Some characterizations of slant submanifolds of trans-Sasakian manifolds

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Abstract. The object of the present paper is to study slant submanifolds of trans-Sasakian manifolds with second fundamental form satisfying certain conditions. Totally umbilical slant submanifolds of trans-Sasakian manifolds have been considered. Nature of slant submanifolds of trans-Sasakian manifolds as Ricci soliton has also been analyzed.

Key words: trans-Sasakian manifolds; second fundamental form; recurrent; slant submanifold; totally geodesic; totally umbilical.

1 Introduction

As a unification of cosymplectic, Sasakian and Kenmotsu structure, the notion of trans-Sasakian structure arose from the works of Blair and Oubina [1], [2], [19]. Trans-Sasakian structures have further been analyzed by Marrero [18]. Marrero proved that a trans-Sasakian structure of dimension $\geq 5$ is either $\alpha$-Sasakian or $\beta$-Kenmotsu. Three-dimensional trans-Sasakian manifolds have been studied in the paper [8].

The theory of slant submanifolds was introduced by Chen [6] as a generalization of holomorphic and totally real submersions in Kaehlerian geometry. The almost contact analogue of such properties can be found in the work of Lotta [17]. Slant submanifolds of Sasakian manifolds have been characterized by Cabrerizo et al [3]. Slant submanifolds of trans-Sasakian manifolds have been studied by R. S. Gupta [14]. Invariant submanifolds of trans-Sasakian manifolds have been characterized in [10], [24], [26]. Pseudo slant submanifolds of trans-Sasakian manifolds have been studied in [9]. Umbilical submanifolds of Kenmotsu manifolds have been considered in [28], [31]. In [30], invariant submanifolds of Kenmotsu manifolds have been studied. The Kenmotsu case has been generalized to trans-Sasakian case in the papers [10], [24]. In [24], some necessary and sufficient conditions for invariant submanifolds of trans-Sasakian manifolds to be totally geodesic have been obtained. Since totally geodesic submanifolds are the simplest submanifolds, a natural tendency of research
in submanifold theory is to establish relation between totally geodesic submanifolds and other submanifolds. In [16], necessary and sufficient conditions for invariant submanifolds of Sasakian manifolds to be totally geodesic have been deduced. In [11], necessary and sufficient conditions for submanifolds of Sasakian manifolds to be totally geodesic have been obtained. Invariant submanifolds of trans-Sasakian manifolds have been studied in [7]. The anti invariant case is treated in [27].

Considering the above works, in the present paper we would like to study slant submanifolds of trans-Sasakian manifolds whose second fundamental form satisfy certain conditions. The present paper is organized as follows:

After the introduction and preliminaries, slant submanifolds of trans-Sasakian manifolds with second fundamental form satisfying some parallel and recurrent conditions have been studied in Section 3. Totally umbilical slant submanifolds of trans-Sasakian manifolds have been considered in Section 4. Nature of slant submanifolds of trans-Sasakian manifolds as Ricci soliton has been analyzed in the last section.

2 Preliminaries

Let $\tilde{M}$ be an almost contact metric manifold of dimension $2n + 1$, that is, a $(2n + 1)$-dimensional differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$. By definition $\phi, \xi, \eta$ are tensor fields of type $(1, 1), (1, 0), (0, 1)$, respectively, and $g$ is a Riemannian metric such that [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all differentiable vector fields $X, Y$ on $\tilde{M}$. Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi).$$

