Duality theory for the vector rational multi-time variational problem on manifolds based on (ρ, b)-geodesic quasiinvexity

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Dedicated to Prof. Dr. Constantin Udrişte on the occasion of his 75-th birthday

Abstract. We consider a multi-time scalar variational problem (SVP), a multi-time vector (or multiobjective) variational problem (VVP) and a multi-time vector fractional variational problem (VFP). For (SVP) we present necessary optimality conditions and we define the notion of normal optimal solution. For the two vector variational problems we define the notions of efficient (that is, Pareto) solution and of normal efficient solution, and using these two notions we also present necessary efficiency conditions for (VVP) and (VFP). Moreover, in the following, we developed a duality of Mond-Weir-Zalmai type for the fractional problem (VFP) through weak, direct and converse duality theorems. We also give a generalization to this duality. While presenting this duality, we use the notion of (ρ, b)-geodesic quasiinvexity, which we define for the variational problems on Riemannian manifolds.

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1 Introduction

Beginning with Valentine [16] in 1937, during the years, the variational problems with constraints had different developing steps (see Geoffrion[3], Preda[12]). In 2007 Mititelu and Stancu-Minasian [8] considered the following vector (or multiobjective)

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fractional variational problem:

$$(MSP): \begin{cases} \text{Maximize} & \left(\frac{\int_a^b f_1(t,x,\dot{x})dt}{\int_a^b k_1(t,x,\dot{x})dt}, \dots, \frac{\int_a^b f_p(t,x,\dot{x})dt}{\int_a^b k_p(t,x,\dot{x})dt}\right) \\ \text{subject to:} & \begin{cases} x(a) = a_0,\ x(b) = b_0, \\ g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0,\ \forall t \in I, \end{cases} \end{cases}$$

where $I = [a,b] \subset \mathbb{R}$, $x = (x_1, \dots, x_n) : I \to \mathbb{R}^n$ is a piecewise smooth function having the derivative denoted by $\dot{x} = \left(\frac{dx}{dt}\right) = \left(\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right)$, and the following mappings are of class C^2

$$f_i, k_i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ (i = \overline{1,p}),\ g : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m,\ h : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q.$$

Mititelu [5] and Mititelu & Stancu-Minasian [8] established for (MSP) necessary efficiency conditions (of Pareto minimum type) and using generalized quasiinvex functions, developed a duality theory through weak, direct and converse duality theorems.

Recently, in 2006, Udrişte [13] studied a variational problem in the multi-dimensional (multi-time) framework, establishing optimality conditions – a multi-time principle. In 2009 Pitea, Udrişte and Mititelu [11] considered the multi-time vector variant of the problem (MSP) in geometrical language, using curvilinear integrals, established necessary optimality conditions and developed a duality theory for this problem. In 2008 Mititelu [5] derived for (VFP) efficiency conditions which were further extended in 2011 by Mititelu and Postolache [7, 8] for the multi-time problems (SVP), (VVP) and (VFP); they defined on a measurable set Ω, necessary optimality, respectively efficiency conditions, and for (VFP), a duality of Mond-Weir-Zalmai type, using (ρ, b)-quasiinvex functions.

In this paper we determine for (VFP) (in Section 4) a duality of Mond-Weir-Zalmai type through weak, direct and converse duality theorems, using (ρ, b)-geodesic quasiinvex functions, defined in this paper. Also in Section 5 we define a generalized Mond-Weir-Zalmai duality for (VFP), all in a geometrical language (as in [11]), but using multiple integrals).

Let (T, h) and (M, g) be two Riemannian manifolds of dimensions m and n, respectively, where M is assumed to be complete. Denote by $t = (t^1, \dots, t^m) = (t^\alpha)$ the elements of a measurable set Ω in T and let be $x = (x^1, \dots, x^n) = (x^k) \in \mathbb{R}^n$ the elements of M. Let $J^1(T, M) = \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm}$ be the first order jet bundle associated to T and M, and consider the functions $x : \Omega \to M, X : J^1(T, M) \to \mathbb{R}$ and

$$f = (f_r)(f_1, \dots, f_p) : J^1(T, M) \to \mathbb{R}^p, \qquad k = (k_r) = (k_1, \dots, k_p) : J^1(T, M) \to \mathbb{R}^p,$$
$$g = (g_\alpha) = (g_1, \dots, g_m) : J^1(T, M) \to \mathbb{R}^m, \quad h = (h_\beta) = (h_1, \dots, h_q) : J^1(T, M) \to \mathbb{R}^q,$$

where $m,\ q \in \mathbb{N}^*, i = \overline{1,n}$, $\alpha = \overline{1,m}$ and $\beta = \overline{1,q}.$

The argument of X, f, k, g, h is $j^1x = (t, x, x_\nu)$, the first jet prolongation of x which, computed at $t \in \Omega$ is the pullback $j^1_t x = (j^1 x)(t)$. We denote as well

$$x = x(t) = (x^1(t), \dots, x^n(t)) = (x^k(t)),\ \ t \in \Omega\ ,\ (k = \overline{1,n})$$
$$x_\nu = x_\nu(t) = (x^1_\nu(t), \dots, x^n_\nu(t)) = \left(\frac{\partial x^k}{\partial t^\nu}(t)\right)\ ,\ (k = \overline{1,n}\ ,\ \nu = \overline{1,m}).$$

We suppose that $x, X, f, k, g_\alpha, h_\beta \in C^2(\Omega)$, and on $C^s(\Omega)$ $(s \geq 1)$ we use the norm $||x|| = ||x||_\infty + \sum\limits_{k=1}^{n} \left\|x_\nu^k\right\|_\infty$, and denote

$$\Phi = \{x : \Omega \to M \mid x(\,\cdot\,) \text{ is normed and piecewise } C^1 \text{ on } \Omega\},$$

where by $x(\,\cdot\,)$ we denote the mapping $x : \Omega \to M$.

