Chen's inequalities for biwarped product submanifolds in complex space forms

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Abstract. In the present paper, we investigate an optimal inequality for the squared norm of the mean curvature vector for the Biwarped product submanifolds isometrically immersed into complex space forms, in the expressions of constant sectional curvature c, Laplacian of the warping functions and slant function. The equality case is likewise discussed. Some special cases are also discussed.

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1 Introduction

The comprehension of warped product manifolds came into existence after the approach of Bishop and O'Neill [6] on manifolds of negative curvature. Examining the fact that a Riemannian product of manifolds can not have negative curvature, they construct the model of warped product manifolds for the class of manifolds of negative (or non positive) curvature which is defined as follows:

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive differentiable function on N_1. If $\pi : N_1 \times N_2 \to N_1$ and $\eta : N_1 \times N_2 \to N_2$ are the projection maps given by $\pi(p,q) = p$ and $\eta(p,q) = q$ for every $(p,q) \in N_1 \times N_2$, then the *warped product manifold* is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + (\psi \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for all $X, Y \in TM$. The function ψ is called the *warping function* of the warped product manifold". If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. On the grounds that warped product manifolds admit a number of applications in Physics and theory of relativity [4], this has been a topic of extensive research. Warped products provide many basic solutions

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to Einstein field equations [4]. The concept of modelling of space-time near black holes adopts the idea of warped product manifolds [20]. Schwartzschild space-time is an example of warped product $P \times_r S^2$, where the base $P = R \times R^+$ is a half plane $r > 0$ and the fibre S^2 is the unit sphere. Under certain conditions, the Schwartzchild space-time becomes the black hole. A cosmological model to model the universe as a space-time known as Robertson-Walker model is a warped product [24].

Some natural properties of warped product manifolds were studied in [6]. B. Y. Chen ([8], [9]) performed an extrinsic study of warped product submanifolds in a Kaehler manifold. Since then, many geometers have explored warped product manifolds in different settings like almost complex and almost contact manifolds and various existence results have been investigated (see the survey article [16]).

Further, Chen initiated the study of warped product submanifold isometrically immersed into Riemannian manifold, especially in Riemannian space forms (see [11], [12], [23]). Recently, many papers related to warped product submanifold isometrically immersed in the different Riemannian space forms (see the survey article [16]). The topic of research which covers the inequalities for warped product submanifolds in the almost Hermitian as well as almost contact metric manifolds is getting significance. In particular, in [14] Chen discovered a strong relationship between warping function and the squared norm of the mean curvature vector for the warped product submanifold $M_1 \times_f M_2$ isometrically immersed into a Real space form. Basically, he obtained the following result

Theorem 1.1. *Let $\bar{M}(c)$ be a m-dimensional real space form and $\phi : M = M_1 \times_f M_2$ be a isometric immersion of an $n-$dimensional warped product in $\bar{M}(c)$. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 c, \tag{1.1}$$

where $n_i = dimm M_i$, $i = 1, 2$ and Δ is the Laplacian operator of M_1 and H is the mean curvature vector of M^n. Moreover, the equality holds in (1.1) if and only if ϕ is mixed totally geodesic and $n_1 H_1 = n_2 H_2$. Such that H_1 and H_2 are partially mean curvature of M_1 and M_2 respectively".

2 Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g, i.e., $J^2 = -I$ and $g(JX, JY) = g(X, Y)$, for all vector fields X, Y on $\bar{M}$. If J is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, that mean

$$(\bar{\nabla}_X J)Y = 0, \tag{2.1}$$

for all $X, Y \in T\bar{M}$, then $(\bar{M}, J, g, \bar{\nabla})$ is called a *Kaehler manifold*

A Kaehler manifold $\bar{M}$ is called a *complex space form* if it has constant holomorphic sectional curvature denoted by $\bar{M}(c)$. The curvature tensor of the complex space form $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X, Y, Z, W) = &\frac{c}{4}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(X, JZ)g(JY, W) \\ &- g(Y, JZ)g(JX, W) + 2g(X, JY)g(JZ, W)], \end{aligned} \tag{2.2}$$

for any $X, Y, Z, W \in T\bar{M}$".

