

# Some characterizations in Kenmotsu manifolds with a new connection

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**Abstract.** The present study initially identify the generalized symmetric connections  $(\alpha, \beta)$  type, which can be regarded as more generalised forms of quarter and semi-symmetric connections. The quarter and semi-symmetric connections are obtained respectively particularly when  $(\alpha, \beta) = (1, 0)$  and  $(\alpha, \beta) = (0, 1)$  are taken into consideration. Taking that into account, a new generalized symmetric metric connection is attained upon Kenmotsu manifolds. In compliance with the new connection, some results are provided through calculations of tensors belonging to Kenmotsu manifold involving curvature tensor, ricci tensor, projective curvature tensor and coincircular curvature tensor. Ultimately, we obtain the generalized Wintgen inequality for submanifolds in Kenmotsu space form with a generalized symmetric metric connection and discuss the equality cases. We finish the manuscript by presenting an original example.

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## 1 Introduction

A particular metric connection with a torsion different from zero was introduced by Hayden upon a Riemannian manifold [9]. The quarter symmetric connections, being more generalized form of semi-symmetric connections, were suggested by Golab upon a differentiable manifold [7]. As for the present study, the definition below is presented by taking it a step further and generalising the quarter symmetric connections as well.

A linear connection is suggested to be generalized symmetric connection on condition that the torsion tensor connection is presented in the form as follows:

$$(1.1) \quad T(U, V) = \alpha\{u(V)U - u(U)V\} + \beta\{u(V)\varphi U - u(U)\varphi V\},$$

for all vector fields  $U, V$  upon a manifold in which  $\alpha$  and  $\beta$  are smooth functions.  $\varphi$  denotes as a tensor of type  $(1, 1)$  and  $u$  is regarded as a 1– form connected with

the vector field which has a non-vanishing smooth non-null unit. Furthermore, the connection mentioned is suggested to be a generalized metric one when a Riemannian metric  $g$  in  $M$  is available as  $\bar{D}g = 0$ ; or else, it is non-metric.

In the equation (1.1), if  $\alpha = 0$  ( $\beta = 0$ ), then the generalized symmetric connection is called  $\beta$ -quarter-symmetric connection ( $\alpha$ -semi-symmetric connection), respectively. Additionally, the connection is decreased to a semi-symmetric one and a quarter-symmetric one when  $(\alpha, \beta) = (1, 0)$  and  $(\alpha, \beta) = (0, 1)$  are selected. Thus, it can be suggested that generalizing semi-symmetric and quarter symmetric connections paves the way for a generalized symmetric metric connection. Those two connections are of great significance both for the study of geometry and applications in physics. For instance Pahan, Pal and Bhattacharyya studied generalized Robertson-Walker space-time with respect to quarter-symmetric connection [16]. Furthermore, many authors investigated the geometrical and physical aspect of different spaces [3], [10], [12], [27]. The connections including semi-symmetric and semi-symmetric non-metric ones were investigated in a Kenmotsu manifold respectively by the authors [17], [21] [23], [24] and [28].

On the other hand, in 4-dimensional Euclidean space, we have a sharp geometric inequality for surfaces known as Wintgen inequality involving Gauss curvature (intrinsic invariants), normal curvature and square mean curvature (extrinsic invariants). In [26], for any surface  $M^2$  in  $E^4$  the following inequality was established

$$\|\mathcal{H}\|^2 \geq \mathcal{K} + |\mathcal{K}^\perp|,$$

and the equality holds if and only if the ellipse of curvature of  $M^2$  in  $E^4$  is a circle, where  $\mathcal{K}$  denotes the Gauss curvature,  $\mathcal{K}^\perp$  the normal curvature and  $\|\mathcal{H}\|^2$  the squared mean curvature. Later on, an extention was obtained for arbitrary codimension  $m$  in real space forms  $\overline{M}^{m+2}(c)$  [4]

$$\|\mathcal{H}\|^2 + c \geq \mathcal{K} + |\mathcal{K}^\perp|.$$

The generalized Wintgen inequality for submanifolds in real space form was conjectured by De Smet, Dillen, Verstraelen and Vrancken also proved by Ge and Tang [5]. For the last couple of years, DDVV inequality has been studied by for submanifolds in several ambient spaces (see [15]).

In the present paper, we define new connection for Kenmotsu manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection. Section 2 illustrates generalized symmetric connection for a Kenmotsu manifold. As for Section 3, the curvature and Ricci tensors of a Kenmotsu manifold and its scalar curvature are calculated in relation to generalized metric connection. Besides, it is found that first Bianchi identity is provided and Ricci tensor is symmetric with respect to generalized metric connection of type  $(-1, \beta)$  and  $(\alpha, 0)$ . Section 3 indicates that it is a generalized  $\eta$ -Einstein manifold with reference generalized symmetric metric connection when a Kenmotsu manifold looks  $\phi$ -projectively flat concerning generalized metric connection. Also via Section 3, an expression is presented in the interest of concircular curvature tensor with respect to generalized metric connection. In section 4, we derive generalized Wintgen inequalities for submanifolds in Kenmotsu space form with a generalized symmetric metric connection and discuss the equality cases. In Section 5 we provide an example verifying some results of Section 2 and Section 3.

