Normalization of the generalized $\mathcal{K}-$ Mittag-Leffler function and ratio to its sequence of partial sums

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Abstract. In this article we introduce an operator $L^{\gamma,q}_{k,\alpha}(\beta,\delta)(f)(z)$ associated with generalized $\mathcal{K}-$ Mittag-Leffler function in the unit disk $\mathbb{U}=\{z:|z|<1\}$. Further the ratio of normalized $\mathcal{K}-$ Mittag-Leffer function $Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ to its sequence of partial sums $Q(^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)$ are calculated.

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1 Introduction

The (M-L) function was introduced by the Swedish Mathematician Mittag-Leffler [13, 14]. This function may be a classical function dealing with problems in Complex Analysis and so it is important for obtaining solutions of fractional differential and integral equations which are associated e.g., with the kinetic equation, random walks, Levy flights and super diffusive transport problems. Moreover, from its exponential behavior, the deviations of physical phenomena could also be described by physical laws through Mittag-Leffler functions. Many authors have studied on (M-L) functions for its properties, generalization, applications and extension, such as [12], [11], and [15].

2 Preliminaries and definitions

The one-parameter (M-L) function $E_\alpha(z) : (z) \in \mathbb{C}$ see [13] and [14],

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+1)} =: E_{\alpha,1}(z), \tag{2.1}$$

and its two-parameters extension $E_{\alpha,\beta}(z)$ was studied by Wiman [10],

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \quad (\alpha \geq 0), \tag{2.2}$$

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where $\alpha, \beta \in \mathbb{C}, \mathcal{R}(\alpha) > 0$ and $\mathcal{R}(\beta) > 0$.

Further, in 1971, Prabhakar [8] proposed another general form of (M-L) function $E^{\gamma}_{\alpha,\beta}(z)$ as:

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta) n!} z^n, \tag{2.3}$$

for which $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$ and $\mathcal{R}(\gamma) > 0$. where $(\gamma)_n$ is the Pochhammar symbol:

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1, & n = 0 \\ \gamma(\gamma+1)...(\gamma+n-1) & \end{cases}$$

Note that

$$(t)_n = t(t+1)_{n-1}, \quad n \in \mathbb{N}$$

and

$$(t)_n \geq t^n, \qquad n \in \mathbb{N}$$

Srivastava and Tomovski [5] proved that the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by 2.3 is an entire function in the complex z-plane.

Another useful generalization of the Mittag-Leffler function called as $\mathcal{K}-$Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$, introduced in [3] as:

$$E^{\gamma}_{k,\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma k(\alpha n + \beta) n!} z^n, \tag{2.4}$$

where $(\gamma)_{n,k}$ is the $k-$Pochhammer symbol defined as:

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+(n-1)k) \qquad (\gamma \in \mathbb{C}, k \in \mathcal{R}, n \in \mathbb{N}). \tag{2.5}$$

The following extension is due to A. K. Shukla and J. C. Prajapati [17],

$$E^{\gamma,q}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma k(\alpha n + \beta) n!} z^n, \tag{2.6}$$

Remark 2.1. In case $q \in \mathbb{N}$ then $(\gamma)_{nq} = q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma + r - 1}{q} \right)_n$

Further extension of $\mathcal{K}-$Mittag-Leffler function $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ took place which was studied by K. S Gehlot [2] as for $q \in (0,1) \cup \mathbb{N}$,

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n + \beta) n!} z^n, \tag{2.7}$$

where $(\gamma)_{nq,k}$ is defined as (2.5) and the generalized pochhammer symbol is defined in [1] under condition if $q \in \mathbb{N}$.

$$(\gamma)_{nq} = \frac{\Gamma(\gamma + nq)}{\Gamma(\gamma)}, \tag{2.8}$$

Due to the high interest of researchers in (M-L) functions another parameter $\delta \in \mathbb{C}$ was inserted by Salim [18] which has the generalized form of: $E^{\gamma,q}_{k,\alpha}(\beta,\delta)(z)$:

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n+\beta)(\delta)_n} z^n, \tag{2.9}$$

the imposition on parameters are $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0,\ \mathcal{R}(\beta) > 0,\ k \in \mathbb{R},\ \delta$ is non-negative real number and nq is a positive integer.

Let $\mathcal{M}$ represents the class of the normalized functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.10}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Since the Mittag-Leffler function in (2.1) does not belong to the class $\mathcal{M}$ therefore, for (M-L) functions to be the member of class $\mathcal{M}$, Srivastava and his co-authors [16] have considered some normalization on the $E_{\alpha,\beta}$ as:

$$\begin{aligned} E_{\alpha,\beta}(z) &= \Gamma(\beta) z E_{\alpha,\beta}(z) \\ &= z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} z^{n+1}, \end{aligned} \tag{2.11}$$

and the above equality can also be written as

$$E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n,$$

where

$$(z, \alpha, \beta \in \mathbb{C}; \mathcal{R}(\alpha) > 0) \quad (\beta \neq 0, -1, -2, ...).$$

The normalized Mittag-Leffler function $E_{\alpha,\beta}$ contains some well-known functions for particular values belonging to real numbers of α, β and $z \in \mathbb{U}$.