Let M be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are $\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y)$ and $\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$ respectively, for all $X, Y \in TM$ and $\xi \in T^\perp M$. Where ∇ is the induced Levi-civita connection on M, ξ is a vector field normal to M, α is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_ξ is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(\alpha(X, Y), \xi) = g(A_\xi X, Y). \tag{2.3}$$

The equation of Gauss is given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g(\alpha(X, W), \alpha(Y, Z)) - g(\alpha(X, Z), \alpha(Y, W)), \tag{2.4}$$

for all $X, Y, Z, W \in TM$". Where, $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively.
For any $X \in TM$ and $N \in T^\perp M$, JX and JN can be decomposed as follows

$$JX = PX + FX \tag{2.5}$$

and

$$JN = tN + fN, \tag{2.6}$$

where PX (resp. tN) is the tangential and FX (resp. fN) is the normal component of JX (resp. JN).

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space $T_x M$, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} \alpha(e_i, e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i, e_i), h(e_j, e_j)), \tag{2.7}$$

where n is the dimension of M. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $\alpha(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is called totally umbilical.

Chen and Gray [17] studied pointwise slant submanifold in the setting of almost Hermitian manifolds. Basically, they provided the following characterization

A submanifold M of an almost Hermitian manifold $\bar{M}$ is a pointwise slant if and only if for any non zero vector $X \in T_x M$ for each $x \in M$, the angle $\theta(X)$ between JX and tangent space $T_x M$ is independent from the choice of the non zero unit vector X. In this case, the wirtinger angle becomes a real valued function and it is not unique along M".

The scalar curvature of $\bar{M}$ is denoted by $\bar{\tau}(\bar{M})$ and is defined as

$$\bar{\tau}(\bar{M}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}, \tag{2.8}$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\tau}(\bar{M}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{2.9}$$

In a similar way, the scalar curvature $\bar{\tau}(L_x)$ of a $L-$plane is given by

$$\bar{\tau}(L_x) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{2.10}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^{\perp}M$, then we have

$$\alpha^r_{pq} = g(\alpha(e_p, e_q), e_r) \tag{2.11}$$

and

$$\|\alpha\|^2 = \sum_{p,q=1}^{n} g(\alpha(e_p, e_q), \alpha(e_p, e_q)). \tag{2.12}$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold M^n and in the Riemannian space form $\bar{M}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=n+1}^{m} (\alpha^r_{pp}\alpha^r_{qq} - (\alpha^r_{pq})^2). \tag{2.13}$$

For a smooth function ψ on a Riemannian manifold M with Riemannian metric g, the gradient of ψ is denoted by $\nabla\psi$ and is defined as

$$g(\nabla\psi, X) = X\psi, \tag{2.14}$$

for all $X \in TM$.

Let the dimension of M is n and $\{e_1, e_2, \ldots, e_n\}$ be a basis of TM. Then as a result of (2.15), we get

$$\|\nabla\psi\|^2 = \sum_{i=1}^{n} (e_i(\psi))^2. \tag{2.15}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)\psi - e_i e_i \psi\}". \tag{2.16}$$

3 Biwarped product submanifolds of the complex space forms

B. Y. Chen and F. Dillen [18] generalize the definition of warped product submanifold to multiply warped product manifolds as follows.

Let $\{N_i\}$, $i = 1, 2, \ldots, k$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\ldots,k}$ and $\{\psi\}_{i=2,3,\ldots,k}$ are positive valued functions on N_1. Then the product manifold $M = N_1 \times N_2 \times \cdots \times N_k$ endowed with the Riemannian metric g given by

$$g = \pi_1^*(g_1) + \sum_{i=2}^{k} (\psi_i \circ \pi_1)^2 \pi_i^*(g_i)$$

is called multiply warped product manifold and denoted by $M = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ where $\pi_i (i = 1, 2, \ldots, k)$ are the projection maps of M onto N_i respectively. The functions f_i are known as the warping functions [18]. If the warping functions are constants, the warped product is simply Riemannian product of manifolds". As a particular case of multiply warped product manifolds, we can define biwarped product manifolds for $i = 3$. For $i = 2$, multiply warped product manifold reduces to single warped product manifold. Consider the biwarped product manifold $M = N_0 \times_{f_1} N_1 \times_{f_2} N_2$ with the Levi-Civita connection of N_i for $i = 0, 1, 2$. Now, we have the following result for biwarped product submanifold

Lemma 3.1. [3] *Let $M = N_0 \times_{f_1} N_1 \times_{f_2} N_2$ be a biwarped product manifold. Then we have*

$$\nabla_X Z = \nabla_Z X = X(ln f_i) Z \tag{3.1}$$

for $X \in TN_0$ and $Z \in TN_i$, for $i = 1, 2$.

