Extrinsic Anti-MANOVA on 3D projective shape spaces

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Abstract. One considers testing for the equality of k extrinsic antimeans on a compact object space with a manifold structure. We give a chi-square test statistics for an extrinsic anti-MANOVA hypothesis testing problem, based on large samples, and show that the low sample size case leads to a nonparametric bootstrap anti-MANOVA test statistics, which are in particular applied to 3D projective shape data, with an application to 3D image analysis.

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1 Introduction

Fréchet (1948)[6] noticed that for high complexity data, such as the shape of an egg randomly selected from a basket, numbers or vectors do not provide a meaningful representation. To investigate these kind of data, he introduced the notion of *element*, nowadays called *object*. Fréchet's visionary concepts, were nevertheless hardly advance during his lifetime, due to a lack of computational tools. It took many decades, until such data became the bread and butter of modern data analysis. Later on, various types of objects popped up as shapes of configurations extracted from digital images; these were represented as points on *object spaces*, such as projective shape spaces (see [10], [13]), affine shape spaces(see [14], [18]), or Kendall (direct similarity) shape spaces (see [7], [5]). Fréchet defined what we call now the *(second order) Fréchet function*, given by

$$\mathcal{F}_\rho(p) = \mathbb{E}(\rho^2(p, X)), \tag{1.1}$$

that helped introduce notions of mean and total variance of a *random object* X on the object space $(\mathcal{M}, \rho)$, as minimizers and minimum of F_ρ in (1.1). While the Fréchet

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function helped pursuing a proper statistical analysis on object spaces, based on a notion of *mean*, as a location parameter that parallels the notion of mean vector from Multivariate Analysis in the linear case (since the mean vector is a minimizer of the expected square distance to the random vector X), in the case of data on compact object spaces, additional *location parameters* can be considered, including maximizers of $\mathcal{F}$ in (1.1). The maximizers are forming the *Fréchet antimean set*. Note however, that in case $\mathcal{M}$, is a smooth manifold, with a geodesic distance $\rho = \rho_g$, associated with a Riemannian structure g on it, **there are no necessary and sufficient conditions for the existence of a unique maximizer** of $\mathcal{F}_{\rho_g}$, therefore in general, with the possible exception of compact flat Riemannian manifolds, like high dimensional flat tori, or flat Klein bottles, it is preferable to work with a "chord" distance on $\mathcal{M}$, that is induced by the Euclidean distance in $\mathbb{R}^N$ via an embedding $j : \mathcal{M} \to \mathbb{R}^N$. In this case, the Fréchet function becomes

$$\mathcal{F}(p) = \int_{\mathcal{M}} \|j(x) - j(p)\|_0^2 Q(dx), \tag{1.2}$$

where $\|\cdot\|_0$ is the Euclidean norm in $\mathbb{R}^N$, $Q = P_X$ is the probability measure on $\mathcal{M}$, associated with the r.o. X on $\mathcal{M}$. In this setting, if the extrinsic antimean set has one point only, that point is called **extrinsic antimean** of X, and is labeled $\alpha\mu_{j,E}(Q)$, or simply $\alpha\mu_E$, when j and Q are known. This happens iff the mean vector μ of $j(X)$ is *αj-nonfocal*, meaning that its farthest projection on $j(\mathcal{M})$, $P_{F,j}(\mu)$, is well defined (see Patrangenaru et al.(2016)[17]). Also, given $X_1, \ldots, X_n$ i.i.d.r.v.'s from Q, their *extrinsic sample antimean (set)* is the extrinsic antimean (set) of the empirical distribution $\hat{Q}_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ (see eg Patrangenaru et al (2016)[17]).

In Section 2, after introducing the extrinsic antimean for a random object on a compact object space (see also Patrangenaru et al(2016)[17]), one recalls the consistency and asymptotic distribution of the sample extrinsic antimeans. A statistic for two sample tests for extrinsic antimeans on compact manifolds is given in section 4. Section 3 is dedicated to anti-MANOVA on manifolds. The nonparametric anti-MANOVA test developed in this section, is detailed further in the case of 3D projective shape data in subsection 4.2, where the 3D projective shape space of projective shapes of landmark configurations of k-ads, containing a projective frame at given landmark indices, leads to a anti-MANOVA projective shape analysis on the manifold $(\mathbb{R}P^3)^{k-5}$. The asymptotic distributions of a Hotelling like statistic associated with the anti-MANOVA hypotheses testing problem is shown to be key to the data analysis in Theorem 3.1, a result that is further specialized to VW anti-MANOVA in Theorem 4.4. In Section 5 we consider an example of 3D projective shape analysis of faces from digital camera imaging data. The paper concludes with a discussion on future directions in extrinsic object data analysis.

2 From baricenters to extrinsic means and antimeans on compact subsets of Hilbert spaces

Basic Euclidean geometry shows that on an Euclidean space, for a given configuration of points $A_1, \ldots, A_n$, the minimizer of the sum of square distances $MA_1^2 + MA_2^2 +$

$\cdots + MA_n^2$ is the baricenter G of this configuration, assuming the points having an equal probability to occur.

In case that the sampling points from the above distribution lie on a sphere of radius 1, centered at the origin O, there is a unique minimizer M on that sphere of $MA_1^2 + MA_2^2 + \cdots + MA_n^2$, the closest point $G_\perp$ to G of intersection of the line OG with the sphere, unless $G = O$, when the sum of squares is constant.

Moreover, since the sphere is compact, the sum $MA_1^2 + MA_2^2 + \cdots + MA_n^2$ has a maximum on the sphere, and the maximizer is the *antipodal* point to $G_\perp$.

In general, if $x_1, \dots, x_n$ are points from a probability distribution Q on a closed subset $\mathcal{M}$ of a finite dimensional Euclidean space $\mathbb{E}^N$, we define the *extrinsic sample mean on* $\mathcal{M}$, $\bar{x}_E$, as the minimizer on $\mathcal{M}$ of the *Fréchet function* $F(p) = \sum_{i=1}^n \|p - x_i\|^2$. Similarly we define the *extrinsic sample antimean* $a\bar{x}_E$ as the maximizer on $\mathcal{M}$ of the Fréchet function $F(p) = \sum_{i=1}^n \|p - x_i\|$.

Further, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $X : \Omega \to \mathbb{R}^m$, be a random vector, and $g : \mathbb{R}^m \to \mathbb{R}$ be a function in $L^2(\mathbb{P}_X)$. Then $E(g(X)) = \int_{\mathbb{R}^m} g(x) P_X(dx)$. Then mean vector $E(X) = \mu_X$ is the minimizer of the function $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}$, $\mathcal{F}(p) = E(\|X - p\|^2)$.

Moreover, if $(\mathcal{M}, \rho)$ is a separable complete metric space, the *Fréchet mean set* is the minimizer set of the function $\mathcal{F}(p) = E(\rho^2(X, p))$.

2.1 Consistency of sample extrinsic means and anti-means on compact sets

Any compact metric space can be embedded in the Hilbert cube. Therefore one may consider only data on compact subsets of a real Hilbert space $(\mathbb{H}, \langle\ ,\ \rangle)$.

The *extrinsic mean (anti-mean) set* of a random point X on $\mathcal{M} \subseteq \mathbb{H}$ is the Fréchet mean (antimean) set of X, that is the set of minimizers (maximizers) of $\mathcal{F}(p) = E(\|X - p\|^2)$, where $\|v\| = \sqrt{\langle v, v\rangle}$.

The point $\mu_E \in \mathcal{M}$ is the *extrinsic mean (antimean)* of the random point X on $\mathcal{M} \subseteq \mathbb{H}$ if the extrinsic mean (antimean) set of X consists of the point $\{\mu_E\}$*only* [11].