## 2 Generalized symmetric metric connection

In order to call a differentiable  $M$  manifold of dimension  $n = 2m + 1$  as practically contact metric one, a  $(1, 1)$  tensor field, a contravariant vector field, a 1-form and a Riemannian metric  $g$  should be admitted, which satisfy

$$(2.1) \quad \psi\xi = 0,$$

$$(2.2) \quad \eta(\psi U) = 0$$

$$(2.3) \quad \eta(\xi) = 1,$$

$$(2.4) \quad \psi^2(U) = -U + \eta(U)\xi,$$

$$(2.5) \quad g(\psi U, \psi V) = g(U, V) - \eta(U)\eta(V),$$

$$(2.6) \quad g(U, \xi) = \eta(U),$$

for all the vector fields  $U, V$  on  $M$ . When  $g(U, \psi V) = \Phi(U, V)$  is written, the tensor field  $\phi$  denotes as an anti-symmetric  $(0, 2)$  tensor field [2]. When a practically contact metric manifold performs

$$(2.7) \quad (\nabla_U \psi)V = g(\psi U, V)\xi - \eta(V)\psi U,$$

$$(2.8) \quad \nabla_U \xi = U - \eta(U)\xi,$$

$M$  is regarded as a Kenmotsu manifold in which the Levi-Civita connection of  $g$  is present [11].

The following relations can be observed in Kenmotsu manifolds [11]:

$$(2.9) \quad (\nabla_U \eta)V = g(\psi U, \psi V),$$

$$(2.10) \quad g(R(U, V)W, \xi) = \eta(R(U, V)W) = g(U, W)\eta(V) - g(V, W)\eta(U),$$

$$(2.11) \quad R(\xi, U)V = \eta(V)U - g(U, V)\xi,$$

$$(2.12) \quad R(U, V)\xi = \eta(U)V - \eta(V)U,$$

$$(2.13) \quad R(\xi, U)\xi = U - \eta(U)\xi,$$

$$(2.14) \quad S(U, \xi) = -(n-1)\eta(U),$$

$$(2.15) \quad S(\psi U, \psi V) = S(U, V) + (n-1)\eta(U)\eta(V)$$

for all vector fields  $U, V$  and  $W$ , in which  $R$  and  $S$  denote as the curvature and Ricci tensors belonging to  $M$ . A Kenmotsu manifold  $M$  is found to be generalized  $\eta$ -Einstein when the Ricci tensor  $S$  of it is in the form presented as follows;

$$(2.16) \quad S(U, V) = ag(U, V) + b\eta(U)\eta(V) + cg(\psi U, V),$$

for every  $U, V \in \Gamma(TM)$ , in which  $a, b$  and  $c$  denote as scalar functions. In such a way that  $b \neq 0$  and  $c \neq 0$  in the event that  $c = 0$ ,  $M$  is regarded as  $\eta$ -Einstein manifold. When we view  $\bar{D}$  as a linear connection and  $D$  as a Levi-Civita connection of practically contact metric manifold  $M$  in such a way that

$$(2.17) \quad \bar{D}_U V = D_U V + H(U, V),$$

for all vector field  $X$  and  $Y$ . The following is obtained so that  $\bar{D}$  is a generalized symmetric connection of  $D$ , in which  $H$  denotes as a tensor of type  $(1, 2)$ ;

$$(2.18) \quad H(U, V) = \frac{1}{2}[T(U, V) + T'(U, V) + T'(V, X)],$$

where  $T$  and  $T'$  denote as the torsion tensor of  $D$  and  $\bar{D}$ , respectively.

$$(2.19) \quad g(T'(U, V), W) = g(T(W, U), V).$$

Thanks to (1.1) and (2.19), we obtain the following;

$$(2.20) \quad T'(U, V) = \alpha\{\eta(U)V - g(U, V)\xi\} + \beta\{-\eta(U)\phi V - g(\psi U, V)\xi\}.$$

Furthermore, through use of (1.1), (2.18) and (2.20), we get

$$(2.21) \quad H(U, V) = \alpha\{\eta(V)U - g(U, V)\xi\} + \beta\{-\eta(U)\psi V\},$$

**Corollary 2.1.** *For a Kenmotsu manifold, generalized symmetric metric connection  $\bar{\nabla}$  is provided by*

$$(2.22) \quad \bar{D}_U V = D_U V + \alpha\{\eta(V)U - g(U, V)\xi\} - \beta\eta(U)\psi V.$$

If  $(\alpha, \beta) = (1, 0)$  and  $(\alpha, \beta) = (0, 1)$  are chosen, the generalized symmetric metric connection is diminished to a semi-symmetric metric and a quarter symmetric metric one as presented in the following;

$$\begin{aligned} \bar{D}_U V &= D_U V + \eta(V)U - g(U, V)\xi, \\ \bar{D}_U V &= D_U V - \eta(U)\psi V. \end{aligned}$$

By means of (2.22), we have the following proposition;

**Proposition 2.2.** *The following relations are obtained when  $M$  is a Kenmotsu manifold with generalized metric connection:*

$$(2.23) \quad (\bar{D}_U \psi)V = (\alpha + 1)\{g(\psi U, V)\xi - \eta(V)\psi U\},$$

$$(2.24) \quad \bar{D}_U \xi = (\alpha + 1)\{U - \eta(U)\xi\},$$

$$(2.25) \quad (\bar{D}_U \eta)V = (\alpha + 1)\{g(U, V) - \eta(V)\eta(U)\},$$

for every  $U, V \in \Gamma(TM)$ .

