Optimal control problems for diffusion processes with random parameters

M. Lefebvre and A. Moutassim

Abstract. We consider the controlled process $(X(t), Y(t))$, where $X(t)$ is a diffusion process whose infinitesimal parameters can depend on a continuous-time Markov chain $Y(t)$. Our aim is to find the control that minimizes the expected value of a cost function in which the final time is a first-passage time. We obtain explicit solutions to this problem in an important particular case.

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1 Introduction

In [8], Whittle considered stochastic optimal control problems that he called *LQG homing*. A particular one-dimensional problem of this type is the following: let $\{X(t), t \geq 0\}$ be the controlled diffusion process defined by the stochastic differential equation

$$dX(t) = f[X(t)]\, dt + b_0\, u[X(t)]\, dt + \sqrt{v[X(t)]}\, dW(t), \tag{1.1}$$

where $b_0 \neq 0$ is a constant, $u(\cdot)$ is the control variable, $v(\cdot) > 0$ and $\{W(t), t \geq 0\}$ is a standard Brownian motion. The optimizer wants to minimize the expected value of the cost function

$$J(x) = \int_0^{T(x)} \left\{ \frac{1}{2} q_0\, u^2[X(t)] + \lambda \right\} dt + K[X(T(x))], \tag{1.2}$$

where $q_0\ (> 0)$ and $\lambda \neq 0$ are constants, and K is a general termination cost function. The final time $T(x)$ is a random variable defined by

$$T(x) = \inf\{t \geq 0 : X(t) = a \text{ or } b \mid X(0) = x \in [a, b]\}.$$

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That is, $T(x)$ is a *first-passage time* for the controlled process $\{X(t), t \geq 0\}$.

In the case when the parameter λ is positive, there is a cost for continuation in the interval $[a, b]$. Therefore, the optimizer wants the controlled process to leave this interval as soon as possible. However, when λ is negative, there is an instantaneous reward given for survival in $[a, b]$, so that the optimizer wants to maximize the survival time in this interval. In both cases, the quadratic control costs must of course be taken into account.

Actually, Whittle considered the n-dimensional case. He showed that it is sometimes possible to transform the optimal control problem into a purely probabilistic problem by expressing the optimal control u^* in terms of a certain mathematical expectation for the *uncontrolled process* that corresponds to $\{X(t), t \geq 0\}$.

However, the probabilistic problem is itself generally very difficult to solve explicitly, so that most LQG homing problems that have been solved so far were for one-dimensional processes, or in the case when symmetry enabled the authors to reduce the problem considered to a one-dimensional problem. Makasu [7] was able to solve explicitly a two-dimensional problem.

Instead of the cost function defined in (1.2), we could use a risk-sensitive cost criterion; see Kuhn [2].

Recent papers on LQG homing problems are the ones by Ionescu *et al.* [1], Lefebvre and Ionescu [4], and Lefebvre and Zitouni [6].

An application of stochastic optimal control problems is in mathematical finance. In order to make the above model more realistic, in this paper we will assume that the infinitesimal parameters $f(\cdot)$ and $v(\cdot)$ of the controlled process can both depend on a continuous-time Markov chain, so that they are random.

We will consider the particular case when the uncontrolled process is an Ornstein-Uhlenbeck process with random parameters. We will find an exact analytical solution in a particular problem that will then be generalized.

2 Optimal control in the case of random infinitesimal parameters

The LQG homing problem set up in the previous section is extended to the case when the functions $f(\cdot)$ and $v(\cdot)$ in the stochastic differential equation (1.1) can depend on a continuous-time Markov chain $\{Y(t), t \geq 0\}$ having state space $E = \{1, \ldots, k\}$. We thus consider the controlled process $\{(X(t), Y(t)), t \geq 0\}$ defined by

$$dX(t) = f[X(t), Y(t)]\, dt + b_0\, u[X(t), Y(t)]\, dt + \sqrt{v[X(t), Y(t)]}\, dW(t).$$

The optimizer is looking for the control $u^*[X(t), i]$ that minimizes the expected value of the cost function

$$J(x, i) := \int_0^{T(x,i)} \left\{ \frac{1}{2} q_{0,i}\, u^2[X(t), i] + \lambda_i \right\} dt + K_i[X(T(x, i))],$$

in which the final time $T(x, i)$ is the first-passage time

$$T(x, i) := \inf\{t \geq 0 : X(t) = a \text{ or } b \mid X(0) = x \in [a, b], Y(0) = i\},$$

for $i = 1, \ldots, k$.

