Mathematical features of a living system

T. T. Bălan

Abstract. We highlight three general characteristics of a living system, namely: I. To survive, each living system must minimize the difficulty to progress from an event to another; II. Each living system is both continuous – as a material body in space, and discrete – as a set of events; and III. The universe of events, which are accessible to our senses, is not hor-complete, i.e. there may exist hidden events. We treat them in terms of super-additivity and horistology.

M.S.C. 2010: 46C20, 92B05, 93C65.
Key words: Super-additivity; horistology; living systems.

1 Introduction

Writing this rather philosophical note about the living systems is inspired by Schrödinger's work [7], where we find quantum explanations of several phenomena. The huge diversity of aspects as well as the reference to a "four-dimensional pattern" in [7] naturally lead us to a relativist vision of such systems. As a result, we obtain applications of the horistological structures (detailed in [1]), which seem more adequate to study sets of events.

The original contribution of this note is a similitude between the events in the Einsteinean space-time and those in a living system. To follow this parallel we specify new senses to the basic notions - time, state, event, accessibility (instead of causality) and difficulty (instead of proper time).

We suppose the interested reader is acquainted with horistological notions concernibg restrained norms and metrics, super-additivity (S.a.), perspectives, proper preorders, discreteness of the functions and of the sets, emergence etc., contained in [1].

2 Relativist living systems

In this section, we describe the evolution of a living system, generically noted $\mathcal{L}$, in terms of events and we organize the universe of these events in accordance to

BSG Proceedings 26, pp. 1-6. The 2nd International Conference on Applied Mathematics and Numerical Metods (ICAMNM), October 19-20, 2018, Craiova, Romania.

the mathematical structures from [1]. In living systems, the fundamental relativist notions have the following specific meanings:

Time. A detailed analysis of the notion of *time* (see [4] etc.) reveals many types of *times*. Here we operate with *chronological time*, which represents the most exact and objective variant.

The notion of *time* derives from that of *event*. Clearly stated by Heidegger (in [2] etc.), "*Der Raum ist an sich nichts; es gibt keinen absoluten Raum. Er existiert nur durch die in ihm enthaltenen Körper und Energien. (Ein alter Aristotelischer Satz:) Auch die Zeit ist nichts. Sie besteht nur infolge der sich in ihr abspielenden Ereignisse.*" In brief, there is no *space* but only bodies and there is no *time* but only events. Therefore, the *subjective time* would better represent the real *principium individuation* ([2] etc.) of a living system, similarly to the relativist *proper time*, which is subjective too, in the sense of its attachment to particular observers. Unfortunately, it is hard to imagine practical techniques of its measurement from outside.

To mark different moments, i.e. particular values of time, we use real numbers and we note the set of all possible values of time by T. Each living system has two remarkable moments in T:

– an *initial time* $t_{ini}(= beginning, appearance, birth etc.)$ and a

– *final time* t_{fin} (= end, disappearance, death etc.).

The interval in T, which consists of all moments between t_{ini} and t_{fin}, represents the *life span* of $\mathcal{L}$; briefly, we note it LS $(\mathcal{L}) = (t_{ini}, t_{fin})$.

State. The *living systems* are particular cases of general dynamical systems, theoretically described by the evolution of the *internal states* (see [3], [5] etc.). In the case of *relativist systems* (discussed in Section IV.5 of [1] etc.), the states are *spatial positions* of the moving points.

For relativist living systems, the *state* is a set of simultaneous values of its parameters (physical, chemical, informational etc.). If $\mathcal{P}$ is the set of all parameters involved in the evolution of a living system $\mathcal{L}$ and $f_p(t)$ denotes the value of $p \in \mathcal{P}$ at the moment $t \in LS(\mathcal{L})$, then the *state* of $\mathcal{L}$ at this (chronological) moment is the set $s(t) = \{f_p(t) : p \in \mathcal{P}\}$.

Obviously, knowing all parameters $p \in \mathcal{P}$ and measuring their values at the same moment is practically impossible since $\mathcal{P}$ may be infinite (some parameters possibly unknown) and the errors of measurement may interact. Therefore it is more realistic to limit our definition to a finite number of parameters, say $p_1, p_2, ..., p_n$, where $n \in \mathbb{N}$, depending on the power of investigation of an external observer (including us). In the simplest case the state refers to a single scalar parameter p.

Event. Extending the relativist vision, the *events* of a living system are pairs of the form (t = *time*, $s(t)$ = *state*), briefly noted $e = (t, s)$. We note the universe of these events by $W(\mathcal{L})$.