### 3 Projective and Concircular curvature tensors

Considering that  $M$  is a  $n$ -dimensional Kenmotsu manifold, the following can define the curvature tensor  $\bar{R}$  of the generalized metric connection  $\bar{D}$  on  $M$ .

$$(3.1) \quad \bar{R}(U, V)W = \bar{D}_U \bar{D}_V W - \bar{D}_V \bar{D}_U W - \bar{D}_{[U, V]} W,$$

When proposition 2.2 is used, through (2.22) and (3.1), we obtain

$$\begin{aligned} \bar{R}(U, V)W &= R(U, V)W + \{(-\alpha^2 - 2\alpha)g(V, W) + (\alpha^2 + \alpha)\eta(V)\eta(W)\}U \\ &\quad + \{(\alpha^2 + 2\alpha)g(U, W) + (-\alpha^2 - \alpha)\eta(U)\eta(W)\}V \\ &\quad + \left\{ (\alpha^2 + \alpha)[g(V, W)\eta(U) - g(U, W)\eta(V)] \right. \\ &\quad \left. + (\beta + \alpha\beta)[g(U, \psi W)\eta(V) - g(V, \psi W)\eta(U)] \right\}\xi \\ (3.2) \quad &\quad + (\beta + \alpha\beta)\eta(V)\eta(W)\psi U - (\beta + \alpha\beta)\eta(U)\eta(W)\psi V \end{aligned}$$

where

$$(3.3) \quad R(U, V)W = D_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W,$$

denotes as the curvature tensor regarding Levi-connection  $D$ . When (3.2) and the first Bianchi identity are paid attention, we have

$$\begin{aligned} \bar{R}(U, V)W + \bar{R}(V, W)U + \bar{R}(W, U)V &= 2(\beta + \alpha\beta) \left\{ \eta(U)g(\psi V, W) \right. \\ &\quad \left. + \eta(V)g(U, \psi W) + \eta(W)g(V, \psi U) \right\}. \end{aligned}$$

Thus, the following proposition is obtained

**Proposition 3.1.** *Let us consider that  $M$  is an  $n$ -dimensional Kenmotsu manifold together with generalized symmetric metric of type  $(\alpha, \beta)$ . When  $(\alpha, \beta) = (-1, \beta)$  or  $(\alpha, \beta) = (\alpha, 0)$ , it provides the first Bianchi identity of the generalized connection  $\bar{D}$  on  $M$ .*

The following proposition is presented using (2.10), (2.11), (2.12), (2.13) and (3.2).

**Proposition 3.2.** *When  $M$  is an  $n$ -dimensional Kenmotsu manifold with generalized symmetric metric of type  $(\alpha, \beta)$ , we have the following equations:*

$$(3.4) \quad \bar{R}(U, V)\xi = (\alpha + 1)\{\eta(U)V - \eta(V)U + \beta[\eta(V)\psi U - \eta(U)\psi V]\},$$

$$(3.5) \quad \bar{R}(\xi, U)V = (\alpha + 1)\{\eta(V)U - g(U, V)\xi + \beta[\eta(V)\psi U - g(U, \psi V)\xi]\},$$

$$(3.6) \quad \bar{R}(\xi, V)\xi = (\alpha + 1)\{V - \eta(V)\xi - \beta\psi V\},$$

$$\eta(\bar{R}(U, V)W) = (\alpha + 1)\{\eta(V)g(U, W) - \eta(U)g(V, W) + \beta[\eta(V)g(U, \psi W)$$

$$(3.7) \quad - \eta(U)g(V, \psi W)]\}$$

for every  $U, V, W \in \Gamma(TM)$ .

In the following, the Ricci tensor  $\bar{S}$  and the scalar curvature  $\bar{r}$  of a Kenmotsu manifold is presented with generalized symmetric metric connection  $\bar{D}$

$$\begin{aligned} \bar{S}(U, V) &= \sum_{i=1}^n g(\bar{R}(\nu_i, U)V, \nu_i), \\ \bar{r} &= \sum_{i=1}^n \bar{S}(\nu_i, \nu_i), \end{aligned}$$

in which  $U, V \in \Gamma(TM)$ ,  $\{\nu_1, \nu_2, \dots, \nu_n\}$  denote as orthonormal frame. Then by using (2.5) and (3.2) we obtain

$$\begin{aligned} \bar{S}(V, W) &= S(V, W) + \{(2 - n)\alpha^2 + (3 - 2n)\alpha\}g(V, W) + (n - 2)(\alpha^2 + \alpha)\eta(V)\eta(W) \\ (3.8) \quad &\quad - (\beta + \alpha\beta)g(V, \phi W), \end{aligned}$$

in which  $S$  denote as the Ricci tensor with regards to Levi-Civita connection. Due to the fact that Ricci tensor of the Levi-connection is symmetric, (3.8) provides us the following

$$(3.9) \quad \bar{S}(V, W) - \bar{S}(W, V) = -2(\beta + \alpha\beta)g(V, \psi W).$$

Thus, the theorem below is obtained.

**Theorem 3.3.** Consider that  $M$  is an  $n$ -dimensional Kenmotsu manifold. The Ricci tensor  $\bar{S}$  of generalized symmetric metric connection  $\bar{D}$  is symmetric if and only if  $(\alpha, \beta) = (-1, \beta)$  or  $(\alpha, \beta) = (\alpha, 0)$ .

The following theorem is enabled through use of (3.8)

**Theorem 3.4.** When  $M$  is an  $n$ -dimensional Kenmotsu manifold with generalized symmetric metric connection of type  $(\alpha, \beta)$ , The scalar curvature concerning the connection is as follows:

$$(3.10) \quad \bar{r} = r + (n - 2)(1 - n)\alpha^2 - 2(n - 1)^2\alpha,$$

in which  $r$  denotes as scalar curvature of Levi-Civita connection.

**Lemma 3.5.** As in the case of Theorem 3.4, let us suppose that  $M$  is an  $n$ -dimensional Kenmotsu manifold with the connection, then we have

$$(3.11) \quad \bar{S}(V, \xi) = (1 - n)(\alpha + 1)\eta(V),$$

$$(3.12) \quad \bar{S}(\phi V, \phi W) = \bar{S}(V, W) + (n - 1)(1 + \alpha),$$

for every  $V, W \in \Gamma(TM)$ .

