On a classification of super quasi-Einstein manifolds

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Abstract. In this paper, we study the geometric properties of super quasi-Einstein manifolds. We provide some necessary conditions for which such manifolds M are quasi-Einstein, pseudo quasi-Einstein manifolds. Also, we discuss Ricci solitons endowed with torse-forming vector field on super quasi-Einstein manifolds and give some important characterizations for such manifolds.

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1 Introduction

It is well known that Einstein manifolds play an important role in Riemannian geometry as well as in general relativity since they are quite useful in differential geometry. Therefore, the geometry of such manifolds has been the focus of attention of many mathematicians and they have been studied extensively on many context.

As a generalization of Einstein manifolds, quasi-Einstein manifolds appeared while studying of exact solutions of the Einstein field equations. The first study on quasi-Einstein manifolds was given by Chaki and Maity in 2000 [3]. Following this study, Chaki extended such manifolds to generalized quasi-Einstein manifolds in 2001 [1]. Then, the theory of quasi-Einstein manifolds has become popular and have been studied by many authors such as Chaki ([1]-[3]), Kırık et al. [13], Shaikh [18], De et al. ([6], [7]), Debnath and Konar [8], Sarbari [9], Güler et al. [10], Nagaraja [14], Chaturvedi and Gupta [4], Özgür and Sular [17] and Özgür ([15], [16]) and many others.

In 2004, Chaki defined the notion of super quasi-Einstein manifold which is an another generalization of quasi-Einstein manifolds [2]. A non-flat semi-Riemannian manifold M of dimension n $(n > 2)$ is called a super quasi-Einstein manifold if its Ricci tensor S is not identically zero and satisfies

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + c\{A(X)B(Y) + A(Y)B(X)\} \\ &\quad + dD(X,Y), \end{aligned} \tag{1.1}$$

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where a, b, c and d are non-zero scalars, A and B are non-zero $1-$forms and D is a symmetric tensor of type $(0,2)$ with zero trace such that

$$A(X) = g(X,U), \quad B(X) = g(X,V), \tag{1.2}$$

$$D(X,U) = 0. \quad \forall X \in \Gamma(TM) \tag{1.3}$$

A and B are called the associated $1-$forms of the manifold. The vector fields U and V corresponding to the $1-$forms A and B are called the generators of the manifold and are mutually orthogonal. If $d = 0$, then the manifold reduces to a generalized quasi-Einstein manifold and if $c = d = 0$, then the manifold reduces to a quasi-Einstein manifold. An $n-$dimensional super quasi-Einstein manifold is denoted by $S(QE)_n$.

From the equalities (1.1), (1.2) and (1.3), for a super quasi-Einstein manifold we also have

$$S(X,U) = (a+b)A(X) + cB(X), \tag{1.4}$$

$$S(X,V) = aB(X) + cA(X) + dD(X,V), \tag{1.5}$$

$$S(U,U) = (a+b), \tag{1.6}$$

$$S(V,V) = a + dD(V,V), \tag{1.7}$$

$$S(U,V) = c. \tag{1.8}$$

In 2009, Shaikh defined a new class of quasi-Einstein manifolds which is known as pseudo quasi-Einstein manifolds [18]. According to Shaikh, a non-flat semi-Riemannian manifold of dimension n $(n > 2)$ whose non-zero Ricci tensor satisfies

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + dD(X,Y),$$

where a, b and d are non-zero scalars, A is a non-zero $1-$form associated with the unit vector field U defined by $A(X) = g(X,U)$ and D is a symmetric tensor with zero trace defined as (1.3). An $n-$dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$.

On the other hand, the notion of Ricci soliton in Riemannian geometry was introduced by Hamilton in 1988 [11]. A triplet (g, ξ, λ) on a Riemannian manifold M is called a Ricci soliton if there exist a real number λ on M and a vector field $\xi \in \Gamma(TM)$ such that

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.9}$$

where $\pounds_\xi g$ denotes the Lie-derivative of the metric tensor g along vector field ξ, S is the Ricci tensor of M and X, Y are arbitrary vector fields on M. A Ricci soliton on M is denoted by (g, ξ, λ). The vector field ξ is called the potential vector field of Ricci soliton. When the potential vector field ξ is zero or Killing in (1.9), then the Ricci soliton reduces to Einstein manifold. Therefore, it is considered as a natural generalization of Einstein metric. The Ricci soliton is called gradient if the potential vector field ξ is the gradient of a potential function $-f$ (i.e., $\xi = -\nabla f$) and is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

Vector fields have been used for studying diferential geometry of manifolds since they determine most geometric properties of the related object. Also, they have many rich properties and play an important role in the study of Riemannian geometry. That is why, geometric vector fields have been studied by many mathematicians on many context.