Recently, H. M. Tastan [31] studied biwarped submanifolds in the Kaehler manifolds and this was followed by M. A. Khan and K. Khan [21]. Basically, M. A. Khan and K. Khan explored biwarped product submanifold of the type $M = N_T \times_{f_1} N_\perp \times_{f_2} N_\theta$ in the setting of complex space forms. Where N_T, $N_\perp$ and N_θ are the invariant, totally real and pointwise slant submanifolds respectively. Throughout this study we consider $n-$dimensional biwarped product submanifold $M^n = N_T^{n_1} \times_{f_2} N_\perp^{n_2} \times_{f_3} N_\theta$ of a complex space form, where n_1, n_2, n_3 are the dimensions of the invariant, totally real and pointwise slant submanifolds. If $N_\theta^{n_3} = \{0\}$ then the biwarped product submanifold becomes the CR-warped product submanifold. Similarly, if $N_\perp^{n_2} = \{0\}$ then the biwarped product submanifold reduced to pointwise semi-slant warped product submanifold. Moreover, if $N_T^{n_1} = \{0\}$, then the biwarped product submanifold becomes the point wise pseudo-slant warped product submanifold as studied in [30].

For a biwarped product submanifold $M^n = N_T^{n_1} \times_{f_2} N_\perp^{n_2} \times_{f_3} N_\theta^{n_3}$ of a Riemannian manifold from equation (3.5) of [18] one can conclude the following result

$$\frac{\Delta f_2}{f_2} + \frac{\Delta f_3}{f_3} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \kappa(e_i, e_j) + \sum_{i=1}^{n_1} \sum_{k=1}^{n_3} \kappa(e_i, e_k). \tag{3.2}$$

Let $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{n_1} = Je_p, e_{n_1+1}, \ldots, e_{n_2}, e_{n_2+1} = e^1, \ldots, e_{n_2+q} = e^q, e_{n_2+q+1} = e^{q+1} = \sec\theta P e^1, \ldots e_{(n_3=2q)} = e^{n_3} = \sec\theta P e^q\}$ be a local orthonormal frame of vector fields on the biwarped product submanifold $N_T^{n_1} \times_{f_2} N_\perp^{n_2} \times_{f_3} N_\theta^{n_3}$ such that the set $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{n_1} = Je_p\}$ is tangent to N_T, the set $\{e_{n_1+1}, \ldots, e_{n_2}\}$ is tangent to $N_\perp^{n_2}$ and the set $\{e_{n_2+1}, \ldots, e_{n_2+q}, \ldots e^{n_3}\}$ is tangent to $N_\theta^{n_3}$. Moreover, $\{e_{n+1} = Je_{n_1+1}, \ldots e_{n+n_2} = Je_{n_2}, e_{n+n_2+1} = \csc\theta Fe^1, \ldots, e_{n+n_3} = \csc\theta Fe^q, e_{n+n_2+n_3+1} = \bar{e}^1, \ldots, e_m = \bar{e}^k\}$ is a basis for the normal bundle $T^\perp M$, such that the sets $\{e_{n+1} = Je_{n_1+1}, \ldots e_{n+n_2} = Je_{n_2}\}$ is tangent to $JD^\perp$, $\{e_{n+1} = \csc\theta Fe^1, \ldots, e_{n+n_2} = \csc\theta Fe^q\}$ is tangent to FD^θ and $\{\bar{e}^1, \ldots, \bar{e}^k\}$ is tangent to the complementary invariant subbundle μ with even dimension k.