- $E_{2,1} = z cosh(\sqrt{z})$,
- $E_{2,2} = \sqrt{z} sinh(\sqrt{z})$,
- $E_{2,3} = 2[cosh(\sqrt{z}) - 1]$,
- $E_{2,4} = \dfrac{6[sinh(\sqrt{z}) - \sqrt{z}]}{\sqrt{z}}$.

Definition 2.2. For a function $f \in \mathcal{M}$ given by (2.1) and $g \in \mathcal{M}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n\, b_n\, z^n, \quad (a_n \geq 0, z \in \mathcal{U}). \tag{2.12}$$

Salah and Darus [15] generalization of Mittag-Leffler functions defined by Srivastava and Tomovski [5] is given by,

$$ {}_mF^{\gamma,q}_{\alpha,\delta}(z) = \sum_{n=0}^{\infty}\prod_{i=1}^{m}\frac{(\gamma_i)_{q_i^n}}{(\delta_i)_{\alpha_i n}\, n!}z^n, \tag{2.13} $$

Remark 2.3. Note that if $m = 1$ then ${}_mF^{\gamma,q}_{\alpha,\delta}(z) = \Gamma(\beta)E^{\gamma,q}_{\alpha,\delta}(z)$

Definition 2.4. (Taylor Series). The Taylor series of a function $f(z)$ about $z = a$ is

$$ f(z) = f(a) + f'(a)(z-a) + ... = \sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(z-a)^n, \tag{2.14} $$

Remark 2.5. The equality between f(z) and its Taylor series is only valid if the series converges.

Remark 2.6. We choose a geometric series $f(z) = \dfrac{1}{1-z} = 1 + z^2 + z^3 + ... = \sum_{n=0}^{\infty} z^n$, which is Taylor series about $z = 0$. Note that the function $f(z) = \dfrac{1}{1-z}$ exists and is infinitely differentiable everywhere except at $z = 1$ while the series exists in the unit disc $|z| < 1$ and hence analytic.

Now we introduce a function $\phi(z) = \dfrac{z}{1-z} = z + z^2 + z^3 + ... = \sum_{n=0}^{\infty} z^{n+1} \in \mathcal{M}$ of the form

$$ \phi(z) = z + \sum_{n=2}^{\infty} z^n \tag{2.15} $$

which is analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 2.7. The generalized Libera integral operator J_c for a function f(z) belonging to the class $\mathcal{A}$, we define the generalized Libera integral operator J_c [20] by

$$ J_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt = z - \sum_{k=n+1}^{\infty}\left(\frac{c+1}{n+1}\right)a_k\, z^k, \quad (c \geq 0) \tag{2.16} $$

$$ J_1(f) = \frac{2}{z}\int_0^z f(t)dt \quad = z - \sum_{k=n+1}^{\infty}\left(\frac{2}{n+1}\right)a_k\, z^k. \tag{2.17} $$

The operator J_c, when $c = \{1, 2, 3, ...\}$ was introduced by Bernardi [19]. In particular, the operator J_1, was studied earlier by Libera [20] and Livingston [21].

Subsequent to our studies of [4] and [16] we impose some normalization over the most generalized Mittag-Leffler function $E^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ which is defined in 2.9.

3 Main Results

Theorem 3.1. *If $f(z)$ belongs to $\mathcal{M}$ such that $z \in \mathbb{U}$ then $\mathcal{M}$ class of function $f(z)$ and convolution of the generalized $\mathcal{K}-$Mittag-Leffler function $GE^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ is*

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} a_n z^n, \tag{3.1}$$

if the following conditions hold

1. $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$ *and* $\mathcal{R}(\gamma) > 0$, $k \in \mathbb{R}$ *and* $q \in (0,1) \cup \mathbb{N}$.
2. $f(z)$ *is analytic in the open unit disk,* $\mathbb{U} = \{z : |z| < 1\}$.
3. $\mathcal{R}(\alpha_i), \mathcal{R}(\beta_i), \mathcal{R}(\gamma_i), \mathcal{R}(\delta_i) > max\{0, \mathcal{R}(q_i) - 1\}$ *and* $q \in (0,1) \cup \mathbb{N}, \mathcal{R}(q_i) > 0, \gamma \neq 0$.
4. $(\gamma)_{nq,k}$ *is generalized pochhammer symbol as defined in* (2.5).

Proof. The Generalized Mittag-Leffler function [18]

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n+\beta)(\delta)_n} z^n,$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$, $k \in \mathbb{R}$, δ is non-negative real number, nq is a positive integer and $q \in (0,1) \cup \mathbb{N}$.