A vector field φ on a Riemannian manifold M is said to be
• a torse-forming vector field if it satisfies [5]

$$\nabla_X \varphi = fX + \alpha(X)\varphi, \tag{1.10}$$

• or a $\varphi(Ric)-$vector field if it satisfies [12]

$$\nabla_X \varphi = \mu Ric X, \tag{1.11}$$

where ∇ is the Levi-Civita connection, f is a function, α is a $1-$form on M, μ is a constant and Ric is the ricci operator defined by $S(X,Y) = g(RicX, Y)$. If $\mu \neq 0$ and $\mu = 0$ in (1.11), then the vector field φ is called a proper $\varphi(Ric)-$vector field and covariantly constant, respectively. Also, if the $1-$form α vanishes identically in (1.10), the vector field φ is called a concircular. If $\alpha = 0$ and $f = 1$ in (1.10), then φ is called a concurrent vector field. The vector field φ is called a recurrent if it satisfies (1.10) with $f = 0$.

Motivated by the above circumstances, we investigate Ricci solitons on super quasi-Einstein manifolds and also study some geometric vector fields on such manifolds.

The paper is organized as follows: Section 1 is devoted to the introduction and preliminaries. In section 2, we deal with super quasi-Einstein manifolds endowed with some special geometric vector field. Also, we study Ricci solitons on super quasi-Einstein manifolds and obtain some important characterizations which classifies such manifolds.

2 Main results

In this section, we give our main results. We firstly state the following:

Proposition 2.1. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, U, λ) such that the potential vector field U is the generator of M. Then, the integral curves of U are geodesic on M if and only if the manifold M is a pseudo quasi-Einstein.*

Proof. Since M is a Ricci soliton with potential vector field U, from (1.9) we have

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$. Putting $Y = U$ in (2.1) and from (1.4) we get

$$g(\nabla_U U, X) = -2(a + b + \lambda)A(X) - 2cB(X) \tag{2.2}$$

Again, putting $X = U$ in (2.2) and using (1.2), (1.6) gives

$$\lambda = -(a + b). \tag{2.3}$$

Therefore, from (2.2) and (2.3) we obtain

$$g(\nabla_U U, X) = -2cB(X). \tag{2.4}$$

Now, let us suppose that the integral curves of U are geodesic on M. Then, the equation (2.4) becomes

$$cB(X) = 0. \tag{2.5}$$

Taking $X = V$ in (2.5), we find $c = 0$. This implies that M is a pseudo quasi-Einstein manifold.

Conversely, we assume that M is a pseudo quasi-Einstein manifold. Then, we have $c = 0$. Hence, we get

$$g(\nabla_U U, X) = 0,$$

which implies that $\nabla_U U = 0$. This is the desired result. The proof is completed. □

The next theorem provides a characterization for a super quasi-Einstein manifold to be pseudo quasi-Einstein.

Theorem 2.2. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, U, λ) such that the potential vector field U is the generator of M. If U is a torse-forming vector field on M, then the manifold is a pseudo quasi-Einstein.*

Proof. Since the vector field U is a torse-forming on M, from (1.10) we write

$$\nabla_X U = fX + \alpha(X)U \tag{2.6}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (2.6) with U and using (1.2) one has

$$\alpha(X) = -fA(X). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\nabla_X U = f(X - A(X)U). \tag{2.8}$$

It follows from the definition of the Lie-derivative and from (1.2), (2.8) we have

$$\begin{aligned} (\pounds_U g)(X, Y) &= g(\nabla_X U, Y) + g(\nabla_Y U, X) \\ &= 2fg(X, Y) - 2fA(X)A(Y). \end{aligned} \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$. Also, making use of (1.9) and (2.9) we find

$$S(X, Y) = -(\lambda + f)g(X, Y) + fA(X)A(Y). \tag{2.10}$$

Putting $X = U$ and $Y = V$ in (2.10) gives

$$S(U, V) = 0. \tag{2.11}$$

By virtue of (1.8) and (2.11), we get $c = 0$. Thus, M is a pseudo quasi-Einstein manifold. This completes the proof of the theorem. □

By applying the same method as given in the proof of Theorem 2.2 we also have the following result.

