z) &+ \frac{r}{n(n-1)}\{g(X, Z)U - g(U, Z)X\}] \\ +A(Z)[K(X, Y, U) &+ \frac{r}{n(n-1)}\{g(X, U)Y - g(Y, U)X\}].\end{aligned}$$

Thus, we can state the following:

Theorem 2.3. *The necessary and sufficient condition for an Einstein special weakly* W_2*-symmetric manifold to be a special weakly symmetric manifold is that*

$$\begin{aligned}&\{2A(U)Y + A(Y)U\}g(X, Z) - \{2A(U)X + A(X)U\}g(Y, Z) \\ &+ \{A(X)Y - A(Y)X\}g(U, Z) + A(Z)g(X, U)Y - A(Z)g(Y, U)X = 0.\end{aligned}$$

Now let the special weakly W_2- symmetric manifold admit a unit parallel vector field Q, that is

$$\nabla_U Q = 0. \tag{2.19}$$

Applying Ricci identity to (2.19), we get

$$K(X, Y, Q) = 0 \tag{2.20}$$

or,

$$\tilde{K}(X, Y, Z, Q) = 0, \tag{2.21}$$

and therefore

$$Ric(X, Q) = 0. \tag{2.22}$$

Using (2.21) and (2.22) in (1.6), we get

$$\tilde{W_2}(X, Y, Z, Q) = 0. \tag{2.23}$$

Using (1.7) in (2.23), we get

$$P(X, Q) = 0. \tag{2.24}$$

Taking an account of (2.24) and the fact that Q is a unit parallel vector field it follows from (1.8) that

$$r = 0. \tag{2.25}$$

Now from (1.7) and (1.10), we have

$$\begin{aligned}
(\nabla_V P)(X, Q) &= \sum_{i=1}^{n} (\nabla_V \tilde{W_2})(e_i, X, Q, e_i) \\
&= \sum_{i=1}^{n} [2A(V)\tilde{W_2}(e_i, X, Q, e_i) + A(e_i)\tilde{W_2}(V, X, Q, e_i) \\
&\quad + A(X)\tilde{W_2}(e_i, V, Q, e_i) + A(Q)\tilde{W_2}(e_i, X, V, e_i) \\
&\quad + A(e_i)\tilde{W_2}(e_i, X, Q, V)].
\end{aligned} \tag{2.26}$$

Using (1.6), (2.19), (2.22), (2.24) and (2.25) the relation (2.26) takes the form

$$A(Q)Ric(X, V) = 0. \tag{2.27}$$

Since $A(Q) \neq 0$, it follows from (2.27) that

$$Ric(X, V) = 0 \tag{2.28}$$

By virtue of equation (2.28), the relation (1.6) gives

$$W_2(X, Y, Z, V) = K(X, Y, Z, V).$$

This leads to the following:

Theorem 2.4. *If a special weakly W_2- symmetric manifold admits a unit parallel vector field, then it is a special weakly symmetric manifold.*

3 Special weakly Ricci symmetric Riemannian manifolds

Let a Riemannian manifold be W_2-flat, then

$$\tilde{W_2}(X,Y,Z,V)=0. \tag{3.1}$$

By virtue of (3.1) the relation (1.6) reduces to

$$\tilde{K}(X,Y,Z,V)=\frac{1}{n-1}[g(Y,Z)Ric(X,U)-g(X,Z)Ric(Y,U)]. \tag{3.2}$$

Taking covariant derivative of (3.2) with respect to U, we get

$$(\nabla_U\tilde{K})(X,Y,Z,V)=\frac{1}{n-1}[g(Y,Z)(\nabla_U Ric)(X,V)-g(X,Z)(\nabla_U Ric)(Y,V)]. \tag{3.3}$$