Let $\Phi$ be the fundamental 2-form defined by $\Phi(X, Y) = g(X, \phi Y)$, for all differentiable vector fields $X, Y$ on $\tilde{M}$. An almost contact metric structure $(\phi, \xi, \eta, g)$ on $\tilde{M}$ is called trans-Sasakian structure [2], [18] [19] if $(\tilde{M} \otimes \mathbb{R}, J, G)$ belongs to the class $W_4$ [13] where $J$ is the almost complex structure on $\tilde{M} \otimes \mathbb{R}$ defined by

$$J(X, f d/df) = (\phi X - f\xi, \eta(X)d/dt),$$

for a vector field $X$ on $\tilde{M}$, a smooth functions $f$ on $\tilde{M} \otimes \mathbb{R}$ and the product metric $G$ on $\tilde{M} \otimes \mathbb{R}$. This may be expressed by the condition [2]

$$\nabla_X \psi Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for smooth functions $\alpha$ and $\beta$ on $\tilde{M}$, where $\nabla$ is the Levi-Civita connection on $\tilde{M}$. In such case, we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. From (2.1) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X))\xi,$$

$$(\nabla_X \eta) Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$
For a \((2n + 1)\)-dimensional trans-Sasakian manifold of type \((\alpha, \beta)\) we know [21] the following:

\[
\begin{align*}
\tilde{R}(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\
&\quad - (X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X.
\end{align*}
\]

\[
\tilde{S}(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X) - (2n - 1)X\beta - \eta(X)\xi\beta - (\phi X)\alpha.
\]

We also know [18]

\[
2\alpha\beta + \xi\alpha = 0.
\]

Let \(f : (M, g) \to (\tilde{M}, g)\) be an isometric immersion from an \(n\)-dimensional Riemannian manifold \(M\) to a \((2n + 1)\) dimensional trans-Sasakian manifold \(\tilde{M}\). Then we have [4]

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\tilde{\nabla}_X N &= -A_N X + \tilde{\nabla}^\perp_X N,
\end{align*}
\]

for all vector fields \(X, Y\) tangent to \(M\) and normal vector field \(N\) on \(M\), where \(\nabla\) is the Riemannian connection on \(M\) defined by the induced metric \(g\) and \(\nabla^\perp\) is the normal connection on \(T^\perp M\) of \(M\). \(h\) is the second fundamental form of the immersion.

We consider the Codazzi equation

\[
R^N(X, Y)Z = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z).
\]

A submanifold \(M\) of \(\tilde{M}\) is said to be invariant if the structure vector field \(\xi\) is tangent to \(M\) at every point of \(M\) and \(\phi X\) is tangent to \(M\) for every vector \(X\) tangent to \(M\) at every point on \(M\). A submanifold \(M\) of \(\tilde{M}\) is said to be anti invariant if the structure vector field \(\xi\) is tangent to \(M\) at every point of \(M\) and \(\phi X\) is normal to \(M\) for every vector \(X\) tangent to \(M\) at every point on \(M\). The submanifold is called totally geodesic if its second fundamental form vanishes identically on it [16]. In view of (2.2) and (2.4), we get

\[
\nabla_X \xi + h(X, \xi) = -\alpha\phi X + \beta(X - \eta(X)\xi).
\]

Assume that \(\xi\) is tangent to \(M\). For each non-zero vector fields \(X\) tangent to \(M\) at a point \(p\) such that \(X\) is not proportional to \(\xi_p\), we denote by \(\theta(X)\) the Weingarten angle of \(X\), that is the angle between \(\phi X\) and \(T_p M\). The submanifold is called slant if the Weingarten angle \(\theta(X)\) is constant which is independent of the choice of \(p \in M\). The angle \(\theta\) is called slant angle. If \(\theta = \frac{\pi}{2}\), the submanifold is anti invariant and for \(\theta = 0\) it is invariant.

Let \(M\) be a slant submanifold of a \((2n + 1)\) dimensional manifold \(\tilde{M}\) with induced metric \(g\). For any \(X\) tangent to \(M\), we write

\[
\phi X = PX + FX
\]

and for \(N\) normal to \(M\) we write

\[
\phi N = tN + fN.
\]
where \( PX \) and \( FX \), respectively, denote the tangential and normal part of \( \phi X \); \( tN \) and \( fN \), respectively, denote the tangential and normal part of \( \phi N \).