Theorem 2.3. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, V, λ) such that the potential vector field V is the generator of M. If V is a torse-forming vector field on M, then the manifold is a pseudo quasi-Einstein.*

Proposition 2.4. *In a super quasi-Einstein manifold, the generators U and V cannot be concurrent vector fields.*

Proof. Assume that the generator U of a super quasi-Einstein manifold M is a concurrent vector field. Then, we have

$$\nabla_X U = X \tag{2.12}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (2.12) with U and using the fact that $g(U, U) = 1$ and (1.2) one has

$$A(X) = 0 \tag{2.13}$$

Putting $X = U$ in (2.13), we get that $A(U) = 0$. This is a contradiction.

Similarly, assume that the generator V of a super quasi-Einstein manifold M is a concurrent vector field. Then, we write

$$\nabla_X V = X \tag{2.14}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (2.14) with V and using the fact that $g(V, V) = 1$ and (1.2) we have

$$B(X) = 0 \tag{2.15}$$

Setting $X = V$ in (2.15), we find that $B(V) = 0$. This is a contradiction. Thus, the proof is completed. □

The following theorem gives a characterization for a Ricci soliton on super quasi-Einstein manifold to be steady.

Theorem 2.5. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, φ, λ) such that the potential vector field φ is a $\varphi(Ric)-$vector field. Then, M is either a pseudo quasi-Einstein or the Ricci soliton (g, φ, λ) is a steady.*

Proof. It follows from the definition of the Lie-derivative and from (1.11) one has

$$(\pounds_\varphi g)(X, Y) = 2\mu S(X, Y) \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$. Since M admits a Ricci soliton, with the help of (1.9) and (2.16) we find

$$(\mu + 1)S(X, Y) = -\lambda g(X, Y). \tag{2.17}$$

Taking $X = Y = U$ in the equation (2.17) gives

$$(\mu + 1)(a + b) = -\lambda. \tag{2.18}$$

Again taking $X = Y = V$ in the equation (2.17) we get

$$(\mu + 1)(a + dD(V, V)) = -\lambda. \tag{2.19}$$

Also, replacing X by U and Y by V in (2.20) yields

$$(\mu+1)c = 0 \tag{2.20}$$

which implies that $\mu = -1$, or $c = 0$.

If $c = 0$, then M is a pseudo quasi-Einstein manifold. If $\mu = -1$, then from (2.18) and (2.19) we have $\lambda = 0$. This means that the Ricci soliton is steady. Hence, we get the requested result. □

As an immediate consequence of Theorem 2.3, we can give the following corollaries:

Corollary 2.6. *Let M be a super quasi-Einstein manifold admitting an expanding or shrinking Ricci soliton (g, φ, λ) such that the potential vector field φ is a $\varphi(Ric)-$vector field. Then, the manifold M is a pseudo quasi-Einstein.*

Corollary 2.7. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, φ, λ) such that the potential vector field φ is a covariantly constant. Then, the manifold M is a pseudo quasi-Einstein.*

Proposition 2.8. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, U, λ) such that the generator vector field U is a proper $U(Ric)-$vector field. Then, M is a pseudo quasi-Einstein manifold and the Ricci soliton (g, U, λ) is steady.*

Proof. Suppose that the generator vector field U is a proper $U(Ric)-$vector field on M. Then, from (1.11) we have

$$\nabla_X U = \mu Ric X \tag{2.21}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (2.21) with U and using (1.2) one has

$$\mu((a+b)A(X) + cB(X)) = 0. \tag{2.22}$$

Putting $X = U$ in (2.22) and using (1.2), we get

$$\mu(a+b) = 0 \tag{2.23}$$

Since the vector field U is a proper $U(Ric)-$vector field on M, the equation (2.23) is written

$$(a+b) = 0. \tag{2.24}$$

Also, if we use (2.24) in (2.22), then the equation (2.22) reduces to

$$\mu c B(X) = 0. \tag{2.25}$$

Since $\mu \neq 0$, by setting $X = V$ in (2.25) gives

$$c = 0$$

which means that M is a pseudo quasi-Einstein manifold.