Recall that a sequence of random points T_n, $n \in \mathbb{N}$, on $\mathcal{M}$ is a *consistent estimator* of the parameter $\theta \in \mathcal{M}$, if $T_n \to_p \theta$, that is $\forall \varepsilon > 0, \lim_{n\to\infty} P(\|T_n - \theta\| > \varepsilon) = 0$.

Given a sample $X_1, \dots, X_n$ of independent identically distributed random points (i.i.d.r.p.'s) from a distribution Q on $\mathcal{M}$ that has an extrinsic mean μ_E, than for any choice $\bar{X}_{E,n}$ from the *extrinsic sample mean set*, $\bar{X}_{E,n}$ is a *consistent estimator* of the extrinsic mean of Q.

Similarly, given a sample $X_1, \dots, X_n$ i.i.d.r.p.'s from a distribution Q on $\mathcal{M}$ that has an extrinsic antimean $\alpha\mu_E$, than for any choice $a\bar{X}_{E,n}$ from the *extrinsic sample mean set*, $a\bar{X}_{E,n}$ is a *consistent estimator* of the extrinsic antimean of Q.

Example 2.1. (see Wang and Patrangenaru(2018)[19]) Assume Q is a probability distribution on the complex projective space $\mathbb{C}P^{k-2}$ and j is its VW embedding, given by $j([z]) = \frac{1}{z^*z} zz^*$. Let $\{[Z_r], \|Z_r\|, r = 1, \dots, n\}$ be i.i.d.r.o.'s from Q. We say that Q is α VW-nonfocal if Q is αj-nonfocal, w.r.t. the VW embedding j. Then (a) Q is α VW-nonfocal iff λ, the smallest eigenvalue of $E[Z_1 Z_1^*]$ is simple and in this case $\alpha\mu_{j,E}Q = [m]$, where m is an eigenvector of $E[Z_1 Z_1^*]$ corresponding to λ, with $\| m \| = 1$ and (b) The sample VW antimean $\alpha\overline{X}_E = [m]$, where m is an eigenvector of norm 1 of $J = \frac{1}{n}\sum_{i=1}^n Z_i Z_i^*$, $\|Z_i\| = 1, i = 1, \dots, n$, corresponding to the smallest eigenvalue of J.

Using a moving frame approach *à* la analysis of extrinsic means in Bhattacharya and Patrangenaru (2003,2005)[2, 3], assuming j is an embedding of a d-dimensional compact manifold $\mathcal{M}$ in $\mathbb{R}^N$, and Q is a αj-nonfocal probability measure on $\mathcal{M}$ such that $j(Q)$ has finite moments of order 2, one may derive the asymptotic distribution of the extrinsic sample means. Let μ and Σ be the mean and covariance matrix of $j(Q)$ regarded as a probability measure on $\mathbb{R}^N$. Let $\alpha\mathcal{F}$ be the set of αj-focal points of $j(M)$, and let $P_{F,j} : \alpha\mathcal{F}^c \to j(M)$ be the farthest projection on $j(M)$. Assume $x \to (f_1(x), \dots, f_d(x))$ is a local frame field on an open subset of M such that for each $x \in M$, $(d_x j(f_1(x)), \dots, d_x j(f_d(x)))$ are orthonormal vector in $\mathbb{R}^N$. A local frame field $p \to (e_1(p), e_2(p), \dots, e_N(p))$ defined on an open neighborhood $U \subseteq \mathbb{R}^N$ is *adapted to the embedding* j if it is an orthonormal frame field and $\forall x \in j^{-1}(U), e_r(j(x)) = d_x j(f_r(x)), r = 1, \dots, d$.

Let $e_1, e_2, \dots, e_N$ be the canonical basis of $\mathbb{R}^k$ and assume $(e_1(p), e_2(p), \dots, e_N(p))$ is an adapted frame field around $P_{F,j}(\mu) = j(\mu_{\alpha E})$. Then $d_\mu P_{F,j}(e_b) \in T_{P_{F,j}(\mu)} j(M)$ is a linear combination of $e_1(P_{F,j}(\mu)), e_2(P_{F,j}(\mu)), \dots, e_d(P_{F,j}(\mu))$:

$$d_\mu P_{F,j}(e_b) = \sum_{a=1}^{d} (d_\mu P_{F,j}(e_b)) \cdot e_a(P_{F,j}(\mu)) e_a(P_{F,j}(\mu)). \tag{2.1}$$

By the delta method, $n^{1/2}(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu))$ converges weakly to $N_N(0_N, \alpha\Sigma_\mu)$, where $\overline{j(X)} = \frac{1}{n}\sum_{i=1}^{n} j(X_i)$ and

$$\begin{aligned} \alpha\Sigma_\mu &= [\sum_{a=1}^{d} d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu)) e_a(P_{F,j}(\mu))]_{b=1,\dots,N} \\ &\times \Sigma [\sum_{a=1}^{d} d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu)) e_a(P_{F,j}(\mu))]^T_{b=1,\dots,N} \end{aligned} \tag{2.2}$$

Here Σ is the covariance matrix of $j(X_1)$ w.r.t the canonical basis $e_1, e_2, \dots, e_N$. The asymptotic distribution $N_N(0_N, \alpha\Sigma_\mu)$ is degenerate and the support of this distribution is on $T_{P_{F,j}} j(M)$, since the range of $d_\mu P_{F,j}$ is a subspace of $T_{P_{F,j}(\mu)} j(M)$. Note that $d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu)) = 0$ for $a = d+1, \dots, N$. The tangential component $tan(v)$ of $v \in \mathbb{R}^N$, w.r.t. the basis $e_a(P_{F,j}(\mu)) \in T_{P_{F,j}(\mu)} j(M), a = 1, \dots, d$ is given by

$$tan(v) = [e_1(P_{F,j}(\mu))^T v, \dots, e_d(P_{F,j}(\mu))^T v]^T. \tag{2.3}$$

From (2.3), $(d_{\alpha\mu_E} j)^{-1}(tan(P_{F,j}(\overline{(j(X))}) - P_{F,j}(\mu))) = \sum_{a=1}^{d} \overline{X}_j^a f_a$ has the following covariance matrix w.r.t. the basis $f_1(\alpha\mu_E), \dots, f_d(\alpha\mu_E)$:

$$\begin{aligned} \alpha\Sigma_{j,E} &= e_a(P_{F,j}(\mu))^T \alpha\Sigma_\mu e_b(P_{F,j}(\mu))_{1 \le a,b \le d} \\ &= [\Sigma d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu))]_{a=1,\dots,d} \Sigma \\ &\quad \times [\Sigma d_\mu P_{F,j}(e_b) \cdot e_a(P_{F,j}(\mu))]^T_{a=1,\dots,d} \end{aligned} \tag{2.4}$$

Definition 2.2. The matrix $\alpha\Sigma_{j,E}$ given above is the *extrinsic anticovariance matrix* of the αj -nonfocal distribution Q (of X_1) w.r.t. the basis $f_1(\mu_{\alpha E}), \dots, f_d(\mu_{\alpha E})$.