Accessibility. In Relativity, the relation of *accessibility* is traditionally called *causality*, noted K. It is a consequence of the ***Principle of Light***, which limits the speed v of any observer by the speed c of light. Similarly, each living system must obey limitations, essentially imposed by the ***Principle of Homeostasis***. Each scalar parameter has an *admitted band* (*vital interval*, *range of admittance* etc.) $A_p = (a_p^-, a_p^+)$, where a_p^- = *lower admitted limit* and a_p^+ = *upper admitted limit*. The living systems shall maintain the values $f_p(t)$ at each $t \in LS(\mathcal{L})$ in the corresponding admitted band in order to remain alive. Extended to its environment, this tendency of the living systems represents the so-called *negentropy* (primarily remarked by Schrödinger in [7], then unanimously recognized as a characteristic of life). In more details, the living systems assure their homeostatic evolution through feedback reactions, genetic information, conditional reflexes and finally by intelligence. If $t \in LS(\mathcal{L})$ and $f_p(t) \in A_p$ for each $p \in \mathcal{P}$, then $e = (t, s)$ is said to be a *proper event* of $\mathcal{L}$. The set of all proper events of $\mathcal{L}$ represents its *life*, noted $L(\mathcal{L})$.

Inside A_p we may distinguish other intervals: $N_p = (n_p^-,\, n_p^+)$, where n_p^- is the *lower normal limit* and n_p^+ is the *upper normal limit*, is called *normal band* (*normality*, *reference interval* etc.); $P_p^- = (a_p^-,\, n_p^-)$ is the *lower pathological interval*, and $P_p^+ = (n_p^+,\, a_p^+)$ is the *upper pathological interval* of p. Living with hyper-values $f_p(t) \in P_p^+$, respectively hypo-values $f_p(t) \in P_p^-$, is possible, but more difficult (meaning *disease* of $\mathcal{L}$).

Let $e_0 = (t_0, s_0)$ be a current proper event of $\mathcal{L}$. We say that $e^\star = (t^\star, s^\star)$ is *accessible* to $\mathcal{L}$ from e_0 if $e^\star$ is also proper to $\mathcal{L}$ and $t_0 < t^\star$. Consequently, the binary relation (preorder) of *accessibility* of $\mathcal{L}$ takes the form

$$\mathcal{A}(\mathcal{L}) = \{((t_0, s_0), (t^\star, s^\star) \in L(\mathcal{L}) \times L(\mathcal{L}) : t_0 < t^\star\}.$$

Difficulty. Without doubt, staying alive is a demanding process for each living system. Maintaining a favorable (homeostatic, desired) state is continuously affected by external factors as well as by altered internal parameters. According to James Miller (see [3], 13.2.5), "*All adjustment processes have their costs, in energy, in material resources, in information, or in time required for an action*". Thus, the simple maintenance alive, without additional perturbations, needs a minimal effort to endure a background difficulty (idling, ralenti), of rate $\eth_0$ along the unit interval of time. This difficulty is really small and we may neglect it. All we discuss is how living systems react to supplementary difficulties, caused by an intrusive disturbing factor, noted $\Finv$. In this case, to progress from an event to another, $\mathcal{L}$ may follow a roundabout evolution, most frequently intended to diminish difficulties via specific changes of parameters.

Instead of a general concept of difficulty, we focus on a separate influence of $\Finv$ on each particular parameter. We note this difficulty to progress from e_0 to $e^\star$ by $\eth_p(e_0, e^\star)$. As a result to the action of $\Finv$ on p, $\mathcal{L}$ has three possibilities:

(*i*) To maintain p unchanged, i.e. $f_p(t) = \pi$;

(*ii*) To increase the difficulty along $f_p^+ \geq f_p$, e.g. by positive feedback; exceeding a_p^+ disqualifies p for further operations;

(*iii*) To diminish the difficulty, $f_p^- \leq f_p$, by homeostasis, negative feedback etc., in order to come back in the reference interval.

Case (*iii*) is the most favorable to $\mathcal{L}$; it certifies a fight to live.

Let $f_p(t) \in P_p^+$ at all $t \in [t_0, t^*]$. In practice, the most available evaluations of $\eth_p(e_0, e^\star)$ are indirect, by the necessary energy, consumed resources, time etc. However, the simple case (*i*) shows that $\eth_p(e_0, e^\star)$ shall be proportional to both amplitude and duration of the deflection from normality. Briefly,

$$\eth_p(e_0, e^\star) = k(\pi - n_p^+)(t^* - t_0) + \eth_0(t^* - t_0),$$

where k may introduce additional nuances, e.g. units of measurement, $1/(a_p^+ - \pi)$ to express the danger for $\mathcal{L}$ to reach the value a_p^+ etc.