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma k(\beta)} + \frac{(\gamma)_{q,k}}{\Gamma k(\alpha+\beta)\Gamma(\delta)} z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n+\beta)(\delta)_n} z^n. \tag{3.2}$$

The above equation can also be written as

$$\frac{\Gamma k(\alpha+\beta)\Gamma(\delta)}{(\gamma)_{q,k}} \left((E^{\gamma,q}_{k,\alpha,\beta,\delta}) - \frac{(\gamma)_k}{\Gamma k(\beta)} \right) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^n. \tag{3.3}$$

Now we define the function $Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ by

$$Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \frac{\Gamma k(\alpha+\beta)\Gamma(\delta)}{(\gamma)_{q,k}} \left((E^{\gamma,q}_{k,\alpha,\beta,\delta}) - \frac{(\gamma)_k}{\Gamma k(\beta)} \right). \tag{3.4}$$

Then equation (3.4) implies

$$Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = z + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^{n+1}. \tag{3.5}$$

$$Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{(n-1)q,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha(n-1)+\beta)\Gamma(\delta+n-1)} z^n. \tag{3.6}$$

Let $f(z) \in \mathcal{M}$. Denote $L^{\gamma,q}_{k,\alpha\beta,\delta}(f)(z) : \mathcal{M} \longrightarrow \mathcal{M}$ the operator is defined by

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z) = Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) * f(z), \tag{3.7}$$

where the symbol (*) stands for Hadamard product (*or convolution*).

Now applying Hadamard product and using the equations (2.10), (3.6) and (3.7) we obtain the desired result. □

Corollary 3.2. *If we put* $\gamma = q = k = \beta = \delta = 1$ *and* $\alpha = 0$ *in equation* (3.1). *Let the condition of Theorem* (3.1) *be satisfied, then we get the following results,*

$$L^{1,1}_{1,0,1,1}(f)(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z)$$

Corollary 3.3. *If we set* $\gamma = 2$, $q = k = \beta = \delta = 1$ *and* $\alpha = 0$ *in equation* (3.1). *Let the condition of Theorem* (3.1) *be satisfied, then the reduced form is ,*

$$L^{2,1}_{1,0,1,1}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right) a_n z^n = \frac{1}{2}\left\{f(z) + zf'(z)\right\}$$

Corollary 3.4. *If we set out the values of* $\gamma = q = k = \beta = 1$, *and* $\delta = \alpha = 0$ *in equation* (3.1). *Let the condition of Theorem* (3.1) *be satisfied, then the reduced form is ,*

$$L^{1,1}_{1,0,1,0}(f)(z) = z + \sum_{n=2}^{\infty} n a_n z^n = zf'(z)$$

Corollary 3.5. *If we keep* $\gamma = q = k = \beta = 1, \delta = 2$ *and* $\alpha = 0$ *in equation* (3.1). *Let the condition of Theorem* (3.1) *be satisfied, then the reduced form is ,*

$$L^{1,1}_{1,0,1,2}(f)(z) = z + \sum_{n=2}^{\infty} (\frac{2}{n+1}) a_n z^n = \frac{2}{z}\int_0^z f(t)dt.$$

Corollary 3.6. *Note that corollary* (3.5) *is Bernardi type of integral [19]. Particularly, the operator* J_1 *defined in* (2.7), *was studied earlier by Libera [20] and Livingston [21].*

Theorem 3.7. *If* $f(z)$ *belongs to* $\mathcal{M}$ *such that* $z \in \mathbb{U}$ *then* $\mathcal{M}$ *class of function* $\phi(z)$ *and convolution of the generalized* $\mathcal{K}-$*Mittag-Leffler function* $GE^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ *is*

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\alpha+\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^n, \tag{3.8}$$

if the following conditions hold

1. $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$ *and* $\mathcal{R}(\gamma) > 0$, $k \in \mathbb{R}$ *and* $q \in (0,1) \cup \mathbb{N}$.
2. $f(z)$ *is analytic in the open unit disk,* $\mathbb{U} = \{z : |z| < 1\}$.
3. $\mathcal{R}(\alpha_i), \mathcal{R}(\beta_i), \mathcal{R}(\gamma_i), \mathcal{R}(\delta_i) > max\{0, \mathcal{R}(q_i) - 1\}$ *and* $q \in (0,1) \cup \mathbb{N}, \mathcal{R}(q_i) > 0, \gamma \neq 0$.

4. $(\gamma)_{nq,k}$ *is generalized pochhammer symbol as defined in* (2.5)*.*

Proof. The generalized Mittag-Leffler function [18]

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n + \beta)(\delta)_n} z^n,$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \mathcal{R}(\alpha) > 0$, $\mathcal{R}(\beta) > 0$, $k \in \mathbb{R}$, δ is non-negative real number, nq is a positive integer and $q \in (0,1) \cup \mathbb{N}$.

$$E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \frac{(\gamma)_k}{\Gamma k(\beta)} + \frac{(\gamma)_{q,k}}{\Gamma k(\alpha + \beta)(\delta)} z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma k(\alpha n + \beta)(\delta)_n} z^n, \tag{3.9}$$