For any $v = (v_1, \ldots,\ v_n),\ w = (w_1, \ldots,\ w_n) \in \mathbb{R}^n$, we define the following relations between n-tuples: $v = w,\ v < w,\ v \leqq w$ and $v \leq w$:

$$v = w \ \Leftrightarrow\ v_i = w_i,\ \ i = \overline{1,\ n}; \qquad v < w \ \Leftrightarrow\ v_i < w_i,\ \ i = \overline{1,\ n}$$
$$v \leqq w \ \Leftrightarrow\ v_i \leqq w_i,\ \ i = \overline{1,\ n}; \qquad v \ \leq\ w \ \Leftrightarrow\ v \leqq w \text{ and } v \neq w.$$

In this paper we present necessary efficiency conditions and develop a duality theory for the next multi-time vector fractional variational problem:

$$(VFP): \begin{cases} \text{Maximize Pareto} & \left(\frac{\int_\Omega f_1(j_t^1 x)dv}{\int_\Omega k_1(j_t^1 x)dv}, \ldots, \frac{\int_\Omega f_p(j_t^1 x)dv}{\int_\Omega k_p(j_t^1 x)dv} \right) \\ \text{subject to:} & \begin{cases} g_\alpha(j_t^1 x) \leqq 0, \ \ \alpha = \overline{1, m}, \\ h_\beta(j_t^1 x) \ = \ 0, \ \ \beta = \overline{1, q}, \\ x(t)\,|_{\partial\Omega} \ = u(t) \text{ (given)},\ \forall t \in \Omega, \end{cases} \end{cases}$$

where $dv = dt_1 dt_2 \ldots dt_n$. First, we provide necessary optimality conditions for a multi-time scalar variational problem (SVP) and necessary efficiency conditions for a multi-time vector variational problem (VVP), both having the same constraints as (VFP).

2 Necessary optimality conditions for multiple scalar variational problem (SVP)

The first studied model in this paper is the following scalar multi-time variational problem

$$(SVP) \begin{cases} \text{Minimize} & I[x] = \int_\Omega X(j_t^1 x)dv \\ \text{subject to:} & \begin{cases} g_\alpha(j_t^1 x) \leqq 0, h_\beta(j_t^1 x) = 0, \\ x(\,\cdot\,),\ x(t)|_{\partial\Omega} = u(t),\ t \in \Omega. \end{cases} \end{cases}$$

The set of all feasible solutions for this problem (the domain of (SVP)) is defined by

$$\mathbb{D} = \{x(\,\cdot\,) \in \Phi \mid g_\alpha(t, x(t), x_\nu(t)) \leqq 0,\ h_\beta(j_t^1 x) = 0,\ x(t)|_{\partial\Omega} = u(t), \forall t \in \Omega\}.$$

For the scalar multi-time variational problem (SVP), the following result holds true:

Theorem 2.1 (Necessary optimality to (SVP))**.** $[7, Th.2.1]$ *Let $x(\,\cdot\,) \in \mathbb{D}$ be an optimal solution to the (SVP) problem. Then there exist: a scalar $\tau \in \mathbb{R}$ and the* *piecewise smooth functions* $\lambda(t) = (\lambda_i^\alpha(t)) \in \mathbb{R}^{nm}$ *and* $\mu(t) = (\mu^\beta(t)) \in \mathbb{R}^q$, *which satisfy the following conditions:*

$$(SFJ): \begin{cases} \tau \frac{\partial X}{\partial x^k} + \lambda_i^\alpha(t)\frac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k} - D_\nu\left(\tau \frac{\partial X}{\partial x_\nu^k} + \lambda_i^\alpha(t)\frac{\partial g_\alpha}{\partial x_\nu^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x_\nu^k}\right) = 0, \\ \mu^\beta(t) Y_\beta(j_t^1 x) = 0,\ \beta = \overline{1,q}, \\ \tau \geqq 0, \quad (\mu^\beta(t)) \geqq 0,\ t \in \Omega, \end{cases}$$

where $D_\nu(\,\cdot\,) = \frac{d}{dt^\nu}(\cdot)$, $\frac{\partial X}{\partial x^k} := \frac{\partial X}{\partial x^k}(j_t^1 x)$, $\frac{\partial X}{\partial x_\nu^k} = \frac{\partial X}{\partial x_\nu^k}(j_t^1 x)$, *etc.*

Definition 2.1. [4] A function $x^*(\,\cdot\,) \in \mathbb{D}$ is called *normal optimal solution* to (SVP) if $\tau \neq 0$.

3 Necessary efficiency conditions for multiple vector variational problems (VVP) and (VFP)

3.1 Necessary efficiency conditions for (VVP)

In the framework of the problem (SVP) we consider the vector functional

$$I[x(\,\cdot\,)] = \int_\Omega f(j_t^1 x) dv,$$

which, for $I[x(\,\cdot\,)] = (I_1[x(\,\cdot\,)], \dots, I_p[x(\,\cdot\,)])$, can be written on components,

$$(I_1[x(\,\cdot\,)], \dots, I_p[x(\,\cdot\,)]) = \left(\int_\Omega f_1(j_t^1 x) dv, \dots, \int_\Omega f_p(j_t^1 x) dv\right),$$

Consider now the multi-time vector variational problem

$$(VVP): \begin{cases} \text{Minimize Pareto} \quad I[x(\,\cdot\,)] = \int_\Omega f(j_t^1 x) dv \\ \text{subject to:} \qquad \begin{cases} g_\alpha(j_t^1 x) \ = \ 0, \ \ h_\beta(j_t^1 x) \leqq 0, \\ x(\,\cdot\,)\,|_{\partial\Omega} = u(t),\ \forall t \in \Omega\ . \end{cases} \end{cases}$$

In this section we establish necessary efficiency conditions for (VVP). We shall consider in the following that the domain of (VVP) is $\mathbb{D}$, as well.

Definition 3.1. [2] A point $x^*(\,\cdot\,) \in \mathbb{D}$ is said to be an *efficiency solution* (Pareto minimum) to (VVP), if there exists no $x(\,\cdot\,) \in \mathbb{D}$, such that $I[x(\,\cdot\,)] \leq I[x^*(\,\cdot\,)]$.

Theorem 3.1 (Necessary efficiency for (VVP)). $[7, Th.3.1]$ *Consider the vector multi-time variational problem (VVP) and let* $x(\,\cdot\,) \in \mathbb{D}$ *be an efficiency solution to (VVP). Then there exist the vector functions* $\tau \in \mathbb{R}^p$, $\lambda(t) \in \mathbb{R}^m$ *and* $\mu(t) \in \mathbb{R}^q$, *all assumed piecewise smooth functions, which satisfy the conditions*

$$(VFJ): \begin{cases} \tau^r \frac{\partial f_r}{\partial x^k} + \lambda_i^\alpha(t)\frac{\partial g_\alpha^i}{\partial x^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k} - D_\nu\left(\tau^r \frac{\partial f_r}{\partial x_\nu^k} + \lambda_i^\alpha(t)\frac{\partial g_\alpha}{\partial x_\nu^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x_\nu^k}\right) = 0, \\ \lambda^\alpha(t) g_\alpha(j_t^1 x) = 0, \ \ \alpha = \overline{1,m}, \ \ \tau \geqq 0, \quad (\lambda^\beta(t)) \geqq 0,\ t \in \Omega, \end{cases}$$

where $D_\nu(\cdot) = \frac{d}{dt^\nu}(\,\cdot\,)$, $\frac{\partial f}{\partial x^k} = \frac{\partial f}{\partial x^k}(j_t^1 x)$ *etc.*

Definition 3.2. [4] The mapping $x^0(\,\cdot\,) \in \mathbb{D}$ is a *normal efficient solution* to (VVP), if the conditions (VFJ) are fulfilled, with $\tau \ \geq \ 0$, $e'\tau = 1$, $e = (1, \dots, 1) \in \mathbb{R}^p$.