Measuring $\eth_p(e_0, e^\star)$ by such a formula is applicable to the other cases. Thus, in case (*iii*), let us define the *reduced pathological function* $F_p : T \to \mathbb{R}$ by

$$F_p(t) = \max\{f_p^-(t), n_p^+\},$$

and use it to evaluate

$$\eth_p(e_0, e^\star) = k \int_{t_0}^{t*} [F_p(t) - n_p^+]dt + \eth_0(t^* - t_0).$$

If $e^/$ is an intermediate event such that $f_p^-(t^/) < f_p(t^/)$, then

$$\eth_p(e_0, e') = k \int_{t_0}^{t^/} [F_p(t) - n_p^+]dt + \eth_0(t' - t_0) \text{ and}$$

$$\eth_p(e', e^\star) = k \int_{t^/}^{t*} [F_p(t) - n_p^+]dt + \eth_0(t^* - t'),$$

where k may have a similar significance. Their sum gives the difficulty of $\mathcal{L}$ to progress on the roundabout way $e_0 \to e' \to e^\star$.

Slight changes lead to similar formulas when $f_p(t_0) \in P_p^-$.

It is easy to see that $\eth_p : \mathcal{A}(\mathcal{L}) \longrightarrow \mathbb{R}$, and $\eth_p(e_0, e^\star) \geq 0$ holds at each $(e_0, e^\star) \in \mathcal{A}(\mathcal{L})$, hence $\eth_p$ has the characteristics of a *restrained metric* (in the sense of [1]). The reactions of $\mathcal{L}$ to the disturbing factor $\dashv$, (*i*) - (*iii*), correspond to the following relations (especially inequalities) involving $\eth_p$:

(*i*) $\Longrightarrow$ *additivity*, i.e. $\eth_p(e_0, e') + \eth_p(e', e^\star) = \eth_p(e_0, e^\star)$;

(*ii*) $\Longrightarrow$ sub$-$*additivity* (s.a.), i.e. $\eth_p(e_0, e^\star) \leq \eth_p(e_0, e') + \eth_p(e', e^\star)$;

(*iii*) $\Longrightarrow$ super$-$*additivity* (S.a.), i.e. $\eth_p(e_0, e^\star) \geq \eth_p(e_0, e') + \eth_p(e', e^\star)$.

The above relations between difficulties are simple consequences of the properties of a integral. The indefinite metric in the relativist (Einsteinean) universe of events has a similar behavior: (*i*) it is additive on collinear events $e_0, e', e^\star$; (*ii*) it is s.a. if the linear manifold containing $\{e_0, e', e^\star\}$ is semi-definite; and (*iii*) it is S.a. if this manifold is indefinite.

3 Minimazing difficulty

To survive, each living system must realize the super $-$ additivity of the difficulty produced by a disturbing factor.

In fact, in the presence of a perturbing factor $\dashv$, which impedes the evolution from e_0 to $e^\star$, a vital reaction of any living system is to minimize the resulting difficulty. To this purpose, $\mathcal{L}$ has to identify an intermediary event $e^/$ such that $(e_0, e^/)$, $(e^/, e^\star) \in \mathcal{A}(\mathcal{L})$ and

$$\eth_p(e_0, e^/) + \eth_p(e^/, e^\star) \leq \eth_p(e_0, e^\star).$$

Obviously, this is case (*iii*), when $\eth_p$ is S.a. If $\mathcal{L}$ cannot find such a rescue, then $\dashv$ becomes a *fatal* factor for $\mathcal{L}$ and $e^\star$ remains inaccessible, which means *death.*

Like in relativity, where the proper time is an S.a. metric, the subjecive time of a homeostatic living system seems to have the same property (see [4], [3] etc.).

Even if it looks like fiction, we may explain the super-additivity of $\eth_p$ as a dilation of the subjective time between e_0 and $e^\star$, which advertises $\mathcal{L}$ to look for $e^/$. A simple example is visible in feeding processes: If $\dashv$ is a drastic diminishing of the internal energy, then $e^/$ is a saving replenishment; there are plenty of devices capable to recharge their batteries in order to continue their programmed jobs (see Section "Artificial Intelligence" in [6] etc.).