An appropriate technique to solve this stochastic optimal control problem is *dynamic programming.* We define the *value function*

$$F(x,i) := \inf_{u[X(t),i],\ 0 \le t \le T(x,i)} E[J(x,i)].$$

Now, according to Bellman's principle of optimality, whatever the optimizer does in the interval $[0, \Delta t]$, the control chosen in the interval $[\Delta t, T(x,i)]$, from the values of $X(t)$ and $Y(t)$ at time Δt, must be optimal.

Moreover, as is well known, the time spent by a continuous-time Markov chain in a given state i is a random variable having an exponential distribution with parameter denoted by ν_i. When the process leaves state i, it goes to state $j \neq i$ with probability p_{ij}. Therefore, we have that $\sum_{j \neq i} p_{ij} = 1$.

Hence, we can write that

$$\begin{aligned} F(x,i) &= \inf_{u[X(t),i],\ 0 \le t \le \Delta t} E\Bigg[\int_0^{\Delta t} \left\{ \frac{1}{2} q_{0,i} u^2[X(t),i] + \lambda_i \right\} dt \\ &\quad + F\big(x + [f(x,i) + b_0 u(x,i)] \Delta t + \sqrt{v(x,i)}\, W(\Delta t), Y(\Delta t)\big) + o(\Delta t) \Bigg] \\ &= \inf_{u[X(t),i],\ 0 \le t \le \Delta t} E\Bigg[\frac{1}{2} q_{0,i} u^2(x,i) \Delta t + \lambda_i \Delta t \\ &\quad + F\big(x + [f(x,i) + b_0 u(x,i)] \Delta t + \sqrt{v(x,i)}\, W(\Delta t), i\big) (1 - \nu_i \Delta t) \\ &\quad + \sum_{j \neq i} F\big(x + [f(x,j) + b_0 u(x,j)] \Delta t + \sqrt{v(x,j)}\, W(\Delta t), j\big) \nu_i \Delta t p_{ij} \\ &\quad + o(\Delta t) \Bigg]. \end{aligned}$$

The second main step in the derivation of the *dynamic programming equation* is to make use of Taylor's formula. Assuming that F is twice differentiable with respect to x, we deduce that, for any $i \in E$,

$$\begin{aligned} &E\big[F\big(x + [f(x,i) + b_0 u(x,i)] \Delta t + \sqrt{v(x,i)}\, W(\Delta t), i\big)\big] \\ &= F(x,i) + [f(x,i) + b_0 u(x,i)] \Delta t \frac{dF(x,i)}{dx} + \frac{1}{2} v(x,i) \Delta t \frac{d^2 F(x,i)}{dx^2} + o(\Delta t), \end{aligned}$$

in which we used the fact that a standard Brownian motion $W(t)$ starting at 0 is such that $E[W(\Delta t)] = 0$ and $E[W^2(\Delta t)] = V[W(\Delta t)] = \Delta t$.

After simplification, we find that

$$\begin{aligned} 0 &= \inf_{u[X(t),i],\ 0 \le t \le \Delta t} \left\{ \frac{1}{2} q_{0,i} u^2(x,i) \Delta t + \lambda_i \Delta t + [f(x,i) + b_0 u(x,i)] \Delta t F'(x,i) \right. \\ &\quad \left. + \frac{1}{2} v(x,i) \Delta t F''(x,i) + \sum_{j \neq i} [F(x,j) - F(x,i)]\, \nu_i \Delta t p_{ij} + o(\Delta t) \right\}. \end{aligned}$$