3.2 Necessary efficiency conditions for multi-time multiple vector fractional variational problem (VFP)

In this section we recall some definitions and auxiliary results that will be needed later in our discussion about efficiency conditions for (VFP) defined by the following multi-time vector fractional variational problem

$$(VFP): \begin{cases} \text{Minimize} \quad J[x(\cdot)] = \left(\frac{\int_\Omega f_1(j_t^1 x)\,\mathrm{d}v}{\int_\Omega k_1(j_t^1 x)\,\mathrm{d}v}, \dots, \frac{\int_\Omega f_p(j_t^1 x)\,\mathrm{d}v}{\int_\Omega k_p(j_t^1 x)\,\mathrm{d}v} \right) \\ \text{subject to:} \quad \begin{cases} g_\alpha(j_t^1 x) \leqq 0, \ \ h_\beta(j_t^1 x) \ = \ 0, \\ x(\cdot) \in \Phi, \ x(t)|_{\partial\Omega} = u(t), \ \forall t \in \Omega\,. \end{cases} \end{cases}$$

Assume that $\int_\Omega k_r(j_t^1 x)\ \mathrm{dv} \neq 0$ for all $r = \overline{1,p}$. The domain of (VFP) is $\mathbb{D}$, as well.

Definition 3.3. [2] A feasible solution $x^0(\cdot) \in \mathbb{D}$ is said to be an *efficient solution* of (VFP) if there is no $x(\cdot) \in \mathbb{D}$, $x(\cdot) \neq x^0(\cdot)$ such that $J[x(\cdot)] \leq J[x^0(\cdot)]$.

We present now the efficiency necessary conditions for (VFP). Let $x^0(t)$ be an efficient solution to (FVP). Consider the problem

$$(FP)_r(x^0): \begin{cases} \underset{x}{\text{Minimize}} \quad \frac{\int_\Omega f_r(j_t^1 x) dv}{\int_\Omega k_r(j_t^1 x) dv} \\ \text{subject to:} \quad \begin{cases} x(\cdot) \in \Phi, \ x(t)\,|_{\partial\Omega} = u(t), \ \forall t \in \Omega\ , \\ g_\alpha(j_t^1 x) \leqq 0, \quad h_\beta(j_t^1 x) \leqq 0, \\ \frac{\int_\Omega f_j(j_t^1 x) dv}{\int_\Omega k_j(j_t^1 x) dv} \leqq \frac{\int_\Omega f_j(j_t^1 x^0) dv}{\int_\Omega k_j(j_t^1 x^0) dv}, j = \overline{1,p} \backslash \{r\}, \end{cases} \end{cases}$$

where $j_t^1 x^0 = (t, x^0(t), x_\nu^0(t))$. By denoting

$$R_r^0 = \frac{\int_\Omega f_r(j_t^1 x^0) dv}{\int_\Omega k_r(j_t^1 x^0) dv} = \min_x \frac{\int_\Omega f_r(j_t^1 x) dv}{\int_\Omega k_r(j_t^1 x) dv}, \ r = \overline{1,p},$$

the problem $(FP)_r(x^0)$ can be written as

$$(FPR)_r: \begin{cases} \underset{x}{\text{Minimize}} \quad \frac{\int_\Omega f_r(j_t^1 x) dv}{\int_\Omega k_r(j_t^1 x) dv} \\ \text{subject to:} \quad \begin{cases} x(\cdot) \in \Phi, x(t)\,|_{\partial\Omega} = u(t), \ \forall t \in \Omega, \\ g_\alpha(j^1 x) \leqq 0, \ h_\beta(j^1 x) \leqq 0, \\ \int_\Omega [f_j(j^1 x) - R_j^0 k_j(j^1 x)] dv \leqq 0, \ j = \overline{1,p} \backslash \{r\}. \end{cases} \end{cases}$$

Consider now the problem

$$(SPR)_r: \begin{cases} \underset{x}{\min} \quad \int_\Omega [f_r(j^1 x) - R_r^0 k_r(j_t^1 x)] dv \\ \text{subject to:} \quad \begin{cases} x(\cdot) \in \Phi, \ x(t)\,|_{\partial\Omega} = u(t), \ \forall t \in \Omega\ , \\ g_\alpha(j_t^1 x) \leqq 0, \ h_\beta(j_t^1 x) = \ 0, \\ \int_\Omega [f_j(j_t^1 x) - R_j^0 k_j(j_t^1 x)] dv \leqq 0, \ j = \overline{1,p} \backslash \{r\}. \end{cases} \end{cases}$$

Lemma 3.2 (Jaganathan [4]). *$x^0(\cdot) \in D$ is optimal to $(FRP)_r$ if and only if it is optimal to $(SPR)_r$, $r = \overline{1,p}$.*

Theorem 3.3. *$x^0(\cdot) \in D$ is an efficient solution to (VFP) if and only if it is an optimal solution for each of the problems $(SPR)_r$, $r \in \overline{1,p}$.*

Proof. $x^0(\cdot)$ is efficient to (VFP) if and only if it is optimal to $(FRP)_r$, $r = \overline{1,p}$ (according to Chankong and Haimes [2]). Also, $x^0(\cdot)$ is optimal to $(FPR)_r$ iff it is optimal to $(SPR)_r$ (see Lemma 3.2). $\square$

Definition 3.4. The efficient solution $x^0(\cdot) \in \mathbb{D}$ is a normal efficient solution to (VFP) if $x^0(\cdot)$ is optimal to least one of scalar problems $(SPR)_r$, $r = \overline{1,p}$.