Suppose the submanifold is slant. Then we get the following:

\[
(2.5) \quad \nabla_X \xi = -\alpha PX + \beta (X - \eta(X)\xi).
\]

\[
(2.6) \quad h(X, \xi) = -\alpha FX.
\]

\[
R^T(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)PX - \eta(X)PY) - (X\alpha)PY + (Y\alpha)PX
\]

\[
- (X\beta)P^2Y + (Y\beta)P^2X.
\]

\[
(2.7) \quad R^N(X, Y)\xi = 2\alpha\beta(\eta(Y)FX - \eta(X)FY) - (X\alpha)FY + (Y\alpha)FX
\]

\[
- (X\beta)F^2Y + (Y\beta)F^2X.
\]

\[
ST(X, \xi) = 2n(\alpha^2 - \beta^2)/\eta(X) - (2n - 1)X\beta - \eta(X)\xi\beta - (PX)\alpha.
\]

\[
SN(X, \xi) = -(FX)\alpha.
\]

(2.8) Here \( R^T \) and \( R^N \) denote respectively the tangential and normal part of \( \tilde{R} \); \( ST \) and \( SN \) denote respectively the tangential and normal part of the Ricci curvature \( \tilde{S} \).

For a slant submanifold of trans-Sasakian manifold with slant angle \( \theta \) we get [14]

\[
P^2X = \cos^2\theta(X - \eta(X)\xi).
\]

\[
(2.9) \quad F^2X = \sin^2\theta(X - \eta(X)\xi).
\]

Example 2.1. Let us consider \( \mathbb{R}^5 \) with

\[
\eta = \frac{1}{2}(dz - \sum_{i=1}^{2} y^i dx^i), \quad \xi = 2 \frac{\partial}{\partial z},
\]

\[
g = \eta \otimes \eta + \frac{1}{4} \left( \sum_{i=1}^{2} (dx^i \otimes dx^i + dy^i \otimes dy^i) \right),
\]

and

\[
\phi(X_1, X_2, Y_1, Y_2, Z) = (X_2, -X_1, Y_2, -Y_1, y^2 X_1 - y^1 X_2).
\]

Then \( (\mathbb{R}^5, \phi, \xi, \eta, g) \) is a trans-Sasakian manifold. The map \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^5 \) defined by

\[
f(u, v, t) = 2(u \cos \theta, v, u \sin \theta, 0, t)
\]

is a slant submersion. For details, we refer to [3].
3  Slant submanifolds of trans-Sasakian manifolds with the second fundamental form satisfying some conditions

**Definition 3.1.** The second fundamental form $h$ of a submanifold is called parallel if it satisfies

$$\left(\nabla_{W} h\right)(X, Y) = 0$$

for the vector fields $X, Y, W$ tangent to the submanifold.

**Theorem 3.1.** If the second fundamental form of a slant submanifold of a proper trans-Sasakian manifold is parallel, then it is a 1-dimensional manifold with $\text{Ker}\{F\}$ as the section of its tangent bundle.

**Proof.** Consider a slant submanifold of a trans-Sasakian manifold with parallel second fundamental form. From (3.1), we have

$$\nabla_{W} h(X, Y) - h(\nabla_{W} X, Y) - h(X, \nabla_{W} Y) = 0.$$ 

Putting $X = Y = \xi$ and using (2.5) and (2.6) in the above equation, we get $\alpha\beta FW = 0$.

If $\alpha$ and $\beta$ both are non-zero, then $W \in \text{Ker}\{F\}$. Thus, as before, the claim follows. \qed

From the above theorem, we immediately obtain the following:

**Corollary 3.1.** A hypersurface with parallel second fundamental form of a proper trans-Sasakian manifold is not slant.

**Definition 3.2.** The second fundamental form of a submanifold is called recurrent if

$$\left(\nabla_{X} h\right)(Y, Z) = \omega(X)h(Y, Z),$$

where $\omega$ is an 1-form on $M$.