The above equation can also be written as

$$\frac{\Gamma k(\alpha + \beta)(\delta)}{(\gamma)_{q,k}} \left((E^{\gamma,q}_{k,\alpha,\beta,\delta}) - \frac{(\gamma)_k}{\Gamma k(\beta)} \right) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k} \Gamma k(\alpha + \beta) \Gamma(\delta + 1)}{(\gamma)_{q,k} \Gamma k(\alpha n + \beta) \Gamma(\delta + n)} z^n. \tag{3.10}$$

Now we define the function $Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z)$ by

$$Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \frac{\Gamma k(\alpha + \beta)(\delta)}{(\gamma)_{q,k}} \left((E^{\gamma,q}_{k,\alpha,\beta,\delta}) - \frac{(\gamma)_k}{\Gamma k(\beta)} \right). \tag{3.11}$$

Then equation (3.11) implies

$$Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{(\gamma)_{nq,k} \Gamma k(\alpha + \beta) \Gamma(\delta + 1)}{(\gamma)_{q,k} \Gamma k(\alpha n + \beta) \Gamma(\delta + n)} z^n. \tag{3.12}$$

Let $\phi(z) = (\frac{z}{1-z}) \in \mathcal{M}$. Denote $L^{\gamma,q}_{k,\alpha\beta,\delta}(\phi)(z) : \mathcal{M} \longrightarrow \mathcal{M}$ the operator is defined by

$$L^{\gamma,q}_{k,\alpha,\beta,\delta}(\phi)(z) = Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) * \phi(z), \tag{3.13}$$

where the symbol (*) stands for Hadamard product (*or convolution*).

Now applying Hadamard product and using the equations (2.10), (3.12) and (3.13) we obtained the desired result. ☐

Remark 3.1. Note that, $Q^{\gamma,q,\delta}_{k,\alpha,\beta}(z) = \frac{\Gamma k(\beta)}{(\gamma)_k} z E^{\gamma,q}_{k,\alpha,\beta,\delta}(z), \qquad (z \in \mathbb{U}).$

Corollary 3.8. *If we set the values of* $\gamma = q = k = \beta = \delta = 1$ *and* $\alpha = 1$ *in equation* (3.8)*. Let the condition of Theorem* (3.7) *be satisfied, then the reduced form is ,*

$$L^{1,1}_{1,1,1,1}(\phi)(z) = z + \sum_{n=2}^{\infty} \frac{1}{n!} z^n = e^z - 1$$

Corollary 3.9. *If we keep* $\gamma = q = k = \beta = 1, \delta = 2$ *and* $\alpha = 0$ *in equation* (3.8)*. Let the condition of Theorem* (3.7) *be satisfied, then the reduced form is ,*

$$L^{1,1}_{1,0,1,2}(\phi)(z) = z + \sum_{n=2}^{\infty} (\frac{2}{n+1}) z^n = -\frac{2log(1-z)}{z} - 2$$

Corollary 3.10. *If we put $\gamma = k = \beta = 1$ and $q = 0$, $\delta = \alpha = 0$ in equation* (3.8). *Let the condition of Theorem* (3.7) *be satisfied, then the reduced form is ,*

$$L^{1,0}_{1,0,1,0}(\phi)(z) = z + \sum_{n=2}^{\infty} \frac{1}{\Gamma(n)} z^n = ze^z$$

Corollary 3.11. *If we put $\gamma = k = q = \beta = 1$ and, $\delta = \alpha = 0$ in equation* (3.8). *Let the condition of Theorem* (3.7) *be satisfied, then the reduced form is ,*

$$L^{1,1}_{1,0,1,0}(\phi)(z) = z + \sum_{n=2}^{\infty} nz^n = z - \frac{(z-2)}{(z-1)^2} z^2$$

Motivated by the work of Bansal and Prajapat [7] and also following the results of Raducanu [6], we investigate the ratio of $\mathcal{K}-$Mittaq-Leffler function $Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ defined by (3.5) to its sequence of partial sums

$$\begin{cases} (Q^{\gamma,q}_{k,\alpha,\beta,\delta})_0(z) = z \\ (Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z) = z + \sum_{n=1}^{m} P_n z^{n+1}, \quad m \in \mathbb{N}, \end{cases}$$

where

$$P_n = \frac{(\gamma)_{nq,k} \Gamma k(\beta) \Gamma(\delta+1)}{(\gamma)_{q,k} \Gamma k(\alpha n + \beta) \Gamma(\delta + n)}, \quad (\alpha, \beta, \delta > 0).$$

The lower bounds we obtained on ratio like

$$\mathcal{R}\left\{\frac{Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}\right\}, \mathcal{R}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}{Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)}\right\}, \mathcal{R}\left\{\frac{Q'^{\gamma,q}_{k,\alpha,\beta,\delta}(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})'_m(z)}\right\}, \mathcal{R}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})'_m(z)}{(Q'^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}\right\}.$$

4 Ratio of normalized $\mathcal{K}-$ Mittag-Leffler function

In order to verify our results we require the following Lemma.