Theorem 3.4 (Necessary efficiency in (FVP)). $[7, Th.3.2]$ *Let $x(\cdot) \in D$ be a normal efficient solution to the problem (FVP). Then there exist a vector $\tau = (\tau^r) \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda = (\lambda^\alpha(t)) \in \mathbb{R}^m$ and $\mu = (\mu^\beta(t)) \in \mathbb{R}^q$, which satisfy the conditions*

$$(MFJ): \quad \begin{cases} \tau^r \left[\frac{\partial f_r}{\partial x^k} - R_r^0 \frac{\partial k_r}{\partial x^k}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k} - \\ \qquad\qquad -D_\nu\left(\tau^r \left[\frac{\partial f_r}{\partial x^k_\nu} - R_r^0 \frac{\partial k_r}{\partial x^k_\nu}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k_\nu} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k_\nu}\right) = 0, \\ \lambda^\alpha(t) g_\alpha(j^1_t x) = 0, \quad \alpha = \overline{1,m}, \\ \tau \ \geq \ 0, \quad e_r\tau^r = 1, \ (\lambda^\alpha(t)) \geqq 0, \ t \in \Omega\ . \end{cases}$$

We denote

$$F_r(x(\cdot)) = \int_\Omega f_r(j^1 x)dv, \quad K_r(x(\cdot)) = \int_\Omega k_r(j^1 x)dv,$$

and then we have $R_r^0 = F_r(x(\cdot))/K_r(x(\cdot))$, $r = \overline{1,p}$. Taking into account these relations, Theorem 3.4 becomes

Theorem 3.5 (Necessary efficiency in (VFP)). $[7, Th.3.3]$ *Let $x(\cdot) \in D$ be a normal efficient solution to problem (VFP). Then there exist a vector $\tau = (\tau^r) \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda = (\lambda^\alpha(t)) \in \mathbb{R}^m$ and $\mu = (\mu^\beta(t)) \in \mathbb{R}^q$, which satisfy the following conditions*

$$(MFJ)_0: \quad \begin{cases} \tau^r \left[K_r(x)\frac{\partial f_r}{\partial x^k} - F_r(x)\frac{\partial k_r}{\partial x^k}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k} - \\ \quad -D_\nu\left(\tau^r \left[K_r(x)\frac{\partial f_r}{\partial x^k_\nu} - F_r(x)\frac{\partial k_r}{\partial x^k_\nu}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k_\nu} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k_\nu}\right) = 0, \\ \lambda^\alpha(t) g_\alpha(j^1_t x) = 0, \ \alpha = \overline{1,m}, \ \tau \geq 0, \ e_r\tau^r = 1, \ (\lambda^\beta(t)) \geqq 0, \ t \in \Omega, \end{cases}$$

where we denoted $\lambda^\alpha(t) := K_r(x(\cdot))\lambda^\alpha(t)$, $\mu^\beta(t) := K_r(x(\cdot))\mu^\beta(t)$.

Hereafter, we shall use the following concept:

Definition 3.5. The efficient mapping $x^0(\cdot) \in \mathbb{D}$ is a *normal efficient solution* to (VFP), if the conditions (MFJ) or $(MFJ)_0$ are fulfilled, with $\tau \geq 0$, $e'\tau = 1$, $e = (1, \ldots, 1) \in \mathbb{R}^p$.

4 Mond-Weir-Zalmai type duality for (VFP)

In this Section we present the notions of *geodesic invex set* and of (ρ, b)*-geodesic quasiinvex function*. We shall use such functions in studying the duality of optimization problems and deriving a natural generalization of the Mond-Weir-Zalmai duality ([9, 17]).

4.1 Geodesic invex sets and (ρ, b)-geodesic quasiinvex functions

In their paper [1], Barani and Pouryayevali introduced the notions of *geodesic invex set* and of *geodesic invex function on a Riemannian manifold*, as follows:

Definition 4.1. Let $\eta : M \times M \to TM$ be a function, such that for every $x, u \in M$, $\eta(x, u) \in T_u M$ and let $S \subset M$ be a nonempty set. The set S is said to be *a geodesic invex set with respect to* η (in brief, S is an η*-geodesic invex set*), if for any $x, u \in S$, there exists exactly one geodesic $\gamma_{x,u} : [0, 1] \to M$, such that

$$\gamma_{x,u}(0) = u, \;\; \gamma'_{x,u}(0) = \eta(x, u), \text{ and } \gamma_{x,u}(t) \in S, \;\; \forall t \in [0, 1].$$

Example. For $M = \mathbb{R}^n$, $\gamma_{x,u}(t) = u + t\eta(x, u)$.

Definition 4.2. Let the set $S \subset M$ be an η-geodesic invex set. A smooth C^1 function $f : S \to \mathbb{R}$ is called η*-geodesic invex* on S, if

$$f(x) - f(u) \geq df_u(\eta(x, u)), \;\; \forall x, u \in S.$$

Remark. If u is a fixed point, then we say that S is a η-geodesic invex set at u, and f is called η*-geodesic invex function at* u.

Definition 4.3. [14, 15] a) Let $G \subset \Phi$ be a nonempty set, and let $x^0(\cdot), x(\cdot) \in G$. A real function $\varphi(t, \theta), t \in \Omega, \theta \in [0, 1]$ is called *geodesic deformation* in G of the pair of functions $(x^0(\cdot), x(\cdot))$, if it satisfies the following properties: (i) the function $\theta \to \varphi(t, \theta)$ is a geodesic; (ii) $\varphi(t, 0) = x^0(t)$, $\varphi(t, 1) = x(t)$, $\varphi(\cdot, \theta) \in G, \forall \theta \in [0, 1]$. Then we say that $x(\cdot)$ is *a geodesic deformation of* $x^0(\cdot)$, where $x^0(\cdot)$ is assumed to be fixed.

b) The set $G = \{x(\cdot) \mid x : \Omega \to S \subset M\}$ is said to be $\eta-$*geodesic invex* if, for every $x^0(\cdot), x(\cdot) \in G$, there exists exactly one geodesic deformation $\varphi(t, \theta), t \in \Omega, \theta \in [0, 1]$, such that the vector function

$$\eta : \Omega \to \mathbb{R}^n, \;\; \eta(t) = \eta(x^0(t), x(t)) = (\eta^1(t), \ldots, \eta^n(t)) = \left.\frac{\partial \varphi}{\partial \theta}\right|_{\theta=0} \in T_{x^0(t)} M$$

is of class C^1 and satisfies $\eta(t)|_{\partial\Omega} = 0$.