Now, we state the following lemma which is an important tool to discover main result

Lemma 3.2. *Let $s_1, s_2, \ldots, s_n, s_{n+1}$ be $n+1$ be real numbers such that*

$$(\sum_{i=1}^{n} s_i)^2 = (n-1)(\sum_{i=1}^{n} s_i^2 + s_{n+1}), \quad n \geq 2.$$

Then $2s_1s_2 \geq s_3$ holds if and only if

$$s_1 + s_2 = s_3 = \cdots = s_k.$$

Now, we investigate the main result of the paper

Theorem 3.3. *Let $M^n = N_T^{n_1} \times_{f_2} N_\perp^{n_2} \times_{f_3} N_\theta^{n_3}$ be a biwarped product submanifold isometrically immersed in a complex space form $\bar{M}(c)$. Then, the following inequality holds*

$$n_2\frac{\Delta f_2}{f_2} + n_3\frac{\Delta f_3}{f_3} \leq \frac{n^2}{4}\|H\|^2 + \frac{c}{4}(n_1n_2 + n_2n_3 + n_3n_1 + 3n_1 + 3n_3\cos^2\theta), \tag{3.3}$$

where n_1, n_2 and n_3 are the dimensions of $N_T, N_\perp$ and N_θ respectively. Moreover, Δ is the Laplacian operator on $N_T^{n_1}$ and $N_\perp^{n_2}$, H is the mean curvature vector of M^n. The equality case satisfies in (3.3) if and only if M is totally geodesic submanifold and satisfies the following equality

$$n_1H_1 = n_2H_2 + n_3H_3,$$

where H_1, H_2 and H_3 are partially mean curvature vectors on $N_T^{n_1}, N_\perp^{n_2}$ and $N_\theta^{n_3}$ respectively.

Proof. Substituting $X = Z = e_i$ and $Y = W = e_j$ for $1 \leq i, \ j \leq n$ in (2.2) and summing over i, j, we get

$$\sum_{i,j=1}^{n} \bar{R}(e_i, e_j, e_i, e_j) = \frac{c}{4}(n(n-1) + 3n_1 + 3n_3\cos^2\theta),$$

and using this in (2.4), we find that

$$2\tau = \frac{c}{4}n(n-1) + \frac{c}{4}(3n_1 + 3n_3\cos^2\theta) + n^2\|H\|^2 - \|\alpha\|^2. \tag{3.4}$$

Now, assume that

$$\delta = 2\tau - \frac{c}{4}n(n-1) - \frac{c}{4}(3n_1 + 3n_3\cos^2\theta) - \frac{n^2}{2}\|H\|^2. \tag{3.5}$$

Then, from (3.4) and (3.5), we get

$$n^2\|H\|^2 = 2\delta + 2\|\alpha\|^2. \tag{3.6}$$

On using the orthogonal frame $\{e_1, e_2, \ldots, e_n\}$, the last equation takes the form

$$\begin{aligned}(\sum_{r=n+1}^{2m}\sum_{i=1}^{n}\alpha^r_{AA})^2 =&2(\delta + \sum_{r=n+1}^{2m}\sum_{i=1}^{n}(\alpha^r_{ii})^2 \sum_{r=n+1}^{2m}\sum_{i<j=1}^{n}(\alpha^r_{ij})^2\\ &+\sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(\alpha^r_{AB})^2).\end{aligned} \tag{3.7}$$

The above expression can be written as

(3.8)

$$\begin{aligned}\frac{1}{2}(\alpha^{n+1}_{11} + \sum_{A=2}^{n_1}\alpha^{n+1}_{AA} + \sum_{l=n_1+1}^{n_2}\alpha^{n+1}_{ll} + \sum_{m=n_2+1}^{n}\alpha^{n+1}_{mm}) = \delta + (\alpha^{n+1}_{11})^2 + \sum_{A=2}^{n_1}(\alpha^{n+1}_{AA})^2 +\\ + \sum_{l=n_1+1}^{n_2}(\alpha^{n+1}_{ll})^2 + \sum_{m=n_2+1}^{n}(\alpha^{n+1}_{mm})^2\\ - \sum_{2\le B\ne q\le n_1}\alpha^{n+1}_{BB}\alpha^{n+1}_{qq} - \sum_{n_1+1\le l\ne s\le n_2}\alpha^{n+1}_{ll}\alpha^{n+1}_{ss}\\ - \sum_{n_2+1\le m\ne k\le n}\alpha^{n+1}_{mm}\alpha^{n+1}_{kk} + \sum_{A<B=1}^{n}(\alpha^{n+1}_{AB})^2\\ + \sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(\alpha^r_{AB})^2.\end{aligned}$$