Permuting twice the vectors X,Y,U; in equation (3.3), then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}&[(\nabla_U Ric)(X,V)-(\nabla_X Ric)(U,V)]g(Y,Z)+[(\nabla_X Ric)(Y,V)\\&-(\nabla_Y Ric)(X,V)]g(U,Z)+[(\nabla_Y Ric)(U,V)-(\nabla_U Ric)(Y,V)]g(X,Z)=0.\end{aligned} \tag{3.4}$$

Using (1.11), in (3.4), we have

$$\begin{aligned}&[A(U)Ric(X,V)-A(X)Ric(U,V)]g(Y,Z)+[A(X)Ric(Y,V)\\&-A(Y)Ric(X,V)]g(U,Z)+[A(Y)Ric(U,V)-A(U)Ric(Y,V)]g(X,Z)=0.\end{aligned} \tag{3.5}$$

Factoring off Z in (3.5), we get

$$\begin{aligned}&[A(U)Ric(X,V)-A(X)Ric(U,V)]Y+[A(X)Ric(Y,V)-A(Y)Ric(X,V)]U\\&+[A(Y)Ric(U,V)-A(U)Ric(Y,V)]X=0.\end{aligned} \tag{3.6}$$

Putting U=V=e_i in (3.6), where $\{e_i, 1\leq i\leq n\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking sum over i, we get

$$[A(R(X))-rA(X)]Y+[nA(X)R(Y)-nA(Y)R(X)]+[rA(Y)-A(R(Y))]X=0 \tag{3.7}$$

where R is Ricci tensor of type (1,1) defined in (1.4).
Contracting (3.7) with respect to Y, we have

$$rA(X)=A(R(X)). \tag{3.8}$$

By virtue of (1.2), the relation (3.8) reduces to

$$g(X,P)r=g(R(X),P). \tag{3.9}$$

Consequently, the above relation gives

$$Xr=R(X). \tag{3.10}$$

Thus, we can state the following:

Theorem 3.1. *If the scalar curvature r is constant in a W_2-flat special weakly Ricci symmetric Riemannian manifold, then the Ricci tensor of type (1,1) must be vanish.*

4 Example of a special weakly symmetric manifold

In this section we prove the existence of a special weakly symmetric Riemannian manifold by constructing a non-trivial example.

We define a Lorentzian manifold $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{4.1}$$

where i,j=1,2,3,4.

In the metric considered the only non-vanishing components of the Christoffel symbols are (see [13])

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -x^1, \quad \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\frac{x^2}{(x^1)^2}, \quad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{x^1}, \quad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \frac{1}{x^2}. \tag{4.2}$$

The non-zero derivatives of (4.2) as follows:

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -1, \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \frac{2x^2}{(x^1)^3}, \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = -\frac{1}{(x^1)^2}. \tag{4.3}$$

The Riemannian curvature tensor as follows:

$$R^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}$$

The non-zero components of (I) are as follows:

$$\begin{aligned} K^1_{212} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -1, \quad K^2_{313} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \frac{2x^2}{(x^1)^3}, \\ K^2_{122} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = -\frac{1}{(x^1)^2} \end{aligned} \tag{4.4}$$

and the non-zero components of (II) are:

$$K^2_{313} = \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 31 \end{Bmatrix}\begin{Bmatrix} 2 \\ 23 \end{Bmatrix} = -\frac{x^2}{(x^1)^3},$$

$$K^2_{112} = \begin{Bmatrix} m \\ 12 \end{Bmatrix}\begin{Bmatrix} 2 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 11 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 11 \end{Bmatrix}\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{1}{(x^1)^2},$$

Adding components corresponding (I) and (II), we get

$$K^1_{212} = -1, \quad K^2_{313} = \frac{x^2}{(x^1)^3} \quad and \quad K^1_{212} = 0.$$

Thus the non-vanishing components of the Riemannian curvature tensor of type (0,4) up to symmetry are,

$$\tilde{K}_{1332} = -\frac{x^2}{x^1}.$$

The derivative of $\tilde{K}$ with respect to first and second variable,

$$\tilde{K}_{1331,1} = \frac{x^2}{(x^1)^2}, \quad \tilde{K}_{1332,2} = -\frac{1}{x^1} \tag{4.5}$$