*Proof.* Thanks to (2.3), (2.14) and (3.8), we have (3.11). Anymore, the use of (2.5), (2.15) and (3.8) provides us (3.12).  $\square$

**Theorem 3.6.** Suppose that  $M$  is an  $n$ -dimeansional Knemotsu manifold with the connection as in Lemma 7.  $M$  denotes as an  $\eta$ -Einstein manifold regarding generalized symmetric metric connection (for  $\beta \neq 1$  and  $\beta \neq -1$ ) on condition that  $M$  is Ricci semi-symmetric regarding the connection  $\bar{D}$ .

*Proof.* Let  $\bar{R}(U, V)\bar{S} = 0$  be on  $M$  for any  $U, V, W, Z \in \Gamma(TM)$ . Then the following is obtained;

$$(3.13) \quad \bar{S}(\bar{R}(U, V)W, Z) + \bar{S}(W, \bar{R}(U, V)Z) = 0.$$

If we choose  $W = \xi$  and  $U = \xi$  in (3.13), we get

$$(3.14) \quad \bar{S}(\bar{R}(\xi, V)\xi, Z) + \bar{S}(\xi, \bar{R}(\xi, U)Z) = 0.$$

Using (3.5), (3.6) and (3.11) in (3.14), we obtain

$$(3.15) \quad \bar{S}(V, Z) - \beta\bar{S}(\phi V, Z) = (1 - n)\{g(V, Z) + \beta g(V, \phi Z)\}.$$

If one substitutes  $V = \phi V$  in the equation (3.15) and using (3.11), we get

$$(3.16) \quad \begin{aligned} \bar{S}(\phi V, Z) + \beta\bar{S}(V, Z) &= (1 - n)\left\{g(\phi V, Z) + \beta g(V, Z)\right. \\ &\quad \left.+ (\alpha - \beta + 1)\eta(V)\eta(Z)\right\}. \end{aligned}$$

From the (3.15) and (3.16), we obtain

$$(3.17) \quad \bar{S}(V, Z) = \frac{1 - n}{1 - \beta^2}\{(1 + \beta^2)g(V, Z) + (\alpha - \beta + 1)\eta(V)\eta(Z)\}.$$

The equation suggests that  $M$  is an  $n$  Einstein manifold in regard to generalized metric connection.  $\square$

**Corollary 3.7.** *Let us suppose that  $M$  is an  $n-$  dimensional Kenmotsu manifold with generalized symmetric metric connection of type  $(\alpha, \beta)$ . Considering  $M$  is Ricci semi-symmetric in regard to the connection, the statements illustrated below are obtained:*

- (i)  *$M$  is reduced to an Einstein manifold with reference to generalized symmetric metric connection of type  $(\beta - 1, \beta)$ .*
- (ii) *There is no Einstein manifold with the connection of type  $(\alpha, 1)$  and  $(\alpha, -1)$ .*

Let us consider that  $M$  is an  $n-$  dimensional Kenmotsu manifold with generalized connection of type  $(\alpha, \beta)$ . The following definition reveals the projective curvature tensor  $P$  of type  $(1, 3)$  of  $M$  in regard to connection  $\bar{\nabla}$ :

$$(3.18) \quad \bar{P}(U, V)W = \bar{R}(U, V)W - \frac{1}{n-1}\{\bar{S}(V, W)U - \bar{S}(U, W)V\}.$$

**Definition 3.1.** Considering that  $M$  is an  $n-$  dimensional Kenmotsu manifold,  $M$  can be suggested to be  $\xi-$  projectively flat in regard to generalized symmetric metric connection  $\bar{D}$  provided that  $\bar{P}(U, V)\xi = 0$  on  $M$ .

The use of (3.4), (3.11) and (3.18) enables us the following:

$$\bar{P}(U, V)\xi = (\alpha + 1)\beta\{\eta(V)\psi U - \eta(U)\psi V\}.$$

Hence, the theorem presented below is obtained

**Theorem 3.8.** *Let us suppose that  $M$  is an  $n-$  dimensional Kenmotsu manifold. On condition that  $(\alpha, \beta) = (-1, \beta)$  or  $(\alpha, \beta) = (\alpha, 0)$ ,  $M$  denotes as  $\xi-$  projectively flat in regard to generalized connection.*

**Corollary 3.9.** *Let us consider that  $M$  is an  $n-$  dimensional Kenmotsu manifold, which provides the following expressions:*

- (i) *The manifold is suggested to be  $\xi-$  projectively flat in regard to  $\beta-$  quarter symmetric metric connection.*
- (ii) *The manifold is suggested to be  $\xi-$  projectively flat in regard to semi-symmetric metric connection.*

**Definition 3.2.** Supposing that  $M$  is an  $n-$  dimensional Kenmotsu manifold,  $M$  is suggested to be  $\psi-$  projectively flat in regard to the connection  $\bar{D}$  when  $g(\bar{P}(\psi U, \psi V)\phi W, \psi Z) = 0$  on  $M$ .