For dealing with sufficient conditions of efficiency, we shall introduce the notion of (ρ, b)-geodesic quasiinvex functionals, which extends the notion of (ρ, b)-quasiinvex functions [8]. We fix a number $\rho \in \mathbb{R}$, a functional $b : \Omega \times \Omega \to [0, \infty)$ and the distance function $d(x(\cdot), y(\cdot))$ on Φ. We consider the functional $E : \Phi \to \mathbb{R}$,

$$E(x(\cdot)) = \int_\Omega X(j^1_t x) dv.$$

Definition 4.4. Let G be an open η-geodesic subset of Φ. The functional E is called (strictly) (ρ, b)-*geodesic quasiinvex* at $x^0(\,\cdot\,) \in G$ ($x^0(\,\cdot\,)$ being assumed as fixed) with respect to the C^1 vector function

$$\eta : \Omega \to \mathbb{R}^n, \quad \eta(t) = (\eta^1(t), \ldots, \eta^n(t)) \in T_{x^0(t)}M, \quad \eta(t)|_{\partial\Omega} = 0,$$

if $E(x(\,\cdot\,)) \leq E(x^0(\,\cdot\,))$ implies

$$b(x, x^0) \int_\Omega \left[\eta^i \frac{\partial X}{\partial x^i}(j^1_t x^0) + \frac{\partial \eta^i}{\partial t^\nu} \frac{\partial X}{\partial x^i_\nu}(j^1_t x^0\right] \,\mathrm{d}v \;(<) \leqq \; -\rho b(x, x^0) d^2(x, x^0),$$

for any $x(\,\cdot\,) \in G$ such that $x(\,\cdot\,) \neq x^0(\,\cdot\,)$.

Example ([10]). Let $a : [0,1] \times C^1([0,1]) \to \mathbb{R}_+$, $\Omega = [0,1]$, $M = \mathbb{R}^2$, and let $x(\,\cdot\,)$ be a continuous function given by $x : [0,1] \to \mathbb{R}^2$, $x = x(t)$, $t \in [0,1]$, $x(\,\cdot\,) = (x^1(\,\cdot\,), x^2(\,\cdot\,))$. One can verify that the functional

$$A(x(\,\cdot\,)) = \int_0^1 a(t, x(t)) dt$$

is (ρ, b)-geodesic quasiinvex for $\rho \geq 0$ at a point x^0 with respect to the mapping

$$\eta(t) = (\eta^1(t), \eta^2(t)) = (A(x) - A(x^0)) \cdot \left(\frac{\partial a}{\partial x^1}(t, x^0(t)), \frac{\partial a}{\partial x^2}(t, x^0(t)) \right). \tag{4.1}$$

The domain of $A(x(\,\cdot\,))$ is the set $G = \{x \mid x : [0,1] \to \mathbb{R}^2,\ x(\,\cdot\,) \text{ is continuous}\}$, and $M = \mathbb{R}^2$. Since $G = x([0,1])$ is a connected set, it is a geodesic invex set.

Another example is given by the function $x(t) = (x^1(t), x^2(t)) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. We note that $C = x([0, 2\pi])$ is the (closed) circle of center $(0,0)$ and radius equal to 1. Then $A(x(\,\cdot\,))$ defined in (4.1) is (ρ, b)-geodesic quasiinvex on the geodesic invex set C.

4.2 Mond-Weir-Zalmai extended duality

We shall further discuss the Mond-Weir [9]-Zalmai [17] type duality for (VFP).

Consider the functions $y(\,\cdot\,) \in \Phi$; we associate to (VFP) the following multi-time vector fractional variational dual problem

$$(WFD): \begin{cases} \text{Maximize Pareto} & \left(\frac{\int_\Omega f_1(j^1 y) dv}{\int_\Omega k_1(j^1 y) dv}, \ldots, \frac{\int_\Omega f_p(j^1 y) dv}{\int_\Omega k_p(j^1 y) dv} \right) \\ \text{subject to:} & (4.2), \end{cases}$$

where the constraints are given by

$$\begin{cases} \tau^r \left[K_r(y) \frac{\partial f_r}{\partial x^k} - F_r(y) \frac{\partial k_r}{\partial x^k} \right] + \lambda^\alpha(t) \frac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t) \frac{\partial h_\beta}{\partial x^k} - \\ \qquad -D_\nu \left(\tau^r \left[K_r(x) \frac{\partial f_r}{\partial x^k_\nu} - F_r(x) \frac{\partial k_r}{\partial x^k_\nu} \right] + \lambda^\alpha(t) \frac{\partial g_\alpha}{\partial x^k_\nu} + \mu^\beta(t) \frac{\partial h_\beta}{\partial x^k_\nu} \right) = 0 \\ \lambda^\alpha(t) g_\alpha(j^1 y) + \mu^\beta(t) h_\beta(j^1 y) \geqq 0, \quad \alpha = \overline{1,m},\ \beta = \overline{1,q}, \quad (R) \\ \tau = (\tau^r) \geqq 0, \quad e_r \tau^r = 1, \ \ (\lambda^\alpha(t)) \geqq 0,\ y(\,\cdot\,) \in G,\ y(t)|_{\partial\Omega} = u(t),\ t \in \Omega\,. \end{cases} \tag{4.2}$$

We further denote by $\pi(x)$ the value of the problem (VFP) at $x \in \mathbb{D}$, and let be $\delta(y, \lambda, \eta, \nu)$ the value of the dual (MFD) at $(y, \lambda, \eta, \nu) \in \Delta$, where Δ is the domain of (WFD). In what follows we develop a duality theory between (VFP) and (WFD), where the two duals have the same objective function. Namely, we produce a duality of Mond-Weir-Zalmai type, under the reduced form [9, 17]. The constraint (R) in (4.2) is one of Mond-Weir type [9]. Zalmai [17] considered programs with fractional objectives.

Theorem 4.1 (Weak duality)**.** *Let $G \supset \mathbb{D}$ be an $\eta-$geodesic invex set in M (like in definition 4.1). Let $x(\,\cdot\,) \in \mathbb{D}$ and let $(y(\,\cdot\,), \lambda, \mu, \nu) \in \Delta$ be feasible points of the problems (VFP) and (WFD), where $y(\,\cdot\,) \in G$. Assume that there are satisfied the conditions:*

a) *For each $r = \overline{1,p}$, the integral*

$$\int_\Omega \left[K_r(y) f_r(j_t^1 x) - F_r(y) k_r(j_t^1 x)\right] dv$$

is (ρ_r', b)-geodesic quasiinvex at $y(\,\cdot\,)$ with respect to η and d.

b) *The integral $\int_\Omega [\lambda^\alpha(t) g_\alpha(j^1 x) + \mu^\beta(t) h_\beta(j^1 x)]\, dv$ is (ρ, b)-geodesic quasiinvex at $y(\,\cdot\,)$ with respect to η and d.*

c) *One of the functions of a)-b) is strictly (ρ, b)-quasiinvex at $y(\,\cdot\,)$ with respect to η and d.*

d) $\tau^r \rho_r' + \rho \geqq 0$.