When j is fixed, the subscript j in $\alpha\Sigma_{j,E}$ will be omitted. If rank $\alpha\Sigma_\mu = d$, then $\alpha\Sigma_{j,E}$ is invertible and we define the j-standardized sample antimean vector

$$a\overline{Z}_{j,n} =: n^{1/2}\alpha\Sigma_{j,E}^{-1/2}(\overline{X}_j^1, \dots, \overline{X}_j^d)^T. \tag{2.5}$$

We recall the following

Theorem 2.1. *(Patrangenaru et al(2016)[16]) Assume $\{X_r\}_{r=1,\dots,n}$ is a random sample from the αj-nonfocal distribution Q. Let $\mu = E(j(X_1))$ and let $(e_1(p), e_2(p), \dots, e_k(p))$ be an orthonormal frame field adapted to j. Then (a) the tangential component of the extrinsic sample antimean $a\overline{X}_E$ has asymptotically a normal distribution in the tangent space to the d dimensional manifold M at $\alpha\mu_E(Q)$ with mean 0_d and covariance matrix $n^{-1}\alpha\Sigma_{j,E}$, and (b) if $\alpha\Sigma_{j,E}$ is nonsingular, the j-standardized mean vector $\overline{\alpha Z}_{j,n}$ converges weakly to a random vector with a multivariate $N_d(0_d, I_d)$ distribution.*

As a particular case, when j is the inclusion map of a submanifold $\mathcal{M}$ of $\mathbb{R}^k$, we get the following result for α-nonfocal distributions on $\mathcal{M}$:

Corollary 2.2. *Assume $M \subseteq \mathbb{R}^N$ is a d-dimensional closed submanifold of $\mathbb{R}^N$. Let $\{X_r\}_{r=1,\dots,n}$ be i.i.d.r.o's from the nonfocal distribution Q on M, and let $\mu = E(X_1)$ and assume the covariance matrix Σ of $j(Q)$ is finite. Let $(e_1(p), e_2(p), \dots, e_N(p))$ be an orthonormal frame field adapted to M. Let $\alpha\Sigma_E := \alpha\Sigma_{j,E}$, where $j : M \to \mathbb{R}^N$ is an embedding. Then (a) $n^{1/2}tan(j(a\overline{X}_E) - j(\alpha\mu_E))$ converges weakly to $N_d(0_d, \alpha\Sigma_E)$, and (b) if $\alpha\Sigma_\mu$ induces a nonsingular bilinear form on $T_{j(\mu_{\alpha E})}j(M)$, then $\|\overline{AZ}_{j,n}\|^2$ converges weakly to the chi-square distribution χ_d^2.*

The CLT for extrinsic sample antimeans can not be used to construct confidence regions for extrinsic antimeans, since the population extrinsic covariance matrix is a nuisance parameter. We then consider a consistent estimator of $\alpha\Sigma_{j,E}$ as follows. Note that $\overline{j(X)}$ is a consistent estimator of μ, $d_{\overline{j(X)}}P_{F,j} \to_P d_\mu P_{F,j}$, and $e_a(P_{F,j}(\overline{j(X)})) \to_P e_a(P_{F,j}(\mu))$ and

$$S_{j,n} = n^{-1}\sum(j(X_r) - \overline{j(X)})(j(X_r) - \overline{j(X)})^T \tag{2.6}$$

is a consistent estimator of $\alpha\Sigma_\mu$. It follows that

$$\left[\sum_{a=1}^{d} d_{\overline{j(X)}}P_{F,j}(e_b) \cdot e_a(P_{F,j}(\overline{j(X)}))e_a(P_{F,j}(\overline{j(X)}))\right] S_{j,n}$$
$$\left[\sum_{a=1}^{d} d_{\overline{j(X)}}P_{F,j}(e_b) \cdot e_a(P_{F,j}(\overline{j(X)}))e_a(P_{F,j}(\overline{j(X)}))\right]^T$$

is a consistent estimator of $\alpha\Sigma_\mu$, and $tan_{P_{F,j}(\overline{j(X)})}v$ is a consistent estimator of $tan(v)$.

Therefore if we take the components of the bilinear form associated with this matrix w.r.t.
$e_1(P_{F,j}(\overline{j(X)})), e_2(P_{F,j}(\overline{j(X)})), ..., e_d(P_{F,j}(\overline{j(X)}))$, we get a consistent estimator of

$\alpha\Sigma_{j,E}$

$$aS_{E,n} = \\ = [[\sum d_{\overline{j(X)}} P_{F,j}(e_b) \cdot e_a(P_{F,j}(\overline{j(X)}))]_{a=1,\dots,d}] \cdot \\ \cdot S_{j,n}[[\sum d_{\overline{j(X)}} P_{F,j}(e_b) \cdot e_a(P_{F,j}(\overline{j(X)}))]_{a=1,\dots,d}]^T. \tag{2.7}$$

Remark 2.3. As a result, if we assume that $j : M \to \mathbb{R}^N$ is an embedding of $\mathcal{M}$ in $\mathbb{R}^N$ and $\{X_r\}_{r=1,\dots,n}$ is are i.i.d.r.o.'s from the αj-nonfocal distribution Q, and $\mu = E(j(X_1))$, $j(X_1)$ has finite second order moments, and $\alpha\Sigma_{j,E}$ of X_1 is nonsingular, then if $(e_1(p), e_2(p), \dots., e_N(p))$ be an orthonormal frame field adapted to j, it follows that $aS_{E,n}$ is (2.7), then for n large enough, $aS_{E,n}$ is nonsingular with probability converging to one, and (a)

$$n^{\frac{1}{2}} aS_{E,n}{}^{-\frac{1}{2}}(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu)) \tag{2.8}$$

converges weakly to a $N(0_d, I_d)$ distributed r.vector, so that

$$n\|aS_{E,n}{}^{-\frac{1}{2}} tan(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu))\|^2 \tag{2.9}$$

converges weakly to a χ^2_d distributed r.v., and (b) the statistic

$$n^{\frac{1}{2}} aS_{E,n}{}^{-\frac{1}{2}} tan_{P_{F,j}(\overline{j(X)})}(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu)) \tag{2.10}$$

converges weakly to a $N(0_d, I_d)$ r. vector, so that

$$n\|aS_{E,n}{}^{-\frac{1}{2}} tan_{P_{F,j}(\overline{j(X)})}(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu))\|^2 \tag{2.11}$$

converges weakly to χ^2_d distributed r.v.

Corollary 2.3. *Under the hypothesis above, a confidence region for $\alpha\mu_E$ at asymptotic level $1-\alpha$ is given by (a) $C_{n,\alpha} := j^{-1}(U_{n,\alpha})$, where*

$$U_{n,\alpha} = \{\mu \in j(M) : n\|aS_{E,n}{}^{-\frac{1}{2}} tan(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu)\|^2 \le \chi^2_{d,1-\alpha}\},$$

or by (b) $D_{n,\alpha} := j^{-1}(V_{n,\alpha})$, where

$$V_{n,\alpha} = \{\mu \in j(M) : n\|aS_{E,n}{}^{-\frac{1}{2}} tan_{P_{F,j}(\overline{j(X)})}(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu)\|^2 \le \chi^2_{d,1-\alpha}\}.$$

At this point we recall the steps that are necessary to obtain a bootstrapped statistic from a pivotal statistic. If $\{X_r\}_{r=1,\dots,n}$ is a random sample from the unknown distribution Q, and $\{X_r^*\}_{r=1,\dots,n}$ is a random sample from the empirical $\hat{Q}_n$, conditionally given $\{X_r\}_{r=1,\dots,n}$, then the statistic

$$T(X, Q) = n\|aS_{E,n}{}^{-\frac{1}{2}} tan(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu))\|^2$$

given above has the bootstrap analog

$$T(X^*, \hat{Q}_n) = n\|aS^*_{E,n}{}^{-\frac{1}{2}} \\ tan_{P_{F,j}(\overline{j(X)}))}(P_{F,j}(\overline{j(X^*)}) - P_{F,j}(\overline{j(X)}))\|^2. \tag{2.12}$$