Assume that $s_1 = \alpha^{n+1}_{11}$, $s_2 = \sum_{A=2}^{n_1}\alpha^{n+1}_{AA}$ and $s_3 = \sum_{l=n_1+1}^{n_2}\alpha^{n+1}_{ll} + \sum_{m=n_2+1}^{n}\alpha^{n+1}_{mm}$. Now applying Lemma 3.2 in (3.2), we derive

(3.9)

$$\begin{aligned}\frac{\delta}{2} + \sum_{A<B=}^{n}(\alpha^{n+1}_{AB})^2 + \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{A,B=1}^{n}(\alpha^r_{AB})^2 \le \sum_{2\le B\ne q\le n_1}\alpha^{n+1}_{BB}\alpha^{n+1}_{qq} + \sum_{n_1+1\le l\ne s\le n_2}\alpha^{n+1}_{ll}\alpha^{n+1}_{ss}\\ + \sum_{n_2+1\le m\ne k\le n}\alpha^{n+1}_{mm}\alpha^{n+1}_{kk},\end{aligned}$$

with equality holds in (3.9) if and only if

$$\sum_{A=2}^{n_1}\alpha^{n+1}_{AA} = \sum_{B=n_1+1}^{n_2}\alpha^{n+1}_{BB} + \sum_{C=n_2+1}^{n}\alpha^{n+1}_{CC}. \tag{3.10}$$

On the other hand, from (3.2), we have

$$\begin{aligned}n_2\frac{\Delta f_2}{f_2} + n_3\frac{\Delta f_3}{f_3} = \tau - \sum_{1\le A<B\le n_1}\kappa(e_A\wedge e_B) - \sum_{n_1+1\le l<q\le n_2}\kappa(e_l\wedge e_q)\\ - \sum_{n_2+1\le m<k\le n}\kappa(e_m\wedge e_k).\end{aligned} \tag{3.11}$$

Then from (2.13) and the scalar curvature for the complex space form (2.2), we get

(3.12)
$$n_2\frac{\Delta f_2}{f_2}+n_3\frac{\Delta f_3}{f_3}=\tau-\frac{n_1(n_1-1)c}{8}-\frac{3n_1c}{4}-\sum_{r=n+1}^{2m}\sum_{1\leq A\neq B\leq n_1}(\alpha^r_{AA}\alpha^r_{BB}-(\alpha^r_{AB})^2)$$
$$-\frac{n_2(n_2-1)c}{8}-\sum_{r=n+1}^{2m}\sum_{n_1+1\leq l\neq q\leq n_2}(\alpha^r_{ll}\alpha^r_{qq}-(\alpha^r_{lq})^2)-\frac{n_3(n_3-1)c}{8}$$
$$-\frac{3n_3c}{4}\cos^2\theta-\sum_{r=n+1}^{2m}\sum_{n_2+1\leq m\neq k\leq n}(\alpha^r_{mm}\alpha^r_{kk}-(\alpha^r_{mk})^2).$$

Now, from (3.9) and (3.12), we have

$$n_2\frac{\Delta f_2}{f_2}+n_3\frac{\Delta f_3}{f_3}\leq\tau-\frac{n(n-1)c}{8}+\frac{(n_1n_2+n_2n_3+n_3n_1)c}{4}$$
$$-\frac{3n_1c}{4}-\frac{\delta}{2}-\frac{3n_3c}{4}\cos^2\theta. \tag{3.13}$$

Using (3.5) in the last inequality we derive

$$n_2\frac{\Delta f_2}{f_2}+n_3\frac{\Delta f_3}{f_3}\leq\frac{n^2}{4}\|H\|^2+\frac{c}{4}(n_1n_2+n_2n_3+n_3n_1+\frac{3n_1}{4}+\frac{3n_3}{4}\cos^2\theta), \tag{3.14}$$

which is the required inequality. The equality sign holds if and only if the unused terms in (3.9) and (3.10) imply that

$$\sum_{r=n+1}^{2m}\sum_{B=1}^{n_1}\alpha^r_{BB}=\sum_{r=n+1}^{2m}\sum_{A=n_1+1}^{n}\alpha^r_{AA}=0,$$

and $n_1H_1=n_2H_2+n_3H_3$, where H_1, H_2 and H_3 are partially mean curvature vectors of $N_T^{n_1}, N_\perp^{n_2}$ and $N_\theta^{n_3}$ respectively.