The last step is to divide both sides of the above equation by Δt and let Δt decrease to 0. We then obtain the dynamic programming equation (DPE)

$$\begin{aligned} 0 &= \inf_{u(x,i)} \left\{ \frac{1}{2} q_{0,i} u^2(x,i) + \lambda_i + [f(x,i) + b_0 u(x,i)] F'(x,i) + \frac{1}{2} v(x,i) F''(x,i) \right. \\ &\qquad \left. + \sum_{j \neq i} \nu_i p_{ij} [F(x,j) - F(x,i)] \right\}. \end{aligned}$$

Once the dynamic programming equation has been obtained, it is a simple task to express the optimal control $u^*(x,i)$ in terms of the value function: differentiating the DPE with respect to $u(x,i)$, we deduce that

$$u^*(x,i) = -\frac{b_0}{q_{0,i}} F'(x,i).$$

Notice that we only need to find the derivative $F'(x,i)$ of the value function to obtain the optimal control.

Finally, substituting the above expression into the DPE, we get the following system of non-linear 2nd-order differential equations:

$$\begin{aligned} 0 &= \lambda_i + f(x,i) F'(x,i) - \frac{1}{2} \frac{b_0^2}{q_{0,i}} [F'(x,i)]^2 + \frac{1}{2} v(x,i) F''(x,i) \\ &\quad + \sum_{j \neq i} \nu_i p_{ij} [F(x,j) - F(x,i)], \end{aligned}$$

for $i = 1, \ldots, k$.

Our objective is to obtain explicit solutions to the previous system in important particular cases. To make this problem as simple as possible, we assume that $k = 2$. The system is then reduced to

$$\begin{aligned} 0 &= \lambda_1 + f(x,1) F'(x,1) - \frac{1}{2} \frac{b_0^2}{q_{0,1}} [F'(x,1)]^2 + \frac{1}{2} v(x,1) F''(x,1) \\ &\quad + \nu_1 [F(x,2) - F(x,1)], \\ 0 &= \lambda_2 + f(x,2) F'(x,2) - \frac{1}{2} \frac{b_0^2}{q_{0,2}} [F'(x,2)]^2 + \frac{1}{2} v(x,2) F''(x,2) \\ &\quad + \nu_2 [F(x,1) - F(x,2)]. \end{aligned}$$

In Lefebvre and Moutassim [5], the authors assumed that $f(x,1) \equiv 1$, $f(x,2) \equiv -1$ and $v(x,i) \equiv 1$. That is, they assumed that the process $\{X(t), t \geq 0\}$ is a controlled Wiener process whose infinitesimal mean is random, but whose infinitesimal variance is constant. Using symmetry, they were able to obtain approximate solutions that they compared with precise numerical solutions.

In the present paper, we will consider the case when $\{X(t), t \geq 0\}$ is a controlled Ornstein-Uhlenbeck process having random infinitesimal parameters.

3 A controlled Ornstein-Uhlenbeck process

An Ornstein-Uhlenbeck process has the following infinitesimal parameters:

$$f(x,i) = -\alpha_i x \quad \text{and} \quad v(x,i) \equiv v_i,$$

where $\alpha_i > 0$ and $v_i > 0$, for $i = 1, 2$. We assume first that

$$f(x,1) = -x, \quad f(x,2) = -2x \quad \text{and} \quad v(x,1) = v(x,2) \equiv 1,$$

so that only the infinitesimal mean of the controlled process is random. Moreover, we take $a = -1$ and $b = 1$ in the definition of $T(x,i)$. Then, by symmetry, we can state that $F(x,i)$ is an *even* function. Finally, we choose $b_0 = q_{0,i} = \lambda_i = 1$, $\nu_i = \frac{3}{2}$ and $K_i[X(T(x,i))] \equiv k_i$, for $i = 1, 2$.

The system that we must solve is then

$$\begin{aligned} 0 &= 1 - xF'(x,1) - \frac{1}{2}\left[F'(x,1)\right]^2 + \frac{1}{2}F''(x,1) + \frac{3}{2}\left[F(x,2) - F(x,1)\right], \\ 0 &= 1 - 2xF'(x,2) - \frac{1}{2}\left[F'(x,2)\right]^2 + \frac{1}{2}F''(x,2) + \frac{3}{2}\left[F(x,1) - F(x,2)\right]. \end{aligned}$$

The boundary conditions are $F(-1,i) = F(1,i) = k_i$, for $i = 1, 2$.