Here $aS^*_{E,n}$ is obtained from $aS_{E,n}$ substituting $X^*_1, ..., X^*_n$ for $X_1, ...X_n$, and $T(X^*, \hat{Q}_n)$ is obtained from $T(X, Q)$ by substituting $X^*_1, ..., X^*_n$ for $X_1, ..., X_n, \overline{j(X)})$ for μ and $aS^*_{E,n}$ for $aS_{E,n}$.
The same procedure can be used for the vector valued statistic

$$V(X, Q) = n^{\frac{1}{2}} aS_{E,n}{}^{-\frac{1}{2}} tan(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu)), \tag{2.13}$$

and as a result we get the bootstrapped statistic

$$V^*(X^*, \hat{Q}_n) = n^{\frac{1}{2}} aS^*_{E,n}{}^{-\frac{1}{2}} \\ tan_{P_{F,j}(\overline{j(X^*)}))}(P_{F,j}(\overline{j(X^*)}) - P_{F,j}(\overline{j(X)})). \tag{2.14}$$

We then obtain the following results:

Theorem 2.4. *Let $\{X_r\}_{r=1,...,n}$ be i.i.d.r.o.'s from the αj-nonfocal distribution Q, which has a nonzero absolutely continuous component w.r.t. the volume measure on M induced by j. Let $\mu = E(j(X_1))$ and assume the extrinsic covariance matrix $\Sigma_{j,E}$ is nonsingular and let $(e_1(p), e_2(p), ..., e_k(p))$ be an orthonormal frame field adapted to j. Then the distribution function of*

$$n\|aS_{E,n}{}^{-\frac{1}{2}} tan(P_{F,j}(\overline{j(X)}) - P_{F,j}(\mu))\|^2 \tag{2.15}$$

can be approximated by the bootstrap distribution function of

$$n\|aS^*_{E,n}{}^{-\frac{1}{2}} tan_{P_{F,j}(\overline{j(X)}))}(P_{F,j}(\overline{j(X^*)}) - P_{F,j}(\overline{j(X)}))\|^2 \tag{2.16}$$

with a coverage error $0_p(n^{-2})$.

A practical method of finding a nonpivotal confidence region for the extrinsic antimean, consists of considering a chart, defined around all the bootstrap antimeans $P_{F,j}(\overline{j(X^*)}$ that the antimeans of resamples to $\mathbb{R}^d$; such a confidence region in terms of simultaneous confidence intervals.

3 Extrinsic Anti-MANOVA on Compact Manifolds

Consider an embedding $j : \mathcal{M} \to \mathbb{R}^N$, of a compact manifold $\mathcal{M}$ of dimension d. For a=1, ..., g, let $X_{a,1}, \dots, X_{a,n_a}$ be i.i.d.r.o.'s on $\mathcal{M}$ with the probability measure $Q_a = P_{X_{a,1}}$ being αj-nonfocal. Let $\alpha\mu_{a,E}$ be the extrinsic antimean of Q_a, and $a\bar{X}_{a,E}$ be the sample extrinsic antimean of $X_{a,1}, \dots, X_{a,n_a}$. Define the *pooled extrinsic antimean* with weights $\lambda = (\lambda_1, \dots, \lambda_g)$, denoted by $\alpha\mu_E(\lambda)$ given by

$$j(\alpha\mu_E) = P_{F,j}(\lambda_1 j(\alpha\mu_{1,E}) + \cdots + \lambda_g j(\alpha\mu_{g,E})). \tag{3.1}$$

Likewise, the *pooled sample extrinsic antimean* , denoted by $a\bar{X}_E \in \mathcal{M}$ is given by

$$j(a\bar{X}_E) = P_{F,j}(a\overline{j^{(p)}(X)}), \tag{3.2}$$

where $a\overline{j^{(p)}(X)} = \frac{n_1}{n} j(a\bar{X}_{1,E}) + \cdots + \frac{n_g}{n} j(a\bar{X}_{g,E})$. Here it is assumed that $a\bar{X}_{a,E}$, the extrinsic sample antimean for the a-th sample is well defined, and $n = \sum_{a=1}^g n_a$; The weights are in this case $\hat{\lambda}_a = \frac{n_a}{n}, a = 1, \dots, g$. Under the null hypothesis

$$H_0 : \alpha\mu_{1,E} = \cdots = \alpha\mu_{g,E}, \tag{3.3}$$

and the usual alternative, for $b = 1, \ldots, g$, we consider:

$aS_b = (n_b)^{-1}\Sigma_{i=1}^{n_b}(j(X_{b,i}) - j(a\bar{X}_{bE}))(j(X_{b,i}) - j(a\bar{X}_{bE}))^T$ as a consistent estimator of $\alpha\Sigma_b$. Also note that $\tan_{j(a\bar{X}_E)}\nu$ is a consistent estimator of $\tan_{P_{F,j}(\mu)}\forall\nu \in \mathbb{R}^N$.

It follows that the *extrinsic sample anticovariance matrix* $aS_{b,E}$, given by

$$aS_{b,E} = \left[\left[\sum_{a=1}^{d} d_{\overline{aj^{(p)}(X)}} P_{F,j}(e_a) \cdot e_i(j(a\bar{X}_E))\ e_i(j(a\bar{X}_E))\right]_{i=1,\ldots,d}\right] \cdot\ S_{n_b}$$
$$\left[\left[\sum_{a=1}^{d} d_{\overline{aj^{(p)}(X)}} P_{F,j}(e_a) \cdot e_i(j(a\bar{X}_E)) e_i(j(a\bar{X}_E))\right]_{i=1,\ldots,d}\right]^T$$

is a consistent estimator for $\alpha\Sigma_{b,E}$

Theorem 3.1. *Assume $j : \mathcal{M} \to \mathbb{R}^N$ is an embedding of the compact manifold $\mathcal{M}$. For $a = 1, ..., g$, let $\{X_{a,i}\}_{i=1,\ldots,n_a}$ be i.i.d.r.o.'s from the j-nonfocal distributions $\mathcal{Q}_a$ on $\mathcal{M}$. Let $\mu_a = E(j(X_{a,1}))$ and assume the extrinsic anticovariance matrices $\alpha\Sigma_{a,E}$ of $X_{a,1}$ are nonsingular. We also let $(e_1(p), ..., e_N(p))$, for $p \in \mathcal{M}$ be an orthonormal frame field adapted to j defined in an open neighborhood of the pooled extrinsic antimean and of the set of extrinsic population antimeans. Assume that $\frac{n_a}{n} \to \lambda_a > 0,$ as $n \to \infty, \forall a = 1, \ldots, k$. Then under (3.3),*

$$\sum_{a=1}^{g} n_a\ \tan_{j(\alpha\mu_E)}(j(a\bar{X}_{a,E}) - j(a\bar{X}_E))^T aS_{b,E}^{-1}\ \tan_{j(\alpha\mu_E)}(j(a\bar{X}_{a,E}) - j(a\bar{X}_E)) \to_d \chi^2_{gd}. \tag{3.4}$$

and

$$\sum_{b=1}^{g} n_b\ \tan_{j(a\bar{X}_E)}(j(a\bar{X}_{b,E}) - j(a\bar{X}_E))^T aS_{b,E}^{-1}\ \tan_{j(a\bar{X}_E)}(j(a\bar{X}_{b,E}) - j(a\bar{X}_E)) \to_d \chi^2_{gd}. \tag{3.5}$$

For a proof, see Lee and Patrangenaru(2019)[9].

Corollary 3.2. *Under the null hypothesis in (3.3), confidence regions for $\alpha\mu_E$ of asymptotic level $1 - c$ are given by $C_{n,c}^{(g)}$ and $D_{n,c}^{(g)}$ as follows*

- $C_{n,c}^{(g)} = j^{-1}(U_{n,c})$ *where*
  $U_{n,c} = \{j(\nu) \in j(\mathcal{M}) : \sum_{a=1}^{g} n_a \left\|aS_{a,E}^{-1/2}\ \tan_{j(\nu)}(j(a\overline{X}_{a,E}) - j(\nu))\right\|^2 \leq \chi^2_{gd,1-c}\}$
- $D_{n,c}^{(g)} = j^{-1}(V_{n,c})$ *where*
  $V_{n,c} = \{j(\nu) \in j(\mathcal{M}) : \sum_{a=1}^{g} n_a \left\|aS_{a,E}^{-1/2}\ \tan_{j(\bar{X}_E)}(j(a\overline{X}_{a,E}) - j(\nu))\right\|^2 \leq \chi^2_{gd,1-c}\}$

where $a\overline{X}_{a,E}$ is the pooled extrinsic sample antimean.