Lemma 4.1. *Let $\alpha \geq 1$, $\gamma \geq 1$, and $\delta \geq 1$. Then the function $Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)$ satisfies the two inequalities enlisted below;*

$$\left|Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)\right| \leq \frac{\beta^2\delta + \beta^2 + 2\beta\delta - \beta\gamma + \beta + \delta + 1}{\beta^2\delta + \beta^2 + \beta\delta - \beta\gamma}, \quad z \in \mathbb{U} \tag{4.1}$$

$$\left|Q'^{\gamma,q}_{k,\alpha,\beta,\delta}(z)\right| \leq \frac{(\beta^2\delta + \beta^2 - \beta\gamma + 4\beta\delta + 3\beta - \gamma + 3\delta + 2)}{\beta^2\delta + \beta^2 + \beta\delta - \beta\gamma}, \quad z \in \mathbb{U} \tag{4.2}$$

Proof. By using hypothesis that, $\Gamma(\alpha n + \beta) \leq \Gamma k(\alpha n + \beta)$ and so we can write,

$$\begin{aligned} \frac{(\gamma)_{nq,k} \Gamma k(\beta) \Gamma(\delta+1)}{(\gamma)_{q,k} \Gamma k(\alpha n + \beta) \Gamma(\delta + n)} &\leq \frac{(\gamma)_{nq,k} \Gamma k(\beta) \Gamma(\delta+1)}{(\gamma)_{q,k} \Gamma k(n + \beta) \Gamma(\delta + n)} \\ &= \frac{(\gamma)_n(\delta)}{(\gamma)(\beta)_n(\delta)_n}, \qquad n \in \mathbb{N}, \end{aligned} \tag{4.3}$$

therefore, for $z \in \mathbb{U}$ picking equation (3.5) using (4.3) and a note followed by equation (2.3) we have,

(4.4)

$$\begin{aligned}
\left|Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^{n+1}\right| \\
&\leq 1 + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} \leq 1 + \sum_{n=1}^{\infty} \frac{(\gamma)_n(\delta)}{(\gamma)(\beta)_n(\delta)_n} \\
&\leq 1 + \frac{\delta}{\gamma}\sum_{n=1}^{\infty} \frac{(\gamma)_n}{(\beta)_n(\delta)_n} \\
&= 1 + \frac{1}{\beta}\sum_{n=1}^{\infty} \frac{(\gamma+1)_{n-1}}{(\beta+1)_{n-1}(\delta+1)_{n-1}} \\
&\leq 1 + \frac{1}{\beta}\sum_{n=1}^{\infty} \frac{(\gamma+1)^{n-1}}{(\beta+1)^{n-1}(\delta+1)^{n-1}} \\
&= 1 + \frac{1}{\beta}\sum_{n=0}^{\infty} \left\{\frac{(\gamma+1)}{(\beta+1)(\delta+1)}\right\}^n, \qquad for \quad \left|\frac{(\gamma+1)}{(\beta+1)(\delta+1)}\right| < 1 \\
&\leq 1 + \frac{1}{\beta}\left(\frac{(\beta+1)(\delta+1)}{\beta\delta+\beta+\delta-\gamma+2}\right) \\
&\leq \frac{\beta^2\delta+\beta^2+2\beta\delta-\beta\gamma+\beta+\delta+1}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}
\end{aligned}$$

and hence the inequality in (4.1) is proved.

Using the derivative of equation (3.5) and once more the triangle inequality, for $z \in \mathbb{U}$, we obtain

$$\left|Q'^{\gamma,q}_{k,\alpha,\beta,\delta}(z)\right| = \left|1 + \sum_{n=1}^{\infty} \frac{(n+1)(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} z^n\right|$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{n(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\gamma+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)} + \sum_{n=1}^{\infty} \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\gamma+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)}. \tag{4.5}$$

For $\beta \geq 1$, $\delta \geq 1$ and $\gamma \geq 1$ we obtain

$$\begin{aligned}
\frac{n(\gamma)_n\Gamma(\beta)(\delta)}{(\gamma)\Gamma(n+\beta)(\delta)_n} &= \frac{n(\gamma)_n(\delta)}{\gamma(\beta)_n(\delta)_n} \\
&= \frac{n(\gamma+1)_{n-1}}{\beta(\beta+1)_{n-1}(\delta+1)_{n-1}} \\
&= \frac{n(\gamma+1)_{n-2}(\gamma+n-1)}{\beta(\beta+1)_{n-2}(\beta+n-1)(\delta+1)_{n-2}(\delta+n-1)} \\
&\leq \frac{(\gamma+1)_{n-2}}{\beta(\beta+1)_{n-2}(\delta+1)_{n-2}}. \qquad (4.6)
\end{aligned}$$