We look for a solution of the form

$$F(x,i) = c_i(1 - x^2) + k_i,$$

where c_i is a positive constant, for $i = 1, 2$. Substituting into the above system, we obtain that

$$\begin{aligned} 0 &= 1 + 2c_1x^2 - \frac{1}{2}(-2c_1x)^2 + \frac{1}{2}(-2c_1) + \frac{3}{2}\left[(c_2 - c_1)(1 - x^2) + (k_2 - k_1)\right], \\ 0 &= 1 + 4c_2x^2 - \frac{1}{2}(-2c_2x)^2 + \frac{1}{2}(-2c_2) + \frac{3}{2}\left[(c_1 - c_2)(1 - x^2) + (k_1 - k_2)\right]. \end{aligned}$$

Adding the two equations, we find that

$$0 = 2 + 2(c_1 + 2c_2)x^2 - 2(c_1^2 + c_2^2)x^2 - (c_1 + c_2).$$

Next, assume that

$$c_2 = 2 - c_1.$$

Then we find that we must choose

$$c_1 = \frac{3}{2} \quad \left(\Longrightarrow \quad c_2 = \frac{1}{2}\right).$$

With these values, the system reduces to

$$\begin{aligned} 0 &= -2 + \frac{3}{2}(k_2 - k_1), \\ 0 &= 2 + \frac{3}{2}(k_1 - k_2). \end{aligned}$$

Hence, we deduce that the proposed solution is valid as long as

$$k_2 - k_1 = \frac{4}{3}.$$

3.1 Generalization

Let

$$c^2 := \frac{b_0^2}{2\,q_0}.$$

The system that we must solve, when $K_i[X(T(x,i))] \equiv k_i$, is

$$\begin{aligned} 0 &= \lambda_1 - \alpha_1\, x\, F'(x,1) - c^2\,[F'(x,1)]^2 + \frac{v_1}{2}\, F''(x,1) \\ &\quad + \nu_1\,[F(x,2) - F(x,1)]\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 &= \lambda_2 - \alpha_2\, x\, F'(x,2) - c^2\,[F'(x,2)]^2 + \frac{v_2}{2}\, F''(x,2) \\ &\quad + \nu_2\,[F(x,1) - F(x,2)]\,, \end{aligned} \tag{3.2}$$

subject to the boundary conditions $F(-1,i) = F(1,i) = k_i$, for $i = 1, 2$.

To solve the coupled differential equations (3.1) and (3.2), we will assume that the solutions can be expressed as

$$F(x,1) = a_2\, x^2 + a_0, \tag{3.3}$$

$$F(x,2) = b_2\, x^2 + b_0. \tag{3.4}$$

Notice that these functions are even and that we must have

$$a_2 + a_0 = k_1 \quad \text{and} \quad b_2 + b_0 = k_2. \tag{3.5}$$

Substituting into (3.1) and (3.2), we get the system

$$\lambda_1 + v_1\, a_2 + \nu_1(b_0 - a_0) = 0, \tag{3.6}$$

$$-2\alpha_1\, a_2 - 4c^2\, a_2^2 + \nu_1(b_2 - a_2) = 0, \tag{3.7}$$

$$\lambda_2 + v_2\, b_2 + \nu_2(a_0 - b_0) = 0, \tag{3.8}$$

$$-2\alpha_2\, b_2 - 4c^2\, b_2^2 + \nu_2(a_2 - b_2) = 0. \tag{3.9}$$

Using Eqs. (3.5), (3.6) and (3.8), the coefficients a_2 and b_2 are determined by

$$a_2 = -\frac{\theta + \lambda_1 v_2 - \nu_1 v_2 (k_1 - k_2)}{\xi}, \tag{3.10}$$

$$b_2 = -\frac{\theta + \lambda_2 v_1 + \nu_2 v_1 (k_1 - k_2)}{\xi}, \tag{3.11}$$