Moreover, from (3.9) it is evident that

$$\alpha^r_{AB}=0,\quad \alpha^r_{AC}=0,$$

for each $1\leq A\leq n_1,\quad n_1+1\leq B\leq n_2,\quad n_2+1\leq C\leq n$, and $n+1\leq r\leq 2m$. This shows that M is mixed totally geodesic. It is easy to see that converse part is true for Bi-warped product submanifold in a complex space form.

□

If $N_\theta^{n_3}=\{0\}$, then the biwarped product submanifold becomes the CR-warped product submanifolds studied by Chen in [9]. Now we have the following corollary

Corollary 3.4. *Let $M^n=N_T^{n_1}\times_{f_2}N_\perp^{n_2}$ be a CR-warped product submanifold isometrically immersed in a complex space form $\bar{M}(c)$. Then, the following inequality holds*

$$n_2\frac{\Delta f_2}{f_2}\leq\frac{n^2}{4}\|H\|^2+\frac{c}{4}(n_1n_2+3n_1), \tag{3.15}$$

where n_1 and n_2 are the dimensions of $N_T^{n_1}$ and $N_\perp^{n_2}$ respectively. Moreover, Δ is the Laplacian operator on $N_T^{n_1}$, H is the mean curvature vector of M^n. The equality case satisfies in (3.15) if and only if M is totally geodesic submanifold and satisfies the following equality

$$n_1 H_1 = n_2 H_2,$$

where H_1 and H_2 are partially mean curvature vectors on $N_T^{n_1}$ and $N_\perp^{n_2}$ respectively.

If $N_\perp^{n_2} = \{0\}$, then the biwarped product submanifold becomes the pointwise semi-slant submanifold of complex space form. Then inspired by the study of B. Sahin [28], we can conclude the following corollary

Corollary 3.5. *Let $M^n = N_T^{n_1} \times_{f_3} N_\theta^{n_3}$ be a point wise semi slant warped product submanifold isometrically immersed in a complex space form $\bar{M}(c)$. Then, the following inequality holds*

$$n_3 \frac{\Delta f_3}{f_3} \leq \frac{n^2}{4}\|H\|^2 + \frac{c}{4}(n_1 n_3 + 3n_3 \cos^2\theta), \tag{3.16}$$

where n_1 and n_3 are the dimensions of $N_T^{n_1}$ and $N_\theta^{n_3}$ respectively. Moreover, Δ is the Laplacian operator on $N_T^{n_1}$, H is the mean curvature vector of M^n. The equality case satisfies in (3.16) if and only if M is totally geodesic submanifold and satisfies the following equality

$$n_1 H_1 = n_3 H_3,$$

where H_1 and H_3 are partially mean curvature vectors on $N_T^{n_1}$ and $N_\theta^{n_3}$ respectively.

Similarly, if $N_T^{n_1} = 0$, then the biwarped product submanifold tends to pointwise pseudo-slant warped product submanifold studied in [30]. Now, we have the following corollary

Corollary 3.6. *Let $M^n = N_\perp^{n_2} \times_{f_3} N_\theta^{n_3}$ be a point wise pesuodo slant warped product submanifold isometrically immersed in a complex space form $\bar{M}(c)$. Then, the following inequality holds*

$$n_3 \frac{\Delta f_3}{f_3} \leq \frac{n^2}{4}\|H\|^2 + \frac{c}{4}(n_2 n_3 + 3n_3 \cos^2\theta), \tag{3.17}$$

where n_2 and n_3 are the dimensions of $N_\perp^{n_2}$ and $N_\theta^{n_3}$ respectively. Moreover, Δ is the Laplacian operator on $N_\perp^{n_2}$, H is the mean curvature vector of M^n. The equality case satisfies in (3.17) if and only if M is totally geodesic submanifold and satisfies the following equality

$$n_2 H_2 = n_3 H_3,$$

where H_2 and H_3 are partially mean curvature vectors on $N_\perp^{n_2}$ and $N_\theta^{n_3}$ respectively.

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