Taking into account, inequalities (4.6), (4.7) and a note followed by equation (2.3) we have,

$$\begin{aligned}
\left|Q'^{\gamma,q}_{k,\alpha,\beta,\delta}(z)\right| &= \left|1+\sum_{n=1}^{\infty}\frac{n(\gamma)_n\Gamma(\beta)\Gamma(\delta+1)}{(\gamma)\Gamma(n+\beta)\Gamma(\delta+n)}+\sum_{n=1}^{\infty}\frac{(\gamma)_n\Gamma(\beta)\Gamma(\delta+1)}{(\gamma)\Gamma(n+\beta)\Gamma(\delta+n)}\right|\\
&\le 1+\sum_{n=1}^{\infty}\frac{n(\gamma)_n(\delta)}{\gamma(\beta)_n(\delta)_n}+\sum_{n=1}^{\infty}\frac{(\gamma)_n(\delta)}{\gamma(\beta)_n(\delta)_n}\\
&\le 1+\frac{1}{\beta}+\frac{1}{\beta}\sum_{n=2}^{\infty}\frac{(\gamma+1)_{n-2}}{(\beta+1)_{n-2}(\delta+1)_{n-2}}+\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(\gamma+1)_{n-1}}{(\beta+1)_{n-1}(\delta+1)_{n-1}}\\
&\le 1+\frac{1}{\beta}+\frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{(\gamma+1)}{(\beta+1)(\delta+1)}\right)^n+\frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{(\gamma+1)}{(\beta+1)(\delta+1)}\right)^n\\
&\le 1+\frac{1}{\beta}+\frac{2}{\beta}\left(\frac{(\beta+1)(\delta+1)}{\delta\beta+\beta+\delta+\gamma+2}\right)\\
&\le \frac{(\beta^2\delta+\beta^2-\beta\gamma+4\beta\delta+3\beta-\gamma+3\delta+2)}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma} \qquad (4.7)
\end{aligned}$$

and hence the inequality in (4.2) is proved. □

In the sequel we involve a well-known result $\mathcal{R}\left(\frac{1+u(z)}{1-u(z)}\right)>0, z\in\mathbb{U}$ if and only if $|u(z)|<1$ and $u(z)$ be analytic function in $\mathbb{U}$. We noted in the paper of Silverman [9, page 223, theorem.1] to set

$$\frac{(1+A(z))}{(1+B(z))}=\frac{(1+u(z))}{(1-u(z))},$$

so that, $u(z)=\dfrac{(A(z)-B(z))}{(2+A(z)+B(z))}$.

Theorem 4.2. *Let* $\{\alpha,q,k,\delta\}\ge 1$ *and* $\beta\ge\dfrac{(\gamma+1)+\sqrt{(-\gamma-1)^2+4(\delta+1)^2}}{2(\delta+1)}$. *Then*

$$\mathcal{R}\left\{\frac{Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}\right\}\ge\frac{\beta^2\delta+\beta^2-\beta\gamma-\beta-\delta-1}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma} \qquad (4.8)$$

and

$$\mathcal{R}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}{Q^{\gamma,q}_{k,\alpha,\beta,\delta}(z)}\right\}\ge\frac{\beta^2\delta+\beta^2-\beta\gamma+\beta\delta}{\beta^2\delta+\beta^2-\beta\gamma+2\beta\delta+\beta+\delta+1} \qquad (4.9)$$

Proof. Using equation (4.1) of Lemma (4.1) we can write

$$1+\sum_{n=1}^{\infty}P_n\le\frac{\beta^2\delta+\beta^2+2\beta\delta-\beta\gamma+\beta+\delta+1}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma},$$

where

$$P_n=\frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)},\quad(\alpha,\beta,\delta>0).$$

The above inequality is equivalent to

$$\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{\beta\delta+\beta+\delta+1}\sum_{n=1}^{\infty}P_n \leq 1.$$

In order to prove our result inequality (4.8), we consider $u(z)$ defined by

$$\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{\beta\delta+\beta+\delta+1}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}\right\}-\frac{\beta^2\delta+\beta^2-\beta\gamma-\beta-\delta-1}{\beta\delta+\beta+\delta+1}=\frac{(1+u(z))}{(1-u(z))}$$

For the sake of simplicity let

$$\psi(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{\beta\delta+\beta+\delta+1}$$

and

$$\xi(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2-\beta\gamma-\beta-\delta-1}{\beta\delta+\beta+\delta+1}$$

$$\frac{(1+u(z))}{(1-u(z))}=\frac{1+\sum_{n=1}^{m}P_nz^n+\psi(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}{1+\sum_{n=1}^{m}P_nz^n,} \tag{4.10}$$

where the value of,

$$u(z)=\frac{\psi(\beta,\delta,\gamma)\sum_{n=1}^{\infty}P_n}{2+2\sum_{n=1}^{m}P_nz^n+\psi(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}$$

and

$$|u(z)|<\frac{\psi(\beta,\delta,\gamma)\sum_{n=1}^{\infty}P_n}{2-2\sum_{n=1}^{m}P_n-\psi(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}.$$

The inequality $|u(z)|<1$ holds if and only if

$$2\psi(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n\leq 2-2\sum_{n=1}^{m}P_n,$$

which is equivalent to

$$\sum_{n=1}^{m}P_n+\psi(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n\leq 1. \tag{4.11}$$