By means of (3.18), we obtain

$$(3.19) \quad \bar{P}(\psi U, \psi V)\phi W = \bar{R}(\psi U, \psi V)\phi W - \frac{1}{n-1}\{\bar{S}(\psi V, \psi W)U - \bar{S}(\psi U, \psi W)\psi V\}.$$

When (3.12) and (3.19) are used,  $\psi-$  projectively flatness indicates

$$(3.20) \quad \begin{aligned} \bar{K}(\psi U, \psi V, \psi W, \psi Z) &= \frac{1}{n-1}\left\{[\bar{S}(V, W) + (n-1)(1+\alpha)]g(\psi U, \psi Z) \right. \\ &\quad \left. + [\bar{S}(U, W) + (n-1)(1+\alpha)]g(\psi V, \psi Z)\right\}. \end{aligned}$$

When we consider that  $\{\nu_1, \nu_2, \dots, \nu_{n-1}, \xi\}$  is a local orthogonal basis of the vector fields in  $M$  and that  $\{\psi\nu_1, \psi\nu_2, \dots, \psi\nu_{n-1}, \xi\}$  is a local orthogonal basis as well, the

following is obtained via putting  $U = Z = \nu_i$  and summing up in regard to  $i = 1, 2, \dots, n - 1$ .

$$(3.21) \quad \begin{aligned} \sum_{i=1}^{n-1} \bar{K}(\psi\nu_i, \psi V, \psi W, \psi\nu_i) &= \frac{1}{n-1} \left\{ \sum_{i=1}^{n-1} [\bar{S}(V, W) + (n-1)(1+\alpha)]g(\psi e_i, \psi e_i) \right. \\ &\quad \left. + [\bar{S}(\nu_i, W) + (n-1)(1+\alpha)]g(\psi V, \psi\nu_i) \right\}. \end{aligned}$$

We know that

$$(3.22) \quad \sum_{i=1}^{n-1} g(\psi\nu_i, \psi\nu_i) = n-1,$$

and from (3.12)

$$(3.23) \quad \sum_{i=1}^{n-1} \bar{S}(\nu_i, W)g(\psi V, \psi\nu_i) = \bar{S}(V, W) + (n-1)(1+\alpha) \left( 1 - \sum_{i=1}^{n-1} g(\psi V, \psi\nu_i) \right).$$

Thus from (3.22) and (3.23), the equation (3.21) becomes

$$\sum_{i=1}^{n-1} \bar{K}(\psi\nu_i, \psi V, \psi W, \psi\nu_i) = \frac{n-2}{n-1} \bar{S}(V, W) + (n-1)(1+\alpha).$$

Moreover we have

$$(3.24) \quad \sum_{i=1}^{n-1} \bar{K}(\psi\nu_i, \psi V, \psi W, \psi\nu_i) = \bar{S}(\psi V, \psi W) - g(\bar{R}(\xi, \psi V)\psi W, \xi).$$

The use of (3.5), (3.12) and (3.24), we obtain

$$(3.25) \quad \begin{aligned} \sum_{i=1}^{n-1} \bar{K}(\psi\nu_i, \psi V, \psi W, \psi\nu_i) &= \bar{S}(V, W) \\ &\quad + (\alpha+1)\{n-1 + g(\psi V, \psi W) + \beta g(\psi V, W)\} \end{aligned}$$

Thus, through use of (3.24) and (3.25), we have

$$\bar{S}(V, W) = (1-n)(\alpha+1)\{g(V, W) - \eta(V)\eta(W) + \beta g(\psi V, W)\}$$

Accordingly, the following theorem is obtained

**Theorem 3.10.** *The manifold denotes as generalized  $\eta$  Einstein in regard to generalized connection  $\bar{D}$  when a Kenmotsu manifold  $\psi$ - projectively flat concerning the connection  $\bar{D}$ .*

**Corollary 3.11.** *Let us consider that  $M$  is an  $n$ -dimensional Kenmotsu manifold. When the manifold is  $\psi$ -projectively flat In relation to the connection  $\bar{\nabla}$ , the following expressions are obtained:*

- (i) *The manifold is  $\eta$ -Einstein manifold in regard to  $\beta$ -quarter symmetric connection.*
- (ii) *It is Ricci flat regarding generalized symmetric metric connection of type  $(-1, \beta)$ .*
- (iii) *It is Ricci flat concerning semi-symmetric connection.*

When  $M$  is considered to be  $n-$  dimensional Kenmotsu manifold, the concircular curvature tensor of  $M$  in relation the connection  $\bar{D}$  can be illustrated through

$$(3.26) \quad \bar{C}^*(U, V)W = \bar{R}(U, V)W - \frac{\bar{r}}{n(n-1)}\{g(V, W)U - g(U, W)V\}$$

The use of (3.2), (3.10) and (3.26) provides us the following:

$$\begin{aligned} g(\bar{C}^*(U, V)W, \xi) &= g(C^*(U, V)W, \xi) + \{(-3+n)\alpha^2 + (-4+2n)\alpha\}g(V, W)\eta(U) \\ &\quad + \{(3-n)\alpha^2 + (4-2n)\alpha\}g(U, W)\eta(V) \\ &\quad + (\alpha + \alpha^2)\{g(V, W)\eta(U) - g(U, W)\eta(V)\} \\ (3.27) \quad &\quad + (\beta + \alpha\beta)\{-g(V, \psi W)\eta(U) + g(U, \psi W)\eta(V)\}, \end{aligned}$$

where  $C^*$  denotes as concircular curvature tensor of  $M$  in relation to Levi-Civita connection. When  $\bar{C}^* = C^*$  is considered, we obtain the following by putting  $U = \xi$  of the equation (3.27);

$$(3.28) \quad \{(-2+n)\alpha^2 + (-3+2n)\alpha\}\{g(V, W) - \eta(V)\eta(W)\} = (\beta + \alpha\beta)g(V, \psi W).$$

The following theorem is obtained through (3.8), (3.27) and (3.28)