**Theorem 3.2.** If the second fundamental form of a slant submanifold of a proper trans-Sasakian manifold is recurrent, then it is a 1-dimensional manifold with $\text{Ker}\{F\}$ as the section of its tangent bundle.

**Proof.** Consider a slant submanifold of a trans-Sasakian manifold with recurrent second fundamental form. From (3.2), we have

$$\nabla_{W} h(X, Y) - h(\nabla_{W} X, Y) - h(X, \nabla_{W} Y) = \omega(X)h(Y, Z).$$

Putting $X = Y = \xi$ and using (2.5) and (2.6) in the above equation, we get $\alpha\beta FW = 0$.

If $\alpha$ and $\beta$ both are non-zero, then $W \in \text{Ker}\{F\}$. Thus, as before, the claim follows. \qed
The above theorem also gives

**Corollary 3.2.** A hypersurface with recurrent second fundamental form of a proper trans-Sasakian manifold is not slant.

**Definition 3.3.** The second fundamental form of a submanifold is called semi parallel if

\[(3.4) \quad \tilde{R}(X, Y)h(U, V) = 0\]

for the vector fields \(X, Y, U, V\) tangent to the submanifold.

**Theorem 3.3.** If the second fundamental form of a slant submanifold of a trans-Sasakian manifold of type \((\alpha, \beta)\) is semi parallel, then the submanifold is a 1-dimensional manifold with \(\text{Ker}\{F\}\) as the section of its tangent bundle, provided \(\alpha \neq 0\) and \(\alpha^2 \neq \beta^2\).

**Proof.** Consider a slant submanifold of trans-Sasakian manifold with semi parallel second fundamental form. Then, expanding (3.4), we get

\[(3.5) \quad \tilde{R}^N(X, Y)h(U, V) - h(R^T(X, Y)U, V) - h(U, R^T(X, Y)V) = 0.\]

In (3.5) putting \(Y = U = V = \xi\) and using (2.5), (2.6) and (2.7), we get after simplification

\[\alpha(\alpha^2 - \beta^2)FX = 0.\]

The above equation concludes the proof. \(\Box\)

From the above theorem, we immediately get the following:

**Corollary 3.3.** A hypersurface with semi parallel second fundamental form of a trans-Sasakian manifold is not slant, provided \(\alpha \neq 0\) and \(\alpha^2 \neq \beta^2\).

**Definition 3.4.** The second fundamental form of a submanifold is called pseudo parallel if

\[(3.6) \quad \tilde{R}(X, Y)h(U, V) = f Q(g, h)(X, Y, U, V)\]

for the vector fields \(X, Y, U, V\) tangent to the submanifold. Here \(f\) is a smooth function on the submanifold and \(Q\) is the pseudo symmetry operator defined in the sense of Deszcz [12].

**Theorem 3.4.** If the second fundamental form of a slant submanifold of a trans-Sasakian manifold of type \((\alpha, \beta)\) is pseudo parallel, then the submanifold is a 1-dimensional manifold with \(\text{Ker}\{F\}\) as the section of its tangent bundle, provided \(\alpha \neq 0\) and \(\alpha^2 \neq \beta^2 + 2f\).

**Proof.** Consider an anti invariant submanifold of trans-Sasakian manifold with pseudo parallel second fundamental form. Then expanding (3.6), we get

\[(3.7) \quad R^N(X, Y)h(U, V) - h(R^T(X, Y)U, V) = \begin{cases} f(g(Y, h(U, V))X - g(X, h(U, V))Y + h(g(Y, U)X - g(X, Y)V, V) + h(U, g(Y, V)X - g(X, V)Y). \end{cases}\]
In (3.7) putting $Y = U = V = \xi$ and using (2.5), (2.6) and (2.7), we get after simplification

$$\alpha(\alpha^2 - \beta^2 - 2f)FX = 0,$$

which concludes the proof.