To verify (4.11) it is sufficient to show that its left-hand side is bounded above by

$$\psi(\beta,\delta,\gamma)\sum_{n=1}^{\infty}P_n$$

which is equivalent to

$$\xi(\beta,\delta,\gamma)\sum_{n=1}^{m}P_n \geq 0.$$

The last inequality holds true for $\beta \geq \dfrac{(\gamma+1)+\sqrt{(-\gamma-1)^2+4(\delta+1)^2}}{2(\delta+1)}; \quad \delta \neq -1.$

In a similar method we prove the second inequality (4.9) of our theorem. Consider the function $u(z)$ given by

$$\frac{\beta^2\delta+\beta^2+2\beta\delta-\beta\gamma+\beta+\delta+1}{\beta\delta+\beta+\delta+1}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})(z)}\right\}-\frac{\beta^2\delta+\beta^2-\beta\gamma+\beta\delta}{\beta\delta+\beta+\delta+1}$$

$$=\frac{(1+u(z))}{(1-u(z))}$$

Again for the sake of simplicity we put,

$$\Theta(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2+2\beta\delta-\beta\gamma+\beta+\delta+1}{\beta\delta+\beta+\delta+1},$$

and

$$\Delta(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2-\beta\gamma+\beta\delta}{\beta\delta+\beta+\delta+1}$$

$$\frac{(1+u(z))}{(1-u(z))}=\frac{1+\sum_{n=1}^{m}P_nz^n+\Theta(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}{1+\sum_{n=1}^{m}P_nz^n}, \tag{4.12}$$

where the value of,

$$u(z)=\frac{-\Theta(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}{2+2\sum_{n=1}^{m}P_nz^n-\Theta(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}$$

and

$$|u(z)|<\frac{\Theta(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}{2-2\sum_{n=1}^{m}P_n-\Theta(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}.$$

The inequality $|u(z)| < 1$ holds if and only if

$$\sum_{n=1}^{m} P_n + \Theta(\beta,\delta,\gamma) \sum_{n=m+1}^{\infty} P_n \leq 1 \tag{4.13}$$

Since the left-hand side of (4.13) is bounded above by,

$$\Theta(\beta,\delta,\gamma) \sum_{n=m+1}^{\infty} P_n,$$

which is equivalent to

$$\Delta(\beta,\delta,\gamma) \sum_{n=1}^{m} P_n \geq 0. \tag{4.14}$$

And hence the inequality (4.9) holds true for $\beta \geq \dfrac{(\gamma+1)+\sqrt{(-\gamma-1)^2+4(\delta+1)^2}}{2(\delta+1)}$; if $\{\alpha, q, k, \delta\} \geq 1$. □

Corollary 4.3. *If we put $\gamma \geq 1$, $\delta \geq 1$ and $\beta \geq \dfrac{1+\sqrt{5}}{2}$, then the inequalities in (4.8) and (4.9) holds true which is in fact the result given by Raducanu [6, page 3, theorem.2.1].*

Theorem 4.4. *Let $\alpha \geq 1$, $q \geq 1$, $k \geq 1$, and $\beta \geq \mathcal{H}(\gamma,\delta)$ where*

$$\mathcal{H}(\gamma,\delta) = \frac{(\gamma+2\delta+3)+\sqrt{(-(\gamma+2\delta+3))^2-4(\delta+1)(\gamma-3\delta-2)}}{2\delta+2}; \qquad \delta \neq -1,$$

Then

$$\mathcal{R}\left\{\frac{Q'^{,\gamma,q}_{k,\alpha,\beta,\delta}(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})'_m(z)}\right\} \geq \frac{\beta^2\delta+\beta^2-\beta\gamma-2\beta\delta-3\beta+\gamma-3\delta-2}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma} \tag{4.15}$$

and

$$\mathcal{R}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})'_m(z)}{Q'^{,\gamma,q}_{k,\alpha,\beta,\delta}(z)}\right\} \geq \frac{\beta^2\delta+\beta^2-\beta\gamma+\beta\delta}{\beta^2\delta+\beta^2-\beta\gamma+4\beta\delta+3\beta-\gamma+3\delta+2} \tag{4.16}$$

Proof. From (4.2) and making the use of (4.5) we can write

$$1+\sum_{n=1}^{\infty} P_n \leq \frac{(\beta^2\delta+\beta^2-\beta\gamma+4\beta\delta+3\beta-\gamma+3\delta+2)}{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma},$$

where

$$P_n = \frac{(\gamma)_{nq,k}\Gamma k(\beta)\Gamma(\delta+1)}{(\gamma)_{q,k}\Gamma k(\alpha n+\beta)\Gamma(\delta+n)}, \quad (\alpha,\beta,\delta > 0).$$

The above inequality is equivalent to

$$\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{3\beta\delta+3\beta+3\delta-\gamma+2}\sum_{n=1}^{\infty} P_n \leq 1$$