**Theorem 3.12.** *In a Kenmotsu manifold, the following is attained when concircular curvature tensor is invariant under generalized metric connection*

$$\begin{aligned} \bar{S}(V, W) &= S(V, W) - \{(-4+2n)\alpha^2 + (-6+4n)\alpha\}g(V, W) + \{(2n-4)\alpha^2 + (3n-5)\alpha\}\eta(V)\eta(W) \\ &\quad \text{for every } V, W \in \Gamma(TM). \end{aligned}$$

**Corollary 3.13.** *Suppose that  $M$  is an  $n-$  dimensional Kenmotsu manifold. When concircular curvature tensor is invariant under the connection  $\bar{\nabla}$ , the expressions presented below are obtained:*

- (i) *If the manifold is Ricci flat, it is  $\eta-$  Einstein regarding generalized symmetric metric connection.*
- (ii) *Ricci tensor is invariant with reference to  $\beta-$  quarter symmetric metric connection.*

## 4 Generalized Wintgen Inequality

Let  $N$  be  $m$ -dimensional submanifold of Kenmotsu manifold  $M$  with induced metric  $g$ . Let  $\nabla$  and  $\nabla^\perp$  denote the induced connection on the tangent bundle  $TN$  and  $TN^\perp$  of  $N$ , respectively. We write the formulas of Gauss and Weingarten as follows

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TN)$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad \forall X \in \Gamma(TN), \forall V \in \Gamma(TN^\perp)$$

where  $h$  gives the second fundamental form and  $A_V$  denotes the shape operator. We have the following relation between  $A_V$  and  $h$

$$g(A_V X, Y) = g(h(X, Y), V) \quad \forall X, Y \in \Gamma(TN), \forall V \in \Gamma(TN^\perp).$$

The equation of Gauss is given by [25]

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W))$$

for all vector fields  $X, Y, Z \in TN$ .

For a local orthonormal tangent frame  $\{E_1, \dots, E_m\}$  of the tangent bundle  $TN$  and a local orthonormal normal frame  $\{E_{m+1}, \dots, E_n\}$  of the normal bundle  $T^\perp N$  of  $N$  in  $M$ , we give the mean curvature vector by

$$\mathcal{H} = \sum_{i=1}^m \frac{1}{m} h(E_i, E_i),$$

and

$$\|h\|^2 = \sum_{i,j=1}^m g(h(E_i, E_j), h(E_i, E_j))^2.$$

We write the scalar curvature  $\tau$  at any point  $p \in N$  by

$$(4.1) \quad \tau = \sum_{1 \leq i < j \leq m} R(E_i, E_j, E_j, E_i)$$

Let  $\mathcal{K}$  be the sectional curvature function on  $N$ , then we define the normalized scalar curvature  $\rho$  as

$$\rho = \frac{2\tau}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \mathcal{K}(e_i \wedge e_j)$$

and the scalar normal curvature  $\mathcal{K}_N$  is expressed as [15]

$$\mathcal{K}_N = \sum_{1 \leq \alpha < \beta \leq n} \sum_{1 \leq i < j \leq m} \left( \sum_{k=1}^m h_{jk}^r h_{ik}^s - h_{jk}^r h_{ik}^s \right)^2.$$

The scalar normal curvature is written as [15]

$$\rho_N = \frac{2}{m(m-1)} \sqrt{\mathcal{K}_N}$$

We recall that a Kenmotsu manifold  $M$  is known as Kenmotsu space form if its sectional curvature is constant  $c$ (say) and its curvature tensor  $R$  is given by

$$(4.2) \quad \begin{aligned} R(X, Y)Z &= \frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\}. \end{aligned}$$

Taking into consideration (3.2), we can define the curvature tensor  $\bar{R}$  of the generalized

metric connection  $\bar{D}$  on Kenmotsu space form  $M(c)$  as follows

$$\begin{aligned}
 \bar{R}(U, V)W &= \left( \frac{c-3}{4} - \alpha^2 - 2\alpha \right) \{g(V, W)U - g(U, W)V\} \\
 &\quad + \left( \frac{c+1}{4} - \alpha^2 - 2\alpha \right) \{\eta(U)\eta(W)V - \eta(V)\eta(W)U \\
 &\quad + \eta(V)g(U, W)\xi - \eta(U)g(V, W)\xi\} \\
 &\quad + \frac{c+1}{4} \{g(U, \psi W)\psi V - g(V, \psi W)\psi U + 2g(U, \psi V)\psi W\} \\
 &\quad + (\beta + \alpha\beta) \{g(U, \psi W)\eta(V)\xi - g(V, \psi W)\eta(U)\xi \\
 (4.3) \quad &\quad + \eta(V)\eta(W)\psi U - \eta(U)\eta(W)\psi V\}.
 \end{aligned}$$

Now, we prove the generalized Wintgen inequality for submanifolds in a Kenmotsu space form endowed with generalized metric connection.

**Theorem 4.1.** *Let  $N$  be  $m$ -dimensional submanifold of a Kenmotsu space form  $M(c)$ . Then*

$$\begin{aligned}
 \rho_N &\leq \|\mathcal{H}\|^2 - 2 \left\{ \rho - \frac{1}{2(m-1)} \left( (2m+1)(c-3) - (c+1) \right) \right\} \\
 (4.4) \quad &\quad - \frac{2\alpha}{m} \left[ (m-2)\alpha + 2(m-1) \right]
 \end{aligned}$$

Moreover, the equality holds in the relation (4.4) if and only if with respect to some orthonormal tangent frame  $\{E_1, \dots, E_m\}$  and orthonormal normal frame  $\{E_{m+1}, \dots, E_n\}$ , the shape operator  $A$ , takes the following form:

$$(4.5) \quad A_{m+1} = \begin{pmatrix} t_1 & t_4 & 0 & \dots & 0 & 0 \\ t_4 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_1 \end{pmatrix},$$

$$(4.6) \quad A_{m+2} = \begin{pmatrix} t_2 + t_4 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_2 - t_4 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_2 \end{pmatrix},$$

$$(4.7) \quad A_{m+3} = \begin{pmatrix} t_3 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_3 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_3 \end{pmatrix}, \quad A_{m+4} = \dots = A_n = 0,$$

where  $t_1, t_2, t_3$  and  $t_4$  are real functions on  $N$ .