Let us consider the one forms as follows:

$$A_i(x) = \begin{cases} -\frac{1}{3x^1}, & \text{if } i = 1 \\ -\frac{1}{3x^2}, & \text{if } i = 2 \\ 0, & otherwise \end{cases} \tag{4.6}$$

at any point $x \in \mathbb{R}^4$. In our ($\mathbb{R}^4$,g), (1.2) reduces with these one-forms to the following equations:

$$\tilde{K}_{1332,1} = 2A_1\tilde{K}_{1332} + A_1\tilde{K}_{1332} + A_3\tilde{K}_{1132} + A_3\tilde{K}_{1312} \tag{4.7}$$

and

$$\tilde{K}_{1332,2} = 2A_2\tilde{K}_{1332} + A_1\tilde{K}_{2332} + A_3\tilde{K}_{1232} + A_3\tilde{K}_{1322}. \tag{4.8}$$

It can be easily proved that the equations (4.7) and (4.8) are true.
So, the manifold under consideration is a special weakly symmetric Riemannian manifold.
Thus the following theorem holds:

Theorem 4.1. *Let* $(\mathbb{R}^4, \mathrm{g})$ *be a 4-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{4.9}$$

where i,j=1,2,3,4. Then $(\mathbb{R}^4, \mathrm{g})$ *is a special weakly symmetric Riemannian manifold.*

References

[1] L. Tamassy, T. Q. Binh, *On weakly symmetries of Einstein and Sasakian manifold*, Tensor, N. S., 18, 5 (1993), 140-148.

[2] L. Tamassy, T. Q. Binh, *Weakly symmetric and weakly projective symmetric Riemannian manifold*, Coll. Math. Soc. J. Bolyai, 50 (1989), 663-570.

[3] H. Singh and Q. Khan, *On special weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen, 58(3) (2001), 523-536.

[4] Q. Khan, *On conharmonically and special weakly Ricci symmetric Sasakian manifold*, Novi Sad J. Math., 34(1) (2004), 71-77.

[5] U. C. De, and A. Sarkar, *On a type of P-Sasakian manifolds*, Math. Reports, 11(61) (2009), 139-144.

[6] K. Matsumoto, S. Ianus and I. Mihai, *On P-Sasakian manifolds which admit certain tensor fields*, Publ. Math. Debrecen, 38(1986), 61-65.

[7] G. P. Pokhariyal, *Study of a new curvature tensor in a Sasakian manifold*, Tensor . S., 36 (1982), 222-225.

[8] G. P. Pokhariyal, *Curvature tensors on A-Einstein Sasakian manifols*, Balkan J. Geom. Appl., 6 (2001), 45-50.
[9] A. Yildiz and U. C. De, *On a type of Kenmotsu manifolds*, Diff. Geom. Dynamical Systems, 12 (2010), 289-298.
[10] A. A. Shaikh, S. K. Jana and S. Eyasmin, *On weakly* W_2*-symmetric manifolds*, Sarajevo J. Math., 3 (15) (2007), 73-91.
[11] G. P. Pokhariyal and R. S. Mishra, *The curvature tensor and their relativistic significance*, Yokohoma Math. J., 18 (1970), 105-108.
[12] D. Ferus, *A remark on Codazzi tensors on constant curvature space*, Lecture notes Math. 838, Global Differential geometry and Global Analysis, Springer-Verlag, New York, 1981.
[13] U.C. De, A.A. Shaikh and J. Sengupta, *Tensor Calculus, Second Edition*, Narosa Publ. Pvt. Ltd., New Delhi (2007).

Authors' address:

Mohabbat Ali, Quddus Khan, Mohd Vasiulla
Department of Applied Sciences and Humanities,
Jamia Millia Islamia, New Delhi, 110025, India.
E-mail addresses: ali.math509@gmail.com , qkhan@jmi.ac.in , vsmlk45@gmail.com