where

$$\theta := \nu_2\,\lambda_1 + \nu_1\,\lambda_2$$

and

$$\xi := \nu_1 v_2 + \nu_2 v_1 + v_1 v_2 \ (> 0).$$

If we substitute a_2 and b_2 into (3.7) and (3.9), we obtain the system

$$\begin{aligned} 0 &= \xi\,[2\alpha_1\,(\nu_2\lambda_1 + \nu_1\lambda_2) + (2\alpha_1 + \nu_1)\,(\lambda_1 v_2 - k\,\nu_1 v_2) - \nu_1\lambda_2 v_1 - k\,\nu_1\nu_2 v_1] \\ &\quad - 4c^2\,(\theta + \lambda_1 v_2 - k\,\nu_1 v_2)^2\,, \end{aligned} \tag{3.12}$$

$$0 = \xi\left[2\alpha_2\left(\nu_2\lambda_1+\nu_1\lambda_2\right)+\left(2\alpha_2+\nu_2\right)\left(\lambda_2 v_1+k\,\nu_2 v_1\right)-\nu_2\lambda_1 v_2+k\,\nu_1\nu_2 v_2\right]$$
$$-4c^2\left(\theta+\lambda_2 v_1+k\,\nu_2 v_1\right)^2, \tag{3.13}$$

where $k := k_1 - k_2$.

A necessary, but in general not sufficient, condition to have solutions of the system (3.12), (3.13) is that the value of c^2 in these solutions is positive. Thus, from (3.12), we should have

$$2\alpha_1\left(\nu_2\lambda_1+\nu_1\lambda_2\right)+\left(2\alpha_1+\nu_1\right)\left(\lambda_1 v_2-k\,\nu_1 v_2\right)-\nu_1\lambda_2 v_1-k\,\nu_1\nu_2 v_1>0. \tag{3.14}$$

Similarly, from (3.13),

$$2\alpha_2\left(\nu_2\lambda_1+\nu_1\lambda_2\right)+\left(2\alpha_2+\nu_2\right)\left(\lambda_2 v_1+k\,\nu_2 v_1\right)-\nu_2\lambda_1 v_2+k\,\nu_1\nu_2 v_2>0. \tag{3.15}$$

From the inequation (3.14), we can determine the following upper bound for k:

$$k < \frac{\lambda_1(2\alpha_1(\nu_2+v_2)+\nu_1 v_2)+\lambda_2\nu_1(2\alpha_1-v_1)}{\nu_1(v_2\nu_1+v_1\nu_2+2\alpha_1 v_2)}. \tag{3.16}$$

Moreover, we can transform the inequation (3.15) to get a lower bound for k:

$$k > -\frac{\lambda_1\nu_2(2\alpha_2-v_2)+\lambda_2(2\alpha_2(\nu_1+v_1)+\nu_2 v_1)}{\nu_2\left(v_1\nu_2+\nu_1 v_2+2\alpha_2 v_1\right)}. \tag{3.17}$$

From these two bounds, we find, after some calculations, that we can write that

$$-\lambda_1\frac{\nu_2}{\nu_1}<\lambda_2. \tag{3.18}$$

Hence, we deduce that for $\lambda_i<0$, $i=1,2$, we should have

$$0<\frac{\lambda_2}{\lambda_1}<-\frac{\nu_2}{\nu_1}<0,$$

which is impossible and thus the problem has no solutions. For $\lambda_i>0$, $i=1,2$, the condition should be

$$\frac{\lambda_2}{\lambda_1}>-\frac{\nu_2}{\nu_1}. \tag{3.19}$$

The condition is then satisfied and the problem has solutions given by the quadratic functions (3.3) and (3.4) for any parameters $\alpha_1,\alpha_2,v_1,v_2$.

Given the parameters α_1, α_2, v_1, v_2, ν_1, ν_2 and λ_1, λ_2 satisfying the expression in (3.19), the system (3.12), (3.13) is in fact a system of two non-linear equations with respect to k and c.