*Proof.* Let us choose  $\{E_1, \dots, E_m\}$  and  $\{E_{m+1}, \dots, E_n\}$  as local orthonormal frame and local orthonormal normal frame on  $N$  respectively. Then, using Gauss equation with (4.3), we have

$$(4.8) \quad \begin{aligned} 2\tau = & 2m(2m+1)\frac{(c-3)}{4} + 2m\frac{(c+1)}{4} + (m-2)(1-m)\alpha^2 \\ & -2(m-1)^2\alpha + \sum_{\alpha=m+1}^n \sum_{1 \leq i < j \leq m} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2] \end{aligned}$$

where we have used (4.1).

We also observe that

$$(4.9) \quad \begin{aligned} n^2\|\mathcal{H}\|^2 = & \sum_{\alpha=m+1}^{n-m} \left( \sum_{i=1}^m h_{ii}^\alpha \right)^2 = \frac{1}{m-1} \sum_{\alpha=m+1}^{n-m} \sum_{1 \leq i < j \leq m} (h_{ii}^\alpha - h_{jj}^\alpha)^2 \\ & + \frac{2m}{m-1} \sum_{\alpha=m+1}^{n-m} \sum_{1 \leq i < j \leq m} h_{ii}^\alpha h_{jj}^\alpha. \end{aligned}$$

On the other hand, from [14] we have

$$(4.10) \quad \begin{aligned} & \sum_{\alpha=m+1}^{n-m} \sum_{1 \leq i < j \leq m} (h_{ii}^\alpha - h_{jj}^\alpha)^2 + 2m \sum_{\alpha=m+1}^{n-m} \sum_{1 \leq i < j \leq m} (h_{ij}^\alpha)^2 \geq \\ & 2m \left[ \sum_{m+1 \leq \alpha < \beta \leq n-m} \sum_{1 \leq i < j \leq m} \left( \sum_{k=1}^m (h_{jk}^\alpha h_{ik}^\beta - h_{ik}^\alpha h_{jk}^\beta) \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

Taking into account (4.9) and (4.10) and in view of (4.2), we get

$$(4.11) \quad m^2\|\mathcal{H}\|^2 - m^2\rho_N \geq \frac{2m}{m-1} \sum_{\alpha=m+1}^{n-m} \sum_{1 \leq i < j \leq m} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2].$$

Finally, combining (4.8) and (4.11), we find

$$(4.12) \quad \begin{aligned} \rho_N \leq & \|\mathcal{H}\|^2 - 2 \left\{ \rho - \left( \frac{2m+1}{m-1} \right) \left( \frac{c-3}{2} \right) - \frac{1}{(m-1)} \left( \frac{c+1}{2} \right) \right\} \\ & - \frac{2(m-2)}{m} \alpha^2 - \frac{4(m-1)}{m} \alpha \end{aligned}$$

whereby proving the required result.

Moreover, equality sign holds in the inequality (4.4) if and only if with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operator takes the forms of (4.5), (4.6) and (4.7).  $\square$

**Remarks :**

- We note that a submanifold  $N$  for which equality holds in Theorem 4.1 is called Wintgen ideal submanifold [8].
- From (1.1), if  $\alpha = 0$  ( $\beta = 0$ ), then the generalized symmetric connection is reduced to  $\beta$ -quarter-symmetric connection ( $\alpha$ -semi-symmetric connection), respectively. Also, the connection is reduced to a semi-symmetric one and a quarter-symmetric one when  $(\alpha, \beta) = (1, 0)$  and  $(\alpha, \beta) = (0, 1)$  are selected. Hence, using (4.4) we can obtain inequalities for above discussed cases.

## 5 An example

A 3-dimensional manifold  $M = \{(x, y, z) \in R^3 : x \neq 0\}$  is considered, in which  $(x, y, z)$  are regarded as the standard coordinates in  $R^3$ . Suppose that  $\nu_1, \nu_2, \nu_3$  are linearly independent global frame on  $M$  as presented below

$$(5.1) \quad \nu_1 = x \frac{\partial}{\partial z}, \quad \nu_2 = x \frac{\partial}{\partial y}, \quad \nu_3 = -x \frac{\partial}{\partial x}.$$

Consider that  $g$  is a Riemannian metric as presented below

$$g(\nu_1, \nu_2) = g(\nu_1, \nu_3) = g(\nu_2, \nu_3) = 0, \quad g(\nu_1, \nu_1) = g(\nu_2, \nu_2) = g(\nu_3, \nu_3) = 1,$$