For $a = 1, ..., g$, let $\{X_{a,i}\}_{i=1,...,n_a}$ be i.i.d.r.o's from the αj-nonfocal distributions $\mathcal{Q}_a$. Let $\{X^*_{a,r}\}_{r=1,...,n_a}$ be random resamples with repetition from the empirical $\hat{Q}_{n_a}$ conditionally given $\{X_{a,i}\}_{i=1,...,n_a}$. The confidence regions $C^{(g)}_{n,c}$ and $D^{(g)}_{n,c}$ described in Corollary 3.2 have corresponding bootstrap analogues as given below.

Corollary 3.3. *The* $(1-c)100\%$ *bootstrap confidence regions for* $\alpha\mu_E$ *with* $d = gp$ *are given by* $C^{*(g)}_{n,c} = j^{-1}(U^*_{n,c})$ *and*

$$U^*_{n,c} = \{j(\nu) \in j(\mathcal{M}) : \sum_{a=1}^{g} n_a \left\| aS^{-1/2}_{a,E}\ \tan_{j(\nu)}(j(a\overline{X}_{a,E}) - j(\nu)) \right\|^2 \le c^{*(g)}_{1-c}\}, \tag{3.6}$$

where $c^{*(g)}_{1-c}$ *is the upper* $100(1-c)\%$ *point of the values*

$$\sum_{a=1}^{g} n_a \left\| aS^{*}_{a,E}{}^{-1/2}\ \tan_{j(a\bar{X}_E)}(j(a\overline{X^*}_{a,E}) - j(a\bar{X}_E)) \right\|^2 \tag{3.7}$$

among all bootstrap resamples, and $D^{*(g)}_{n,c} = j^{-1}(V^*{}_{n,c})$*, with*

$$V^*{}_{n,c} = \{j(\nu) \in j(\mathcal{M}) : \sum_{a=1}^{g} n_a \left\| aS^{-1/2}_{a,E}\ \tan_{j(\bar{X}_E)}(j(a\overline{X}_{a,E}) - j(\nu)) \right\|^2 \le d^{*(g)}_{1-c}\} \tag{3.8}$$

where $d^{*(g)}_{1-c}$ *is the upper* $100(1-c)\%$ *point of the values*

$$\sum_{a=1}^{g} n_a \left\| aS^{*}_{a,E}{}^{-1/2}\ \tan_{j(\alpha\bar{X}^*{}_E)}(j(a\overline{X^*}_{a,E}) - j(\alpha\bar{X}_E)) \right\|^2, \tag{3.9}$$

and $a\bar{X}^*{}_E$ *is the extrinsic pooled bootstrap sample antimean given by*

$$j(a\bar{X}^*{}_E) = P_j\left(\frac{n_1}{n} j(a\bar{X}^*_{1,E}) + \cdots + \frac{n_g}{n} j(a\bar{X}^*_{g,E})\right). \tag{3.10}$$

Both confidence regions given by (3.8) *and* (3.6) *have coverage error* $O_p(n^{-2})$.

Note that

$$aS^*_{a,E} = \left[\left[\sum_{a=1}^{d} d_{\overline{aj^{(p)}(X^*)}} P_j(e_b) \cdot e_i(j(\alpha\bar{X}^*_E))\ e_i(j(\alpha\bar{X}^*_E))\right]_{i=1,...,p}\right] \cdot\ S^*_{n_a}$$
$$\left[\left[\sum_{a=1}^{m} d_{\overline{aj^{(p)}(X^*)}} P_j(e_b) \cdot e_i(j(\alpha\bar{X}^*_E)) e_i(j(\alpha\bar{X}^*_E))\right]_{i=1,...,p}\right]^T$$

where $S^*_{n_a} = (n_a)^{-1}\Sigma_{i=1}^{n_a}(j(X^*_{a,i}) - j(\alpha\bar{X}^*_E))(j(X^*_{a,i}) - j(\alpha\bar{X}^*_E))^T$.
In terms of nonparametric bootstrap approximations, for hypothesis testing, we will rely on the following result obtained by substituting

$$X^{(g)} = (X_{1,a_1})_{a_1=1,...,n_1}, \cdots, (X_{g,a_g\,a_g=1,...,n_g})$$

with resamples with repetition

$$X^{*(g)} = (X^*_{1,a_1})_{a_1=1,...,n_1}, \cdots, (X^*_{g,a_g}))_{a_g=1,...,n_g}).$$

Proposition 3.4. *For a $a = 1, ..., g$, let $\{X_{a,i}\}_{i=1,...,n_a}$ i.i.d.r.o.'s from the j-nonfocal distributions $\mathcal{Q}_a$. Let $\mu_a = E(j(X_{a,1}))$ and assume the extrinsic covariance matrice $a\Sigma_{a,E}$ of $X_{a,1}$ is nonsingular, $\forall a = 1, \ldots, g$. Then the distribution of*

$$T_c(X^{(g)}, Q^{(g)}) = \sum_{a=1}^{g} n_a \left\| a\Sigma_{a,E}^{-1/2} \ \tan_{j(\alpha\mu_E)}(j(a\overline{X}_{a,E}) - j(\alpha\mu_E)) \right\|^2$$

can be approximated by the bootstrap distribution of
$T_c(X^{*(g)}, \hat{Q}^{(g)}) = \sum_{a=1}^{g} n_a \left\| aS_{a,E}^{-1/2} \ \tan_{j(a\bar{X}_E)}(j(a\overline{X}^*_{a,E}) - j(a\bar{X}_E)) \right\|^2$.
Similarly, the distribution of
$T_d(X^{(g)}, \hat{Q}^{(g)}) = \sum_{a=1}^{g} n_a \left\| aS_{a,E}^{-1/2} \ \tan_{j(a\bar{X}_E)}(j(a\overline{X}_{a,E}) - j(\alpha\mu_E)) \right\|^2$ *can be approximated by the bootstrap distribution function of*
$T_d(X^{*(g)}, \hat{Q}^{*(g)}) = \sum_{a=1}^{g} n_a \left\| aS^*_{a,E}{}^{-1/2} \ \tan_{j(a\bar{X}^*{}_E)}(j(a\overline{X^*}_{a,E}) - j(a\bar{X}_E)) \right\|^2$ *with coverage error $O_P(n^{-2})$.*

4 VW antimeans on $\mathbb{R}P^m$

In this section we consider the case when $\mathcal{M} = \mathbb{R}P^m$ is the real projective space, set of 1-dimensional linear subspaces of $\mathbb{R}^{m+1}$. $(\mathbb{R}P^m, \rho_0)$ is a compact space with ρ_0 the chord distance induced by the Veronese Whitney (VW) embedding in the space of $(m+1) \times (m+1)$ positive semi-definite symmetric matrices, $j : \mathbb{R}P^m \to S_+(m+1, \mathbb{R})$ given by

$$j([x]) = xx^T, \|x\| = 1 \tag{4.1}$$

We first must recall some properties of the VW embedding. It is an equivariant embedding, this means that it acts on the left on $S_+(m+1, \mathbb{R})$, the set of nonnegative definite symmetric matrices with real coefficients, by

$$T \cdot A = TAT^T, \quad \forall\, T \in SO(m+1), \forall\, A \in S_+(m+1, \mathbb{R})$$
$$j(T \cdot [x]) = T \cdot j([x]), \quad \forall\, [x] \in \mathbb{R}P^m,$$

where $T \cdot [x] = [Tx]$.
Also $j(\mathbb{R}P^m) = \{A \in S_+(m+1, \mathbb{R}) : rank(A) = 1, Tr(A) = 1.\}$ And the set $\mathcal{F}$ of j-focal points of $j(\mathbb{R}P^m)$ in $S_+(m+1, \mathbb{R})$, is the set of matrices in $S_+(m+1, \mathbb{R})$ whose largest eigenvalues are of multiplicity at least 2. The induce distance id defined as follow; for $A,\ B\ \in\ S(m+1, \mathbb{R})$ we define $d_0(A, B) = tr((A - B)^2)$. Recall that if