An immediate consequence of the above theorem is

**Corollary 3.4.** A hypersurface with pseudo parallel second fundamental form of a trans-Sasakian manifold is not slant, provided $\alpha \neq 0$ and $\alpha^2 \neq \beta^2 + 2f$.

## 4 Totally umbilical slant submanifolds of trans-Sasakian manifolds

**Definition 4.1.** A submanifold of a trans-Sasakian manifold is called totally umbilical [5] if its second fundamental form satisfies

$$h(X, Y) = g(X, Y)H$$

for the tangent vector fields $X, Y$ of the submanifold. Here $H$ is the mean curvature vector.

**Theorem 4.1.** A totally umbilical slant submanifold of a trans-Sasakian manifold with $\xi \beta \neq 0$ is a 1-dimensional manifold and it is integral curve of the Reeb vector field $\xi$.

**Proof.** Let $M$ be a totally umbilical proper slant submanifold of a trans-Sasakian manifold. Then from Codazzi equation, we get

$$R^N (X, Y)\xi = \eta(Y)\nabla^\perp_X H - \eta(X)\nabla^\perp_Y H.$$

Putting $Y = \xi$ and using (2.3) and (2.8), in the above equation, we obtain

$$\nabla^\perp_X H = \xi \beta F^2 X.$$

But from the umbilical condition

$$g(U, V)\nabla^\perp_X H = (\tilde{\nabla}_X h)(U, V).$$

Putting $U = V = \xi$ in above

(4.1) $$\nabla^\perp_X H = 0.$$

In view of the above equations, we see that

$$F^2 X = 0.$$

Using (2.9), we get for proper slant submanifold with $\xi \beta \neq 0$

$$X = \eta(X)\xi.$$

Hence $X \in \{\xi\}$ and we conclude the proof.

**Remark 4.1.** The above theorem is also true for Kenmotsu case. In Kenmotsu case the above theorem has been proved in the paper [31] in another way. Our process is simpler than that of the paper [31].
5 Ricci soliton on slant submanifolds of trans-Sasakian manifolds

The notion of Ricci flow [15] was introduced by Hamilton and was later successfully used by Perelman [20] to prove the well known Poincare conjecture. A self similar solution of Ricci flow equation is known as Ricci soliton. Ricci soliton in the context of almost contact geometry was studied by Sharma [29].

Definition 5.1. A submanifold of a $(2n + 1)$-dimensional trans-Sasakian manifold is said to be Ricci soliton if it satisfies the following:

\[ \mathcal{L}_V g(X, Y) + 2S^T(X, Y) + \lambda g(X, Y) = 0. \]

Here $\lambda$ is a constant called the soliton constant. The soliton is called shrinking if $\lambda < 0$, expanding if $\lambda > 0$ and steady if $\lambda = 0$.

Theorem 5.1. A slant submanifold of a trans-Sasakian manifold as a Ricci soliton is expanding, steady or shrinking according as $\xi \beta > (\alpha^2 - \beta^2)$, $\xi \beta = (\alpha^2 - \beta^2)$ or $\xi \beta < (\alpha^2 - \beta^2)$.

Proof. Let a slant submanifold of a trans-Sasakian manifold admits a Ricci soliton. Then putting $X = Y = V = \xi$ in (5.1), we get after simplification

\[ \lambda = 4n(\xi \beta - \alpha^2 + \beta^2). \]

Hence by definition, we obtain the above theorem. \qed

As immediate consequences of the above theorem, we obtain the following result:

Corollary 5.1.

a) A slant submanifold of a Sasakian manifold as a Ricci soliton is shrinking.

b) A slant submanifold of a Kenmotsu manifold as a Ricci soliton is expanding.

c) A slant submanifold of a cosymplectic manifold as a Ricci soliton is steady.

References


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