Then $\pi(x) \leq \delta(y, \lambda, \mu, v)$ is false.

Proof. We suppose that, on the contrary, there exist $x(\,\cdot\,) \in \mathbb{D}$, and $(y(\,\cdot\,), \lambda, \mu, \nu) \in \Delta$ such that $\pi(x) \leq \delta(y, \lambda, \mu, \nu)$, or, in detail,

$$\left(\frac{\int_\Omega f_1(j^1 x)\mathrm{d}v}{\int_\Omega k_1(j^1 x)\mathrm{d}v}, \ldots, \frac{\int_\Omega f_p(j^1 x)\mathrm{d}t}{\int_\Omega k_p(j^1 x)\mathrm{d}t}\right) \leq \left(\frac{\int_\Omega f_1(j^1 y)\mathrm{d}v}{\int_\Omega k_1(j^1 y)\mathrm{d}v}, \ldots, \frac{\int_\Omega f_p(j^1 y)\mathrm{d}v}{\int_\Omega k_p(j^1 y)\mathrm{d}v}\right)$$

or, in on components, $\frac{\int_\Omega f_r(j^1 x)\, \mathrm{d}v}{\int_\Omega k_r(j^1 x)\, \mathrm{d}v} \leqq \frac{\int_\Omega f_r(j^1 y)\, \mathrm{d}v}{\int_\Omega k_r(j^1 y)\, \mathrm{d}v}$, $r = \overline{1,p}$. This relation can be written

$$\int_\Omega [K_r(y) f_r(j_t^1 x) - F_r(y) k_r(j_t^1 x)] dv \leqq 0 \left[\int_\Omega [K_r(y) f_r(j_t^1 y) - F_r(y) k_r(j_t^1 y)] dv\right]. \tag{4.3}$$

According to hypothesis a), (4.3) implies

$$\begin{aligned} & b(x,y) \int_\Omega \left\{ \eta_s \left[K_r(y) \frac{\partial f_r}{\partial y^s} - F_r(y) \frac{\partial k_r}{\partial y^s}\right] + \right. \\ & \left. +(D_\gamma \eta_s)' \left[K_r(y) \frac{\partial f_r}{\partial y^s_\nu} - F_r(y) \frac{\partial k_r}{\partial y^s_\nu}\right]\right\} \mathrm{d}v \leqq -\rho_r' b(x,y) d^2(x,y). \end{aligned} \tag{4.4}$$

After multiplying (4.3) and (4.4) by $\tau^r \geqq 0$ and summing over $r = \overline{1,p}$, it results

$$\tau^r [F_r(x) K_r(y) - K_r(x) F_r(y)] \leqq 0, \tag{4.5}$$

which implies

$$b(x,y)\int_\Omega \Big\{\eta_s\tau^r[K_r(y)\tfrac{\partial f_r}{\partial y^s} - F_r(y)\tfrac{\partial k_r}{\partial y^s}] + (D_\nu\eta_s)\tau^r[K_r(y)\tfrac{\partial f_r}{\partial y^s_\nu} - F_r(y)\tfrac{\partial k_r}{\partial y^s_\nu}]\Big\}$$
$$\leqq -\tau^r\rho'_r b(x,y)d^2(x,y).$$

From the domains $\mathbb{D}$ and Δ, we yield

$$\begin{aligned}&\textstyle\int_\Omega [\lambda^\alpha(t)g_\alpha(j^1_t x) + \mu^\beta(t)h_\beta(j^1_t x)]\,\mathrm{d}v\\ &\quad\leqq \textstyle\int_\Omega [\lambda^\alpha(t)g_\alpha(j^1_t y) + \mu^\beta(t)h_\beta(j^1_t y)]\,\mathrm{d}v,\end{aligned}\tag{4.6}$$

and according to b), from (4.6), it results

$$b(x,y)\int_\Omega \Big\{\eta_s[\lambda^\alpha(t)'\tfrac{\partial g_\alpha}{\partial y^s} + \mu^\beta(t)\tfrac{\partial h_\beta}{\partial y^s}] + (D_\nu\eta_s)[\lambda^\alpha(t)\tfrac{\partial g_\alpha}{\partial y^s_\nu} + \mu^\beta(t)\tfrac{\partial h_\beta}{\partial y^s_\nu}]\Big\}\,\mathrm{d}v$$
$$\leqq \rho b(x,y)d^2(x,y).$$

Summing now side by side the double implications (4.5) and (4.6) (written under the form l.h.s. $\leqq 0$) and taking into account c), we infer that

$$\begin{aligned}&\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y) + [\textstyle\int_\Omega [\lambda^\alpha(t)g_\alpha(j^1_t x) + \mu^\beta(t)h_\beta(j^1_t x)]\,\mathrm{d}v\\ &\quad - \textstyle\int_\Omega [\lambda^\alpha(t)g_\alpha(j^1_t y) + \mu^\beta(t)h_\beta(j^1_t y)]\,\mathrm{d}v \leqq 0,\end{aligned}\tag{4.7}$$

implies

$$b(x,y)\int_\Omega \Big[\eta_s\{\tau^r[K_r(y)\tfrac{\partial f_r}{\partial y^s} - F_r(y)\tfrac{\partial k_r}{\partial y^s}] + \lambda^\alpha(t)\tfrac{\partial g_\alpha}{\partial y^s} + \mu^\beta(t)\tfrac{\partial h_\beta}{\partial y^s}\}$$
$$+(D_\nu\eta_s)\{\tau^r[K_r(y)\tfrac{\partial f_r}{\partial y^s_\nu} - F_r(y)\tfrac{\partial k_r}{\partial y^s_\nu}] + \lambda^\alpha(t)\tfrac{\partial g_\alpha}{\partial y^s_\nu} + \mu^\beta(t)\tfrac{\partial h_\beta}{\partial y^s_\nu}\Big]\,dv$$
$$< -(\tau^r\rho_r + \rho)b(x,y)d^2(x,y).$$

From the second implication of (4.7) it results $b(x,y) > 0$. In brief, this can be written as:

$$\int_\Omega \left[\eta_s\frac{\partial V}{\partial y^s} + (D_\nu\eta_s)\frac{\partial V}{\partial y^s_\nu}\right]\ \mathrm{d}t \ <\ -(\tau^r\rho_r + \rho)d^2(x,y),\tag{4.8}$$

where we denoted

$$V = \tau^r[K_r(y)f_r(j^1y) - F_r(y)k_r(j^1y)] + \lambda^\alpha(t)g_\alpha(j^1y) + \mu^\beta(t)h_\beta(j^1y).$$