$\mu = E(XX^T)$ is the mean of $j(Q)$ in $\mathbb{R}^N$.

$$\mathcal{F}([p]) = \|j([p]) - \mu\|_0^2 + \int_{\mathcal{M}} \|\mu - j([x])\|_0^2 Q(dx) \tag{4.2}$$

And $\mathcal{F}([p])$ is maximized if and only if $\|j([p]) - \mu\|_0^2$ is maximize with respect to $[p] \in \mathcal{M}$.

Proposition 4.1. *[(i)]*

1. *$(\alpha F)^c$, set of αVW-nonfocal points in $S_+(m+1,\mathbb{R})$, is made of matrices in whose smallest eigenvalue has multiplicity 1.*

2. *The projection $P_{F,j} : (\alpha F)^c \to j(\mathbb{R}P^m)$ assigns to each nonnegative definite symmetric matrix A, of rank 1, with a smallest eigenvalue of multiplicity 1, the matrix $j([\nu])$, where $\|\nu\| = 1$ and ν is an eigenvector of A corresponding to that eigenvalue.*

We now have the following;

Proposition 4.2. *Let Q be a distribution on $\mathbb{R}P^m$.*

1. *The VW-antimean set of a random object $[X], X^TX = 1$ on $\mathbb{R}P^m$, is the set of points $p = [v] \in V_1$, where V_1 is the eigenspace corresponding to the smallest eigenvalue $\lambda(1)$ of $E(XX^T)$.*

2. *If in addition $Q = P_{[X]}$ is αVW-nonfocal, then*
$$\alpha\mu_{j,E}(Q) = j^{-1}(P_{F,j}(\mu)) = \gamma(1)$$
where $(\lambda(a), \gamma(a))$, $a = 1,..,m+1$ are eigenvalues in increasing order and the corresponding unit eigenvectors of $\mu = E(XX^T)$.

3. *Let $x_1, \ldots, x_n$ be random observations from a distribution Q on $\mathbb{R}P^m$, such that $\overline{j(X)}$ is α VW-nonfocal. Then the VW sample antimean of $x_1, \ldots, x_n$ is given by;*
$$a\overline{x}_{j,E} = j^{-1}(P_{F,j}(\overline{j(x)})) = g(1)$$
where $(d(a), g(a))$ are the eigenvalues in increasing order and the corresponding unit eigenvectors of $J = \sum_{i=1}^{n} x_i x_i^T$.

4.1 Hypothesis testing for two VW antimean projective shapes

The real projective space, $\mathbb{R}P^m$, is the building block in the geometric structure of the projective shape space; the projective shape space of k-ads $(x_1, \ldots, x_k$ in $\mathbb{R}P^m$ that include a *projective frame* at given fixed indices, can be identified with $(\mathbb{R}P^m)^q$, where $q = k - m - 2$. (see Mardia and Patrangenaru (2005)[10]). This space is embedded via the VW-embedding $j_q : (\mathbb{R}P^m)^q \to (S_+(m+1,\mathbb{R}))^q$ as follows:

$$j_q([x_1], \ldots, [x_q]) = (j([x_1]), \ldots, j_q([x_q])), \tag{4.3}$$

where j is the VW embedding of $\mathbb{R}P^m$, given in (4.1).

Assume that for $a = 1, 2, Q_a$ are αVW-nonfocal. We are now interested in the hypothesis testing problem:

$$H_0: \ \alpha\mu_{1,E} = \alpha\mu_{2,E} \text{ vs.} H_a: \ \alpha\mu_{1,E} \neq \alpha\mu_{2,E}, \tag{4.4}$$

For $m = 3$, the hypothesis (4.4) is equivalent to the following

$$H_0: \ \alpha\mu_{2,E}^{-1} \odot \alpha\mu_{1,E} = 1_q \text{ vs.} H_a: \ \alpha\mu_{2,E}^{-1} \odot \alpha\mu_{1,E} \neq 1_q \tag{4.5}$$

1. Let $n_+ = n_1+n_2$ be the total sample size, and assume $\lim_{n_+\to\infty} \frac{n_1}{n_+} \to \lambda \in (0,1)$. Let φ be the log chart defined in a neighborhood of 1_q (see Helgason (2001)), with $\varphi(1_q) = 0$. Then, under H_0

   (4.6) $$n_+^{1/2}\ \varphi(a\bar{Y}_{n_2,E}^{-1} \odot a\bar{Y}_{n_1,E}) \to_d \mathcal{N}_{3q}(0_{3q}, \tilde{\Sigma}_{j_q}),$$

   for some covariance matrix $\tilde{\Sigma}_{j_q}$.

2. Assume in addition that for $a = 1, 2$ the support of the distribution of $Y_{a,1}$ and the VW anti mean $\alpha\mu_{a,E}$ are included in the domain of the chart φ and $\varphi(Y_{a,1})$ has an absolutely continuous component and finite moment of sufficiently high order. Then the joint distribution

   (4.7) $$aV = n+^{\frac{1}{2}}\varphi(a\bar{Y}_{n_2,E}^{-1} \odot a\bar{Y}_{n_1,E})$$

   can be approximated by the bootstrap joint distribution of

   (4.8) $$aV^* = n+^{1/2}\ \varphi(a\bar{Y^*}_{n_2,E}^{-1} \odot a\bar{Y}_{n_1,E}^*)$$

Now, from proposition 4.2, we get the following result that is used for the computation of the VW sample antimeans:

Proposition 4.3. *follows that given a random sample from a distribution Q on $\mathbb{R}P^m$, if $J_s, s = 1,\dots,q$ are the matrices $J_s = n^{-1}\sum_{r=1}^n X_r^s(X_r^s)^T$, and if for $a = 1,\dots,m+1, d_s(a)$ and $g_s(a)$ are the eigenvalues in increasing order and corresponding unit eigenvectors of J_s, then the VW sample antimean $a\bar{Y}_{n,E}$ is given by*

(4.9) $$a\bar{Y}_{n,E} = ([g_1(1)],\dots,[g_q(1)]).$$

4.2 VW Anti-MANOVA on $(\mathbb{R}P^3)^q$

In this subsection, we specialize the methods presented above in 3 to anti-MANOVA on $P\Sigma_3^k$, the projective shape space of 3D k-ads in $\mathbb{R}P^m$ for which $\pi = ([u_1],\dots,[u_5])$ is a projective frame in $\mathbb{R}P^3$, which is homeomorphic to the manifold $\left(\mathbb{R}P^3\right)^{k-5}$ with $k-5 = q$ (see Patrangenaru et. al (2010)[13]). The embedding on this space is the VW embedding given in (4.3).