When we consider that the  $\eta$  is the 1-form represented as  $\eta(Z) = g(Z, \nu_1)$  for every  $Z \in TM$  and  $\psi$  is the  $(1, 1)$  tensor field presented as  $\psi\nu_1 = \nu_2, \psi\nu_2 = -\nu_1$  and  $\psi\nu_3 = 0$ , we thereby get  $\eta(\nu_3) = 1, \psi^2 Z = -Z + \eta(Z)\nu_3$  and  $g(\psi Z, \psi W) = g(Z, W) - \eta(Z)\eta(W)$  for all  $Z, W \in TM$  through use of linearity of  $\psi$  and  $g$ . Therefore,  $\nu_3 = \xi$  describes an almost contact metric manifold. Considering that  $D$  is the Levi-Civita connection concerning the Riemannian metric  $g$ , The following is obtained;

$$(5.2) \quad [\nu_1, \nu_2] = 0, \quad [\nu_1, \nu_3] = \nu_1, \quad [\nu_2, \nu_3] = \nu_2,$$

By means of using Koszul formula, the following can be calculated in an easy way

$$(5.3) \quad \begin{aligned} D_{\nu_1}\nu_1 &= -\nu_3, & D_{\nu_1}\nu_2 &= 0, & D_{\nu_1}\nu_3 &= \nu_1, \\ D_{\nu_2}\nu_1 &= 0, & D_{\nu_2}\nu_2 &= -\nu_3, & D_{\nu_2}\nu_3 &= 0, \\ D_{\nu_3}\nu_1 &= 0, & D_{\nu_3}\nu_2 &= 0, & D_{\nu_3}\nu_3 &= 0. \end{aligned}$$

The relations presented above remark that  $(D_U\phi)V = g(\phi U, V)\xi - \eta(V)\phi U, \nabla_U\xi = U - \eta(U)\xi$ , for all  $\nu_3 = \xi$ . Thus, the manifold  $M$  denotes as a Kenmotsu

one with the structure  $(\psi, \xi, \eta, g)$ . Besides, the non-vanishing components of the tensor can be calculated by using the relations above as presented in the following;

$$(5.4) \quad \begin{aligned} R(\nu_1, \nu_2)\nu_1 &= \nu_2, \quad R(\nu_1, \nu_2)\nu_2 = -\nu_1, \quad R(\nu_1, \nu_3)\nu_1 = \nu_3 \\ R(\nu_1, \nu_3)\nu_3 &= -\nu_1, \quad R(\nu_2, \nu_3)\nu_2 = \nu_3, \quad R(\nu_2, \nu_3)\nu_3 = -\nu_2 \end{aligned}$$

The equations (5.4) enable us to calculate the components of Ricci tensor in an easy way as in the following: [20]

$$(5.5) \quad S(\nu_1, \nu_1) = -2, \quad S(\nu_2, \nu_2) = -2, \quad S(\nu_3, \nu_3) = -2$$

Now, we can make similar calculations for generalized metric connection. Using (2.22) in the above equations, we get

$$(5.6) \quad \begin{aligned} \bar{D}_{\nu_1}\nu_1 &= -(1+\alpha)\nu_3, & \bar{D}_{\nu_1}\nu_2 &= 0, & \bar{D}_{\nu_1}\nu_3 &= (1+\alpha)\nu_1, \\ \bar{D}_{\nu_2}\nu_1 &= 0, & \bar{D}_{\nu_2}\nu_2 &= -(1+\alpha)\nu_3, & \bar{D}_{\nu_2}\nu_3 &= \alpha\nu_2, \\ \bar{D}_{\nu_3}\nu_1 &= -\beta\nu_2, & \bar{D}_{\nu_3}\nu_2 &= \beta\nu_1, & \bar{D}_{\nu_3}\nu_3 &= 0. \end{aligned}$$

By means of (5.6), we can make calculations of the components of curvature tensor concerning generalized metric connection as follows:

$$(5.7) \quad \begin{aligned} \bar{R}(\nu_1, \nu_2)\nu_1 &= (1+\alpha)^2\nu_2, & \bar{R}(\nu_1, \nu_2)\nu_2 &= -(1+\alpha)^2\nu_1, \\ \bar{R}(\nu_1, \nu_3)\nu_1 &= (1+\alpha)\nu_3 & \bar{R}(\nu_1, \nu_3)\nu_3 &= (1+\alpha)(\beta\nu_2 - \nu_1), \\ \bar{R}(\nu_2, \nu_3)\nu_2 &= (1+\alpha)\nu_3, & \bar{R}(\nu_2, \nu_3)\nu_3 &= -(1+\alpha)(-\beta\nu_1 + \nu_2) \\ \bar{R}(\nu_3, \nu_2)\nu_1 &= -(1+\alpha)\beta\nu_3, & \bar{R}(\nu_3, \nu_1)\nu_2 &= (1+\alpha)\beta\nu_3. \end{aligned}$$

Through (5.7), The components of Ricci tensor mentioned above are presented below:

$$(5.8) \quad \begin{aligned} \bar{S}(\nu_1, \nu_1) &= -(1+\alpha)(2+\alpha), & \bar{S}(\nu_2, \nu_2) &= -(1+\alpha)(2+\alpha), \\ \bar{S}(\nu_3, \nu_3) &= -2(1+\alpha), & \bar{S}(\nu_2, \nu_1) &= -(1+\alpha)\beta. \end{aligned}$$

(5.7) and (5.8) are verified through the equations (3.2) and (3.8), respectively. Moreover, the scalar curvature regarding the Levi-Civita connection and generalized-metric connection are  $r = -6$  and  $\bar{r} = -2(1+\alpha)(3+\alpha)$ . The following expressions are obtained through (3.18):

$$(5.9) \quad \bar{P}(\nu_1, \nu_3)\nu_3 = (1+\alpha)\beta\nu_2, \quad \bar{P}(\nu_2, \nu_3)\nu_3 = -(1+\alpha)\beta E_1, \quad \bar{P}(\nu_1, \nu_2)\nu_3 = 0.$$

which verify Theorem 3.3, Theorem 3.4 and Theorem 3.8.

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