If $\theta+\lambda_1 v_2\neq k\nu_1 v_2$, we can determine c^2 as a function of k according to

$$c^2=\frac{\xi\left[(\nu_1+2\alpha_1)(\lambda_1 v_2-k\nu_1 v_2)+2\alpha_1\theta-\nu_1 v_1(\lambda_2+k\nu_2)\right]}{4(\theta+\lambda_1 v_2-k\nu_1 v_2)^2}. \tag{3.20}$$

Substituting c^2 into Eq. (3.13), we obtain a third-degree polynomial equation with respect to k:

$$r_3\,k^3+r_2\,k^2+r_1\,k+r_0=0, \tag{3.21}$$

where the coefficients r_0, r_1, r_2 and r_3 are given as follows:

$$
\begin{aligned}
r_0 &= \eta(\lambda_1 v_2+\theta)^2-\beta(\lambda_2 v_1+\theta)^2,\\
r_1 &= \nu_2(v_1(2\alpha_2+\nu_2)+\nu_1 v_2)(\lambda_1 v_2+\theta)^2+\nu_1(v_2(2\alpha_1+\nu_1)+\nu_2 v_1)(\lambda_2 v_1+\theta)^2\\
&\quad -2\nu_1 v_2(\lambda_1 v_2+\theta)\eta-2\nu_2 v_1(\lambda_2 v_1+\theta)\beta,\\
r_2 &= v_2 v_1\nu_2(2\alpha_1+\nu_1)(2\nu_1(\lambda_2 v_1+\theta)-\lambda_1\nu_2 v_1)-\nu_2^2 v_1^2(2\alpha_1\theta-\lambda_2\nu_1 v_1)\\
&\quad +2\nu_1\nu_2(\nu_2 v_1^2(\lambda_2 v_1+\theta)-\nu_1 v_2^2(\lambda_1 v_2+\theta))+\nu_1^2 v_2^2(2\alpha_2\theta-\lambda_1\nu_2 v_2)\\
&\quad +v_1 v_2\nu_1(2\alpha_2+\nu_2)(-2\nu_2(\lambda_2 v_1+\theta)+\lambda_2\nu_1 v_2),\\
r_3 &= \nu_1\nu_2^2 v_1^2(v_2(2\alpha_1+\nu_1)+\nu_2 v_1)+\nu_2\nu_1^2 v_2^2(v_1(2\alpha_2+\nu_2)+\nu_1 v_2),
\end{aligned}
$$

in which

$$
\begin{aligned}
\eta &:= 2\alpha_2\theta-\lambda_1\nu_2 v_2+\lambda_2 v_1(2\alpha_2+\nu_2),\\
\beta &:= 2\alpha_1\theta-\lambda_2\nu_1 v_1+\lambda_1 v_2(2\alpha_1+\nu_1).
\end{aligned}
$$

Under the above conditions, the determination of k such that c^2 satisfies (3.20) enables us to deduce the solutions $F(x,1)$ and $F(x,2)$ whose coefficients are calculated according to (3.5), (3.10) and (3.11). An example will be presented next.

An example. Let $\alpha_1 = 1$, $\alpha_2 = 2$, $v_1 = 1$, $v_2 = 2$, $\lambda_1 = \lambda_2 = 1$ and $\nu_1 = \nu_2 = 1$. We have $\theta = 2$, $\eta = 11$ and $\beta = 9$. In this case, the coefficients $(r_i)_{0\le i\le 3}$ of Eq. (3.21) are $r_0 = 95$, $r_1 = -55$, $r_2 = -15$ and $r_3 = 35$. The equation that we have to solve is

$$35\,k^3-15\,k^2-55\,k+95=0.$$

Using a numerical method to solve this non-linear equation, we find that one of its roots is equal to $k \simeq -544/339$, and that the other roots are complex. If we substitute k into Eq. (3.20), we find that $c^2 \simeq 363/746$.

Now, we can choose a value of k_1 and of k_2 such that $k = k_1 - k_2 = -544/339$. In this example, we take $k_1 = 1$ and $k_2 = 883/339$.