The reaction of $\mathcal{L}$ to progress along a devious way $e_0 \to e^/ \to e^*$is genetically codified like an alternative to be followed "just in case". Even hardly accepted, this interpretation of the genetic program highlights an *anticipative* behavior of the living systems, like testing (be prepared to, waiting for) the future, beside obeying the consequences of the past.

4 Structural duality

Each living system has a dual character; it is both continuous – as a material body in space, and discrete – as a set of events.

Here, continuity and discreteness have a structural (theoretical) meaning, since they refer to the topological, respectively horistological structures. In more practical terms, this duality involves a *visible* part - geometric bodies, respectively an *invisible* part - histories, i.e. sets of events.

Because the difficulty $\eth_p$ to progress from an event to another in the presence of $\dashv$ is an S.a. metric (discussed in the previous section), it naturally leads to a horistology, noted χ_p. Passing from $\eth_p$ to its horistological structure corresponds to an extension from a single $e^\star$ from the future of e_0 to an entire perspective $P \in \chi_p(e_0)$. In particular, a hyperbolic perspective of e_0 has the form

$$H(e_0, r) = \{\ell \in \mathcal{A}(\mathcal{L})[e_0] : \eth_p(e_0, \ell) > r\},$$

where $r > 0$ represents the minimal difficulty to reach ℓ from e_0.

To adapt the notion of discrete set to a subset H of events (history) in a living system, we say that an event $\ell \in H$ is *detachable* from H if

$$H \cap \mathcal{A}(\mathcal{L})[\ell] \in \chi_p(\ell).$$

The quantum character of the events in a living system (like the temporized change of states etc.) suggests the discreteness of the histories that are taking place in $\mathcal{L}$. Consequently, the discrete feature of a living system arises from the presence in its functioning of the S.a. metrics $\eth_p$ and of the resulting horistologies, which represent structures of discreteness.

Except informing how able is $\mathcal{L}$ in preserving homeostasis, the opposition s.a. - S.a. has nothing to do with dichotomies like *good* - *bad*, *correct* - *incorrect*, *moral* - *immoral* etc.

5 Non completeness

The universe of events taking place in a living system, which are accessible to our senses, may be not hor – complete.

The well known notion of *topological completeness* of a (s.a.) metric space (X, ρ) means that X has enough points to assure the convergence of the fundamental *nets* (in particular *sequences*). In S.a. metric universes of events, (W, ρ), we meet the same problem, formulated in horistological terms.

Following [1] etc., to describe the positioning of the events in an emergent net, we say that a net $\xi : \mathfrak{D} \to W$ is *hor-fundamental* if

$$\forall d \in \mathfrak{D}\ \exists r > 0 \text{ such that } [n < m < d] \Longrightarrow [\rho(\xi(m), \xi(n)) > r].$$

The universe (W, ρ) of events is *hor-complete* if each hor-fundamental net ξ has *Germ* $\xi \neq \varnothing$. Like in topology, a simple example of non hor-complete horistological world is $\mathbb{Q}$, endowed by the usual S.a. norm $\rfloor \cdot \lfloor = \iota : \mathbb{Q}_+ \to \mathbb{Q}_+$. In the universe of living systems, the failure of hor-completeness may explain why we cannot perceive several primordial events (say "sparks of life").

We may distinguish two types of non hor-completeness: *absolute* (definitive, immutable), respectively *relative* (apparent, changeable).

- The *absolute* non hor-completeness shall be an objective physical property of the universe of events, independent of our knowledge.

- The *relative* non hor-completeness is depending on the stage of our knowledge, reflected in words like (temporarily) ineffable, indefinable, termless.

References

[1] Bălan Trandafir, *Complements of Hyperbolic Mathematics, from Super-Additivity to Structural Discreteness*, Universitaria, Craiova, 2016

[2] Heidegger Martin, *Der Begriff der Zeit*, Max Niemeyer Verlag, Tübingen, 1989

[3] Miller G. James, *Living Systems*, McGraw-Hill,.New York, 1978

[4] Marcus Solomon, *Timpul*, Editura ALBATROS, Bucureşti, 1985

[5] Mesarovich M. D., Takahara Y., *General System Theory* - *Mathematical Foundations*, Acad. Press, New York, 1975

[6] Penrose Roger, *The Emperor's New Mind*, Oxford Univ. Press, 1980

[7] Schrödinger Erwin, *What is Life*, Cambridge University Press, 1944

Author's address:

Trandafir T. Bălan
University of Craiova, Department of Applied Mathematics,
13, "Al. I. Cuza" st., 200585 Craiova, Romania.
E-mail: ttbalan@yahoo.com