We further have $(D_\nu\eta_s)\frac{\partial V}{\partial y^s_\gamma} = D_\nu\left(\eta_s\frac{\partial V}{\partial y^s_\nu}\right) - \eta_s D_\nu\left(\frac{\partial V}{\partial y^s_\nu}\right)$, which leads by integration to

$$\int_\Omega (D_\nu\eta_s)\frac{\partial V}{\partial y^s_\gamma}dv = \int_\Omega D_\nu\left(\eta_s\frac{\partial V}{\partial y^s_\nu}\right)dv - \int_\Omega \eta_s D_\nu\left(\frac{\partial V}{\partial y^s_\nu}\right)dv,$$

and using the gradient formula, we infer

$$\int_\Omega D_\nu\left(\eta_s\frac{\partial V}{\partial y^s_\nu}\right)dv = \int_{\partial\Omega}\left(\eta_s\frac{\partial V}{\partial y^s_\nu}\right)\vec{n}(t)\,d\sigma = 0,$$

where $\vec{n}(t)$is unit vector to surface $\partial\Omega$ at the current point, and $\eta_s(t)\mid_{\partial\Omega}=0$. Then relation (4.8) becomes

$$\int_a^b \eta_s\left[\frac{\partial V}{\partial y^s}-D_\nu\left(\frac{\partial V}{\partial y^s_\nu}\right)\right]\, dv\ < -(\tau^r\rho_r+\rho)d^2(x,y). \tag{4.9}$$

Taking into account the first constraint of problem (WFD), we yield

$$\frac{\partial V}{\partial y^s}-D_\nu\left(\frac{\partial V}{\partial y^s_\nu}\right)=0,$$

and then relation (4.9) simplifies to $\;0<-(\tau^r\rho_r+\rho)d^2(x,y)$.

But using the hypothesis d) of the Theorem, this inequality becomes $0<0$, which is false. Then, from (4.7) it follows

$$\begin{aligned}&\tau^r[F_r(x)K_r(y)-K_r(x)F_r(y)+\int_\Omega[\lambda^\alpha(t)g_\alpha(j^1x)+\mu^\beta(t)h_\beta(j^1x)]\,\mathrm{d}v\\&-\int_\Omega[\lambda^\alpha(t)g_\alpha(j^1y)+\mu^\beta(t)h_\beta(j^1y)]\,\mathrm{d}v\ >\ 0,\end{aligned} \tag{4.10}$$

and taking into account relation (4.6), from (4.10), it results

$$\tau^r[F_r(x)K_r(y)-K_r(x)F_r(y)]>0,$$

which contradicts the relation (4.5). Therefore $\pi(x)\leq\delta(y,\ \lambda,\ \mu,\nu)$ is false. □

According to Definitions 4.4 and 4.1, between $x\in\mathbb{D}$ and $y\in G$ there exists a geodesic $\gamma=(x,y)$ which is completely contained in G. It follows that $\mathbb{D}\subseteq G$.

Theorem 4.2 (Direct duality). *Let $x^0(\cdot)\in\mathbb{D}$ be a normal efficient solution of the primal (VFP) and suppose satisfied the hypotheses of Theorem 4.1. Then there exist the vector $\tau^0\in\mathbb{R}^p$ and the piecewise smooth functions $\lambda^0=(\lambda^\alpha)^0:\Omega\to\mathbb{R}^m$ and $\mu^0=(\mu^\beta)^0:\Omega\to\mathbb{R}^q$, such that $(x^0,\lambda^0,\mu^o,\nu^0)$ is an efficient solution of the dual $(MWFD)$; moreover, $\pi(x^0)\ =\ \delta(x^0,\lambda^0,\mu^0,\nu^0)$.*

Proof. Since $x^0(\cdot)$ is a regular efficient solution to (VFP), according to Theorem 3.4, there exist the vectors $\tau^0=(\tau^r)^0\in\mathbb{R}^p$ and the piecewise smooth functions $\lambda^0=(\lambda^\alpha)^0:\Omega\to\mathbb{R}^m$ and $\mu^0=(\mu^\beta)^0:\Omega\to\mathbb{R}^q$, which satisfy the relations $(MFJ)_0$. It results that $(x^0(\cdot),\tau^0,\lambda^0,\mu^0)\in\Delta$, and, moreover, $\pi(x^0)=\delta(x^0,\tau^0,\lambda^0,\mu^0)$. □

Theorem 4.3 (The converse duality). *Let $(x^0(\cdot),\tau^0,\lambda^0,\mu^0)\in\Delta'$ be an efficient solution to the dual and $(GWFD)$ and assume satisfied the following conditions:*

i) $\bar{x}(\cdot)$ is a normal efficient solution of the primal (VFP).

ii) The hypotheses of Theorem 4.1 are satisfied for $(y(\cdot),\tau,\lambda,\mu)=(x^0(\cdot),\tau^0,\lambda^0,\mu^0)$.

Then $\bar{x}(\cdot)=x^0(\cdot)$ and moreover, $\pi(x^0)=\delta(x^0,\tau^0,\lambda^0,\mu^0)$.

Proof. We shall assume that, on the contrary, $\bar{x}(\cdot)\neq x^0(\cdot)$, and we will obtain a contradiction. Since $\bar{x}(\cdot)$ is a normal efficient solution to (VFP) then, according to Theorem 3.4, there exist a vector $\bar{\tau}\in\mathbb{R}^p$ and the vector functions $\bar{\lambda}=(\bar{\lambda}^\alpha)$: $\Omega\to\mathbb{R}^m$ and $\bar{\mu}=(\bar{\mu}^\beta):\Omega\to\mathbb{R}^q$, which satisfy the conditions $(MFJ)_0$. It results

$\bar{\lambda}^{\alpha}(t)g_{\alpha}(j_t^1\bar{x}) + \bar{\mu}^{\beta}(t)h_{\beta}(j_t^1\bar{x}) = 0$ and hence, $(\bar{x}(\cdot), \bar{\tau}, \bar{\lambda}, \bar{\mu}) \in \Delta$. Moreover,$\pi(\bar{x}) = \delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$. According to Theorem 4.1, relation $\pi(\bar{x}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0)$ is false. It results that the relation $\delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0)$, is false as well. Therefore, the maximal efficiency of $(x^0(\cdot), \tau^0, \lambda^0, \mu^0)$ is contradicted. Then, it follows that the above made assumption $\bar{x}(\cdot) \neq x^0(\cdot)$ is false. It results that $\bar{x}(\cdot) = x^0(\cdot)$ and also $\bar{\tau} = \tau^0, \quad \bar{\lambda} = \lambda^0,\ \bar{\mu} = \mu^0$. Finally, we have $\pi(\bar{x}) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$. □