The coefficients a_0, b_0, a_2 and b_2 for which we obtain an exact solution are

$$a_2 \simeq -\frac{1192}{826}, \quad a_0 \simeq \frac{2017}{826}, \quad b_2 \simeq -\frac{461}{1652} \quad \text{and} \quad b_0 \simeq \frac{1191}{413}.$$

We thus find the following solutions:

$$
\begin{aligned}
F(x,1) &\simeq -\frac{1191}{826}x^2+\frac{2017}{826},\\
F(x,2) &\simeq -\frac{461}{1652}x^2+\frac{1191}{413}.
\end{aligned}
$$

Finally, we want to compare the above solutions with the ones obtained using a numerical method. We chose the finite difference numerical method to solve our coupled differential equations. The vectors $(\tilde{F_i^1})_{1\le i\le n+1}$ and $(\tilde{F_i^2})_{1\le i\le n+1}$, whose elements are given by

$$\tilde{F}_i^1 = F(x_i,1) \quad \text{and} \quad \tilde{F}_i^2 = F(x_i,2) \quad \text{for } i = 1,\dots,n+1,$$

are the solutions of the system

$$\tilde{F}_i^1 = \frac{h^2}{\nu_1 h^2 + v_1} \left(\lambda_1 - \alpha\, x_i \frac{\tilde{F}_{i+1}^1 - \tilde{F}_{i-1}^1}{2h} - \frac{c^2}{4h^2} \left(\tilde{F}_{i+1}^1 - \tilde{F}_{i-1}^1 \right)^2 + v_1 \frac{\tilde{F}_{i+1}^1 + \tilde{F}_{i-1}^1}{2h^2} + \nu_1 \tilde{F}_i^2 \right),$$

$$\tilde{F}_i^2 = \frac{h^2}{\nu_2 h^2 + v_2} \left(\lambda_2 - \alpha\, x_i \frac{\tilde{F}_{i+1}^2 - \tilde{F}_{i-1}^2}{2h} - \frac{c^2}{4h^2} \left(\tilde{F}_{i+1}^2 - \tilde{F}_{i-1}^2 \right)^2 + v_2 \frac{\tilde{F}_{i+1}^2 + \tilde{F}_{i-1}^2}{2h^2} + \nu_2 \tilde{F}_i^1 \right),$$

for $i = 2, \ldots, n$, where h denotes the discretization time step of the interval $[-1, 1]$ and satisfies

$$h = x_j - x_{j-1} = \frac{2}{n}$$

for $j = 2, \ldots, n+1$. Moreover, $(x_i)_{1 \le i \le n+1}$ are the equidistant points of this interval, such that

$$x_1 = -1 < \ldots < x_i = x_1 + (i-1)\, h < \ldots < x_{n+1} = 1.$$

Figure 1 presents the exact and numerical solutions. We see that they practically coincide.

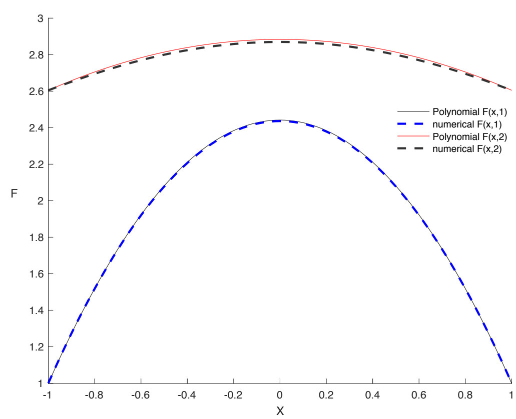

Figure 1: Exact and numerical solutions in the example.

4 Concluding remarks

In this paper, we extended LQG homing problems to the case when the infinitesimal parameters of the controlled diffusion process depend on a continuous-time Markov chain. As mentioned above, a reason for doing so is that it makes the model more realistic in applications such as financial mathematics.

We were able to obtain exact analytical solutions in the special case of a controlled Ornstein-Uhlenbeck process, which is a very important diffusion process for the applications. First a particular problem was solved explicitly, and then it was generalized.

Finally, we could also assume that there are random jumps of the controlled process; see [3]. Jump-diffusion processes are widely used in financial mathematics, in particular.

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