Additionally, from Proposition 4.3, the corresponding farthest projection

(4.10) $$\begin{aligned} P_{j_q,F} &: (S_+(4,\mathbb{R}))^q \backslash \mathcal{F}_q \to j_k\left(\mathbb{R}P^3\right)^q \\ P_{j_q,F}(A_1,\dots,A_q) &= (j([m_1]),\dots,j[m_q])) \end{aligned}$$

where $\forall a = 1,\dots,q, m_a$ is an eigenvectors of norm one of A_a, corresponding to its lowest eigenvalues, which is simple. That is same as saying that if Y is a random object from a distribution Q on $(\mathbb{R}P^3)^q$, where $Y = (Y^1,\dots,Y^q)$, and $Y^s = [X^s] \in \mathbb{R}P^3, s = 1,\dots,q$, then T]the VW antimean of Y is given by

(4.11) $$\alpha\mu_{j_q} = ([\gamma_1(1)],\cdots,[\gamma_q(1)]),$$

where, for $s = \overline{1,q}$, $\lambda_s(r)$ and $\gamma_s(r), r = 1, \ldots, 4$ are the eigenvalues in increasing order and the corresponding eigenvectors of $E\left[X^s(X^s)^T\right]$. Given i.i.d.r.o.'s $Y_1, \ldots, Y_n$ from a distribution Q on $(\mathbb{R}P^3)^q$, with $Y_i = (Y_i^1, \ldots, Y_i^q)$, and $Y_i^s = [X_i^s], X_i^{sT} X_i^s = 1$, their sample VW-antimean is given in (4.9), for $m = 3$. The VW-anticovariance matrix (anticovariance matrix associated with the VW embedding j_q) derived from (2.4) has the entries

$$
\begin{aligned}
aS_{j_q,(s,a),(t,b)} &= n^{-1}(d_s(1) - d_s(a))^{-1}(d_t(1) - d_t(b))^{-1} \times \\
&\sum_{i=1}^{n}(g_s(a) \cdot X_i^s)(g_t(b) \cdot X_i^T)(g_s(1) \cdot X_i^s)(g_t(1) \cdot X_i^T),
\end{aligned} \tag{4.12}
$$

for the pair of indices $(s, a), (t, b), s, t = 1, \ldots, q$ and $a, b = 2, 3, 4$, listed in their lexicographic order, where for $a = 1, \ldots, 4$, $d_s(a), g_s(a)$ are the respectively eigenvalues in increasing order and corresponding unit eigenvectors of

$$
J_s = n^{-1} \sum_{r=1}^{n} X_r^s (X_r^s)^T \tag{4.13}
$$

Assume the VW anticovariance matrix $\alpha\Sigma_{j_q}$ is positive definite, thus given a large sample, with high probability the sample VW anticovariance matrix aS_{j_q} has an inverse. Then, asymptotic distribution of the corresponding Hotelling T^2 type r.v.

$$
T(Y, \alpha\mu_{j_q}) = n\|aS_{j_q}{}^{-1/2} \tan_{j(a\overline{Y}_{j_q})}\left(j_q(a\overline{Y}_{j_q}) - j(\alpha\mu_{j_q})\right)\|^2 \tag{4.14}
$$

is a χ^2_{3q}, and its expression is

$$
\begin{aligned}
T(Y, ([\gamma_1(1)], \cdots, [\gamma_q(1)])) = n \ &\left(\gamma_1(1)^T D_1 \ldots \gamma_q(1)^T D_q\right) \quad aS_{j_q}{}^{-1} \cdot \\
&\cdot \left(\gamma_1(1)^T D_1 \ldots \gamma_q(1)^T D_q\right)^T
\end{aligned} \tag{4.15}
$$

where aS_{j_q} and $D_s = (g_s(2)\ g_s(3)\ g_s(4)) \in \mathcal{M}(4, 3, \mathbb{R})$ are given as in (4.12). We are in the position of giving the explicit expression of the test statistics that are addressing the VW anti-MANOVA hypothesis testing problem:

$$
\begin{aligned}
&H_0 : \ \alpha\mu_{1,E} = \alpha\mu_{2,E} = ... = \alpha\mu_{g,E} = \alpha\mu_E, \\
&H_a : \ \textit{at least one equality } \alpha\mu_{a,E} = \alpha\mu_{b,E}, 1 \le a < b \le g \textit{ does not hold.}
\end{aligned} \tag{4.16}
$$

Let $Y^{(g)} = (Y_{a,1}, \ldots, Y_{a,n_a})_{a=1,\ldots,g}$ be independent r.o.'s from the distributions $Q_a, a = 1, \ldots, g$ on $(\mathbb{R}P^3)^q$. We aim at having an explicit representation of the expression of the second test statistic

$$
T_d(Y^{(g)}, \hat{Q}^{(g)}) = \sum_{a=1}^{g} n_a \|aS_{a,j_q}^{-1/2} \tan_{j_q}(a\bar{Y}_{j_q})\left(j_q(a\bar{Y}_{j_q}) - j_q(\alpha\mu_{j_q})\right)\|^2, \tag{4.17}
$$

from Proposition 3.4, where $\alpha\mu_{j_q}, a\bar{Y}_{j_q}$ are respectively the pooled VW antimean and pooled sample VW antimean for the given data. Note that $\alpha\mu_{a,j_q} = ([\gamma_1^a(1)], \ldots, [\gamma_q^a(1)])$ is the VW antimean for the sample from distribution Q_a (of $Y_{a,1}, \ldots, Y_{a,n_a}$) and

$(\eta_s^a(r), \nu_s^a(r))$, $r = 1, \ldots, 4$, are eigenvalues and corresponding unit eigenvectors of $E(X_{a,1}^s (X_{a,1}^s)^T]$. The corresponding VW sample antimean is given by $a\overline{Y}_{a,j_q} = ([g_1^a(1), \ldots, [g_q^a(1)])$, where for each $s = 1, \ldots, q$ and $r = 1, \ldots, 4$, $(d_s^a(r), g_s^a(r))$ are eigenvalues in increasing order and corresponding unit eigenvectors of $J_s^a = \frac{1}{n_a} \sum_{i=1}^{n_a} X_{a,i}^s (X_{a,i}^s)^T$. Also $\alpha\mu_{j_q}$ is the VW pooled antimean given by

$$j_q(\alpha\mu_{j_q}) = P_{j_q,F}\left(\sum_{a=1}^{g} \lambda_a j_q(\alpha\mu_{a,j_q})\right) \alpha\mu_{j_q} = ([\gamma_1^{(p)}(1)], \ldots, [\gamma_q^{(p)}(1)]), \tag{4.18}$$

where for $s = 1, \ldots, q$, $\gamma_1^{(p)}(1)$ is the eigenvector corresponding to the smallest eigenvalue of the $s - th$ axial component of the pooled matrix with weights $\lambda_a, a = 1, \ldots, g, \sum_a \lambda_a = 1, \lambda_a > 0$ given by

$$\sum_{a=1}^{g} \lambda_a E(X_{a,1} X_{a,1}^T).$$

The pooled VW-sample antimean $a\overline{Y}_{j_q}^{(p)}$ is given by

$$j_q\left(a\overline{Y}_{j_q}\right) = P_{j_q,F}\left(\sum_{a=1}^{g} \frac{n_a}{n} j_q(a\overline{Y}_{a,j_q})\right) \tag{4.19}$$

$$a\overline{Y}_{j_q}^{(p)} = ([\mathbf{g}_1^{(p)}(1)], \ldots, [\mathbf{g}_q^{(p)}(1)]). \tag{4.20}$$

Here for $s = 1, \ldots, q$, $\mathbf{d}_s^{(p)}(r)$ and $\mathbf{g}_s^{(p)}(r) \in \mathbb{R}^4$, $r = 1, 2, 3, 4$, are eigenvalues in increasing order and corresponding unit eigenvectors of the matrix $J^{(p)} = \sum_{a=1}^{g} \frac{n_a}{n} j_k(\overline{Y}_{a,E})$.