In order to prove our result inequality (4.15), define a function $u(z)$ by

$$\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{3\beta\delta+3\beta+3\delta-\gamma+2}\left\{\frac{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})_m(z)}\right\}-\frac{\beta^2\delta+\beta^2-\beta\gamma-2\beta\delta-3\beta+\gamma-3\delta-2}{3\beta\delta+3\beta+3\delta-\gamma+2}$$

$$=\frac{(1+u(z))}{(1-u(z))}$$

For the sake of convenience suppose

$$\Lambda(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2+\beta\delta-\beta\gamma}{3\beta\delta+3\beta+3\delta-\gamma+2}$$

and

$$\Upsilon(\beta,\delta,\gamma)=\frac{\beta^2\delta+\beta^2-\beta\gamma-2\beta\delta-3\beta+\gamma-3\delta-2}{3\beta\delta+3\beta+3\delta-\gamma+2}$$

which produces

$$\frac{(1+u(z))}{(1-u(z))}=\frac{1+\sum_{n=1}^{m}P_nz^n+\Lambda(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}{1+\sum_{n=1}^{m}P_nz^n}, \tag{4.17}$$

where the value of,

$$u(z)=\frac{\Lambda(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}{2+2\sum_{n=1}^{m}P_nz^n+\Lambda(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}$$

and

$$u(z)=\frac{\Lambda(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n}{2-2\sum_{n=1}^{m}P_nz^n-\Lambda(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_nz^n}$$

the provision $|u(z)|<1$ is valid if and only if

$$\sum_{n=1}^{m}(n+1)P_n+\Lambda(\alpha,\beta)\sum_{n=m+1}^{\infty}(n+1)P_n\leq 1. \tag{4.18}$$

The left-hand side of the (4.18) is bounded above by

$$\Lambda(\beta,\delta,\gamma)\sum_{n=1}^{\infty}(n+1)P_n,$$

if

$$\Upsilon(\beta,\delta,\gamma)\sum_{n=1}^{m}(n+1)P_n \geq 0$$

which holds true for

$$\beta \geq \frac{(\gamma+2\delta+3)+\sqrt{(-(\gamma+2\delta+3))^2-4(\delta+1)(\gamma-3\delta-2)}}{2\delta+2}; \qquad \delta \neq -1.$$

The proof of (4.16) follows the same pattern. For this purpose consider the function $u(z)$ given by

$$\Omega(\beta,\delta,\gamma)\left\{\frac{Q'^{,\gamma,q}_{k,\alpha,\beta,\delta}(z)}{(Q^{\gamma,q}_{k,\alpha,\beta,\delta})'_m(z)}\right\} - \Phi(\beta,\delta,\gamma) = \frac{(1+u(z))}{(1-u(z))}$$

where,

$$\Omega(\beta,\delta\gamma) = \frac{\beta^2\delta+\beta^2-\beta\gamma+4\beta\delta+3\beta-\gamma+3\delta+2}{3\beta\delta+3\beta-\gamma-3\delta+2}$$

and

$$\Phi(\beta,\delta,\gamma) = \frac{\beta^2\delta+\beta^2-\beta\gamma+\beta\delta}{3\beta\delta+3\beta-\gamma-3\delta+2}$$

where the value of,

$$u(z) = \frac{-\Omega(\beta,\delta,\gamma)\displaystyle\sum_{n=m+1}^{\infty}P_n}{2+2\displaystyle\sum_{n=1}^{m}P_n z^n-\Omega(\beta,\delta,\gamma)\displaystyle\sum_{n=m+1}^{\infty}P_n z^n}$$

and

$$|u(z)| < \frac{\Omega(\beta,\delta,\gamma)\displaystyle\sum_{n=m+1}^{\infty}P_n}{2-2\displaystyle\sum_{n=1}^{m}P_n-\Omega(\beta,\delta,\gamma)\displaystyle\sum_{n=m+1}^{\infty}P_n}.$$

The inequality $|u(z)|<1$ holds if and only if

$$\sum_{n=1}^{m}P_n+\Omega(\beta,\delta,\gamma)\sum_{n=m+1}^{\infty}P_n \leq 1. \tag{4.19}$$

Since the left-hand side of (4.19) is bounded above by, $\Omega(\beta,\delta,\gamma)\displaystyle\sum_{n=m+1}^{\infty}P_n$, which is equivalent to

$$\Phi(\beta,\delta,\gamma)\sum_{n=1}^{m}P_n \geq 0,$$

and hence the inequality (4.16) holds true for

$$\beta \geq \frac{(\gamma+2\delta+3)+\sqrt{(-(\gamma+2\delta+3))^2-4(\delta+1)(\gamma-3\delta-2)}}{2\delta+2}; \quad 2\delta+2 \neq 0$$

so, the last condition completes the proof. □

Corollary 4.5. *If we put* $\gamma \geq 1$, $\delta \geq 1$ *and* $\beta \geq \frac{3+\sqrt{17}}{2}$, *then the inequalities in* (4.15) *and* (4.16) *holds true which is in fact the result given by Raducanu [6, page 5, theorem.2.2].*

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