5 Generalized Mond-Weir-Zalmai type duality for (VFP)

Starting from the problema (WFD) we define a $(GWFD)$ problem as follows. Let $\{A_1, \ldots, A_s\}(s \leq m)$ be a partition of the set $\{1, \ldots, m\} = \{\alpha\} = A = \{A_\theta\}$, that is, $A_\theta \subseteq A$, $A_\theta \cap A_\zeta = \phi$, if $\theta \neq \zeta$, $\bigcup\limits_{\theta=1}^{s} A_\theta = A$. Let also $\{B_1, \ldots, B_s\}$, $(s \leq q)$, be a partition of the set $\{1, \ldots, q\} = \{\beta\} = B = \{B_\theta\}$, $(s = \max\{p, m, q\})$. Some subsets A_θ or B_θ may be empty.

We associate to (VFP) the next multi-time vector fractional variational dual problem, which is a general Mond-Weir-Zalmai dual program.

$$(GWFD): \begin{cases} \text{Maximize Pareto} & \left(\frac{\int_\Omega f_1(j^1y)dv}{\int_\Omega k_1(j^1y)dv}, \ldots, \frac{\int_\Omega f_p(j^1y)dv}{\int_\Omega k_p(j^1y)dv}\right) \\ \text{subject to:} & (5.1), \end{cases}$$

where

$$(5.1) \quad \begin{cases} \tau^r \left[K_r(y)\frac{\partial f_r}{\partial x^k} - F_r(y)\frac{\partial k_r}{\partial x^k}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k} - \\ \qquad -D_\nu\left(\tau^r\left[K_r(x)\frac{\partial f_r}{\partial x^k_\nu} - F_r(x)\frac{\partial k_r}{\partial x^k_\nu}\right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^k_\nu} + \mu^\beta(t)\frac{\partial h_\beta}{\partial x^k_\nu}\right) = 0 \\ \lambda^{A_\theta}(t)g_{A_\theta}(j^1y) + \mu^{B_\theta}(t)h_{B_\theta}(j^1y) \geqq 0, \ \ \theta = \overline{1,s} \quad (R_\theta) \\ \tau = (\tau^r) \geqq 0, \ e_r\tau^r = 1, \ \ (\lambda^{A_\theta}(t)) \geqq 0, \\ y(\cdot) \in G, \ y(t)\,|_{\partial\Omega} = u(t), \ \ t \in \Omega\ . \end{cases}$$

Denote by $\pi(x)$ the value of the problem (VFP) at $x(\cdot) \in \mathbb{D}$ and let be $\delta(y, \lambda, \eta, \nu)$ the value of the dual $(GWFD)$ at $(y(\cdot), \lambda, \eta, \nu) \in \Delta'$, where Δ' is the domain of $(GWFD)$.

Remark. The relations (R_θ) in (5.1) from $(GWFD)$ can be detailed as follows, where we further use Einstein's summation rule:

$$\lambda^{A_\theta}(t)g_{A_\theta}(j^1y) = \lambda^\alpha(t)g_\alpha(j^1y), \quad \alpha \in A_\theta, \ \theta = \overline{1,s}$$
$$\mu^{B_\theta}(t)h_{B_\theta}(j^1y) = \mu^\beta(t)h_\beta(j^1y), \quad \beta \in B_\theta, \beta = \overline{1,q}.$$

The following results provide the grounds of the duality theory between (VFP) and $(GWFD)$.

Theorem 5.1 (Weak duality)**.** *Let $G \supset \mathbb{D}$ be an η-geodesic invex set in M (like in Definition 4.1). Let $x(\cdot) \in \mathbb{D}$ and let $(y(\cdot), \lambda, \mu, v) \in \Delta'$ be feasible points for the problems (VFP) and (GWFD), and let $y(\cdot) \in G$. Assume that the following conditions are satisfied:*

a) *For each* $r = \overline{1,p}$, *the integral* $\int_\Omega \left[K_r(y)f_r(j_t^1 x) - F_r(y)k_r(j_t^1 x)\right] dv$ *is* (ρ'_r, b)-*geodesic quasiinvex at* $y(\,\cdot\,)$ *with respect to* η *and* d.

b) *Each integral* $\int_\Omega [\lambda^{A_\theta}(t)g_{A_\theta}(j_t^1 x) + \mu^{B_\theta}(t)h_{B_\theta}(j_t^1 x)]\, dv$ *is* (ρ_θ, b)-*geodesic quasiinvex at* $y(\,\cdot\,)$ *with respect to* η *and* d, *where* $\theta = \overline{1,s}$.

c) *One of the functions of a)-b) is strictly* (ρ, b)-*quasiinvex at* $y(\,\cdot\,)$ *with respect to* η *and* d.

d) $\tau^r \rho'_r + \sum\limits_{\theta=1}^{s} \rho_\theta \geqq 0$.

Then the inequality $\pi(x) \leq \delta(y, \lambda, \mu, v)$ *is false.*

Theorem 5.2 (Direct duality)**.** *Let* $x^0(\,\cdot\,)$ *be a normal efficient solution of the primal (VFP) and assume satisfied the hypotheses of Theorem 5.1. Then there exist the vector* $\tau^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\lambda^0 = (\lambda^\alpha)^0 : \Omega \to \mathbb{R}^m$ *and* $\mu^0 = (\mu^\beta)^0 : \Omega \to \mathbb{R}^q$, *such that* $(x^0(\,\cdot\,), \lambda^0, \mu^0, \nu^0)$ *is an efficient solution of the dual (GWFD) and moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \nu^0)$.

Theorem 5.3 (Converse duality)**.** *Let* $(x^0(\,\cdot\,), \tau^0, \lambda^0, \mu^0) \in \Delta$ *be an efficient solution to the dual and (MWFD) and assume satisfied the following conditions:*

i) $\bar{x}(\,\cdot\,)$ is a normal efficient solution of the primal *(VFP).*

ii) The hypotheses of Theorem 5.1 are satisfied with $(y(\,\cdot\,), \tau, \lambda, \mu) = (x^0(\,\cdot\,), \tau^0, \lambda^0, \mu^0)$.

Then $\bar{x}(\,\cdot\,) = x^0(\,\cdot\,)$ *and moreover,* $\pi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

The proofs of Theorems 5.1, 5.2 and 5.3 are similar to those of Theorems 4.1, 4.2 and 4.3, respectively.

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