The following matrices

$$\mathbf{D}_s = (\mathbf{g}_s^{(p)}(2)\ \mathbf{g}_s^{(p)}(3)\ \mathbf{g}_s^{(p)}(4)) \in \mathcal{M}(4, 3 : \mathbb{R}) \tag{4.21}$$

are giving a basis in the tangent space of the pooled sample VW antimean.

Theorem 4.4. *Assume $\{Y_{a,r_a}\}_{r_a=1,\ldots,n_a}$, $a = 1, \ldots, g$ are i.i.d.r.o.'s from the j_q-nonfocal probability measures Q_a on $(\mathbb{R}P^3)^q$ with the VW embedding of j_q leading to nondegenerate j_q-extrinsic anticovariance matrices. Consider the statistic*

$$T_d(Y^{(g)}, a\overline{Y}_{j_q}) = \sum_{a=1}^{g} n_a \left[(\gamma_1^{(p)}(1) - g_1^a(1))^T \mathbf{D}_1 \ldots (\gamma_q^{(p)}(1) - g_q^a(1))^T \mathbf{D}_q\right] aS_{a,j_q}^{-1} \left[(\gamma_1^{(p)}(1) - g_1^a(1))^T \mathbf{D}_1 \ldots (\gamma_q^{(p)}(1) - g_q^a(1))^T \mathbf{D}_q\right]^T, \tag{4.22}$$

where

$$aS_{a,j_q(s,c)(t,b)} = n_a^{-1} (\mathbf{d}_s^{(p)}(1) - \mathbf{d}_s^{(p)}(c))^{-1} (\mathbf{d}_t^{(p)}(1) - \mathbf{d}_t^{(p)}(b))^{-1}$$
$$\times \sum_i (\mathbf{g}_s^{(p)}(c) \cdot X_{a,i}^s)(\mathbf{g}_t^{(p)}(b) \cdot X_{a,i}^t)(\mathbf{g}_s^{(p)}(1) \cdot X_{a,i}^s)(\mathbf{g}_t^{(p)}(1) \cdot X_{a,i}^t)$$

and $s, t = 1, \dots, q$ *and* $c, b = 2, 3, 4$. *If* $\frac{n_a}{n} \to \lambda_a > 0$, *as* $n \to \infty$, *then* $T_d(Y^{(g)}, a\overline{Y}_{j_q})$ *converges weakly to a* χ^2_{3q} *distributed r.v.*

Proof. The asymptotic behaviors of the sample VW-antimeans follow from Theorem 3.1, when applied to the VW embedding j_q of $(\mathbb{R}P^3)^q$ (a.k.a. projective shape space of $q+5$-ads in general position in $\mathbb{R}P^3$) given in (4.3). Indeed from (3.4), we split the difference $j_q(a\bar{X}_{a,j_q}) - j(a\bar{X}_{j_q})$ in the tangent space $T_{j_q(a\bar{X}_{a,j_q})}(j_q(\mathbb{R}P^3)^q)$, w.r.t. the orthogonal basis described in (4.21). □

Corollary 4.5. *A* $(1-c)100\%$ *nonparametric bootstrap confidence region for* $\alpha\mu_{j_q}$ *is given by*

$$D^{*(g)}_{n,c} = j^{-1}(V^*{}_{n,c}), \tag{4.23}$$

where $V^*{}_{n,c} = \{j_k(\nu), T_d(Y^{(g)}, a\overline{Y}_{j_q}, \nu) \le d^{*(g)}_{1-c}\}$ *and*

$$T_d(Y^{(g)}, a\overline{Y}_{j_q}, \nu) = n_a \sum_{a=1}^{g} \left\| aS^{-1/2}_{a,j_q} \tan_{j_q(a\overline{Y}_{j_q})}(j_q(a\overline{Y}_{j_q}) - j_q(\nu)) \right\|^2,$$

with $d^{*(g)}_{1-c}$ *being the upper* $100(1-c)\%$ *point of the values of*

(4.24)
$$T_d(Y^{*(g)}, a\overline{Y}^*_{j_q}, a\overline{Y}_{j_q}) = \sum_{a=1}^{g} n_a \left\| aS^{*}_{a,j_q}{}^{-1/2} \tan_{j_q(a\overline{Y}^*_{j_q})}(j_q(a\overline{Y}^*_{a,j_q}) - j_q(a\overline{Y}_{j_q})) \right\|^2,$$

among the bootstrap resamples, where

$$aS^*_{a,j_q(s,c)(t,b)} = n_a^{-1}(\mathbf{d}^{*(p)}_s(1) - \mathbf{d}^{*(p)}_s(c))^{-1}(\mathbf{d}^{*(p)}_t(1) - \mathbf{d}^{*(p)}_t(b))^{-1} \times$$
$$\sum_i (\mathbf{g}^{*(p)}_s(c) \cdot X^{*s}_{a,i})(\mathbf{g}^{*(p)}_t(b) \cdot X^{*t}_{a,i})(\mathbf{g}^{*(p)}_s(1) \cdot X^{*s}_{a,i})(\mathbf{g}^{*(p)}_t(1) \cdot X^{*t}_{a,i}), b, c = 2, 3, 4.$$

The confidence regions given by (4.23) *has coverage error* $O_p(n^{-2})$.

5 Application to face data analysis

Digital images collected with a high resolution Panasonic-Lumix DMC-FZ200 camera, posted at ani.stat.fsu.edu/∼vic/E-MANOVA, were used to test for a VW mean 3D projective shape difference between five faces (see Yao et al(2017)[20]). 3D surface reconstructions of those faces, with the seven labeled landmarks, and a projective frame are displayed in Figure 5 below.

These are based on Agisoft 3D surface reconstructions (see Yao et al(2017)[20]).

Here we compare the projective shapes of these faces by first conducting a VW anti-MANOVA analysis on $P\Sigma^7_3 = (\mathbb{R}P^3)^2$, testing the hypotheses (4.16), based on the sample sizes on hand: $n_1 = n_2 = n_4 = n_5 = 6$ and $n_3 = 7$, the null hypothesis being rejected if

$$T_d(Y^{(5)}, a\overline{Y}_{j_2}) = \sum_{a=1}^{5} n_a \left\| S^{-1/2}_{\alpha\bar{Y}_a, j_2} \tan_{j_2(a\overline{Y}_{j_2})}(j_2(a\overline{Y}_{a,j_2}) - j_2(a\overline{Y}_{j_2})) \right\|^2$$

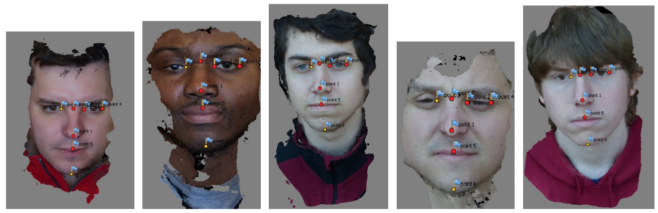

Figure 1: Sample of size one of facial surface reconstructions for each individual face, with corresponding marked landmarks

	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
Test result	Reject	Reject	Reject	No	Reject	Reject	No	No	No	No

Table 1: Results of pairwise VW mean change

is greater than ${d^*}^{(5)}_{1-\alpha}$, where ${d^*}^{(5}_{1-\alpha}$ is the $(1-\alpha)100\%$ cutoff of the corresponding bootstrap distribution in equation (4.24). With $5,000$ bootstrap resamples, we obtain $T_d(y^{(5)}, a\overline{y}_{j_2}) = 26,848.81$, and their corresponding empirical p-value 0.0088. Thus concluding that there exists a statistically significant VW-antimean 3D-projective shape face difference between at least two of the individuals in our data set. We then ran pairwise tests for antimean projective shape changes from subsection 4, and the results are given in Table 1.

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