On the other hand, taking U instead of the vector field φ in the Theorem 2.3. we write

$$(\mu+1)(a+b) = -\lambda. \tag{2.26}$$

Combining (2.24) with (2.26), we get $\lambda = 0$, that is, the Ricci soliton is steady. The proof is completed. □

Proposition 2.9. *Let M be a super quasi-Einstein manifold admitting a Ricci soliton (g, V, λ) such that the generator vector field V is a proper $V(Ric)-$vector field. Then, M is a pseudo quasi-Einstein manifold and the Ricci soliton (g, V, λ) is steady.*

Proof. Suppose that the generator vector field V is a proper $V(Ric)-$vector field on M. Then, from (1.11) we write

$$\nabla_X V = \mu Ric X \tag{2.27}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (2.27) with V and using (1.2) we get

$$\mu(aB(X) + cA(X) + dD(X, V)) = 0. \tag{2.28}$$

Putting $X = U$ in (2.28) and using (1.2), (1.3) we derive

$$\mu c = 0. \tag{2.29}$$

Since V is a proper $V(Ric)-$vector field on M, we find

$$c = 0. \tag{2.30}$$

which implies that M is a pseudo quasi-Einstein manifold. Also, using (2.30) in (2.28), then the equation (2.28) becomes $\mu(aB(X) + dD(X, V)) = 0$. Since $\mu \neq 0$, we have

$$(aB(X) + dD(X, V)) = 0. \tag{2.31}$$

Taking $X = V$ in (2.31) and from (1.2), we have

$$(a + dD(V, V)) = 0. \tag{2.32}$$

On the other hand, taking V instead of the vector field φ in the Theorem 2.3. we write

$$(\mu + 1)(a + dD(V, V)) = -\lambda. \tag{2.33}$$

From (2.32) and (2.33) we obtain $\lambda = 0$ namely, the Ricci soliton is steady. Therefore, the proof is completed. □

The last result of this section is the following:

Theorem 2.10. *Let M be a super quasi-Einstein manifold. If the condition $R.S = 0$ holds on M, then M is either pseudo quasi-Einstein manifold or M is a generalized quasi-Einstein manifold.*

Proof. Let us assume that a super quasi-Einstein manifold M satisfies the condition $R.S = 0$. Then, we have

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0 \tag{2.34}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Setting $W = V$ in (2.34) and using (1.5), one has

$$\begin{aligned} S(Z, R(X, Y)V) &= -aB(R(X, Y)Z) - cA(R(X, Y)Z) \\ &\quad -dD(R(X, Y)Z, V). \end{aligned} \tag{2.35}$$

Similarly, taking $Z = U$ in (2.35) and using (1.2), (1.4) we have

$$(a+b)A(R(X,Y)V) = -aB(R(X,Y)U) - dD(R(X,Y)U,V). \tag{2.36}$$

On the other hand, if we put $W = U$ in (2.34) and use (1.4), we get

$$(a+b)A(R(X,Y)Z) + cB(R(X,Y)Z) + S(Z,R(X,Y)U) = 0. \tag{2.37}$$

Again, putting $Z = U$ in (2.37) and using (1.4) yields

$$(a+b)A(R(X,Y)U) + cB(R(X,Y)U) = 0. \tag{2.38}$$

Since $A(R(X,Y)U) = 0$, the equation (2.38) becomes $cB(R(X,Y)U) = 0$ which gives either $c = 0$ or $B(R(X,Y)U) = 0$. If $c = 0$, then M is a pseudo quasi-Einstein manifold.

If $B(R(X,Y)U) = 0$, then we have $A(R(X,Y)V) = 0$. Then, the equation (2.36) reduces to

$$dD(R(X,Y)U,V) = 0.$$

Since $D(R(X,Y)U,V) \neq 0$, we find that $d = 0$. This implies that M is a generalized quasi-Einstein manifold. This result ends the proof of the theorem. □

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