Multitime Samuelson-Hicks diagonal recurrence

Cristian Ghiu, Raluca Tuligă, Constantin Udrişte, Ionel Ţevy

Abstract. Our original results regarding the multitime diagonal recurrences are applied now to the discrete multitime Samuelson-Hicks models, with constant, respectively multi-periodic, coefficients. The aim is to find bivariate sequences with economic meaning. For constant coefficients case, it was found also the generating function; for multi-periodic coefficients case, we determined the Floquet multipliers associated to the monodromy matrix.

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1 Why discrete multitime recurrences?

The multivariate recurrences are based on multiple sequences and come from areas like analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics, economics etc.

Our point of view regarding the discrete multitime recurrences was initiated in [4]-[8], splitting the ideas in two direction: (i) multiple recurrences, (ii) diagonal recurrences.

Bousquet-Mélou and Petkovšek [1] showed that while in the univariate case solutions of linear recurrences with constant coefficients have rational generating functions (see also [2]), the multivariate case is much richer, even though initial conditions have rational generating functions the corresponding solutions can have generating functions which are not rational, algebraic, or even D-finite GF in two or more variables.

Floquet theory, first formulated for periodic linear ODEs [3], was extended to difference equations ([11], [23], [12], [10]). We have extended this theory to the multitemporal first order PDEs [21] and now to multivariate multi-periodic recurrences, borrowing mathematical ingredients from some papers [14]-[22] involving multitime (multivariate time) dynamical systems. In Floquet theory it is necessary to find explicitly the associated monodromy matrix and its eigenvalues (called Floquet multipliers).

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2 Discrete multitime constant coefficients Samuelson-Hicks model

An element $t = (t^1, \dots, t^m) \in \mathbb{N}^m$ is called *discrete multitime*. A function of the type $x : \mathbb{N}^m \to \mathbb{R}^n$ is called *multivariate sequence*. Also, for convenience, we denote $\mu(t) = \min\{t^1, t^2, \dots, t^m\}$ and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^m$.

Let $m \geq 2$ and $A\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R})$. A linear discrete multitime diagonal recurrence homogeneous system has the form [4]-[8]

$$x(t+\mathbf{1}) = A(t)x(t), \quad \forall t \in \mathbb{N}^m. \tag{2.1}$$

Let $T \in \mathbb{N}^*$. The linear multitime diagonal recurrence (2.1) is called *T-diagonal-periodic* if

$$A(t + T\cdot\mathbf{1}) = A(t), \quad \forall t \in \mathbb{N}^m. \tag{2.2}$$

We assume that $t = (t^1, ..., t^m) \in \mathbb{N}^m$ is a discrete multitime. Having in mind the discrete single-time Samuelson-Hicks model [12], we introduce a *discrete multitime Samuelson-Hicks like model* based on the following ingredients: (i) two parameters, the first γ, called the *marginal propensity to consume*, subject to $0 < \gamma < 1$, and the second α as *decelerator* if $0 < \alpha < 1$, *keeper* if $\alpha = 1$ or *accelerator* if $\alpha > 1$; (ii) the multiple sequence $Y(t)$ means the *national income* and is the main endogenous variable, the multiple sequence $C(t)$ is the *consumption*; (iii) we assume that multiple sequences $Y(t)$, $C(t)$ are non-negative.

We propose a *first order discrete multitime constant coefficients Samuelson-Hicks model* either as first order diagonal recurrence system

$$\begin{pmatrix} Y(t+\mathbf{1}) \\ C(t+\mathbf{1}) \end{pmatrix} = \begin{pmatrix} \gamma+\alpha & -\frac{\alpha}{\gamma} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix},\ t \in \mathbb{N}^m,$$

$$Y(t)|_{t^\alpha=0} = Y_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m),\ C(t)|_{t^\alpha=0} = C_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m) \tag{2.3}$$

$$Y_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m)|_{t^\beta=0} = Y_{0\beta}(t^1, ..., \widehat{t^\beta}, ..., t^m)|_{t^\alpha=0} \tag{2.4}$$

$$C_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m)|_{t^\beta=0} = C_{0\beta}(t^1, ..., \widehat{t^\beta}, ..., t^m)|_{t^\alpha=0} \tag{2.5}$$

or as the second order homogeneous diagonal recurrence equation

$$Y(t+\mathbf{2}) - (\gamma+\alpha)Y(t+\mathbf{1}) + \alpha Y(t) = 0,\ t \in \mathbb{N}^m, \tag{2.6}$$

$$Y(t)|_{t^\alpha=0} = Y_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m),\ Y(t)|_{t^\alpha=1} = Y_{1\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m) \tag{2.7}$$

$$Y_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m)|_{t^\beta=0} = Y_{0\beta}(t^1, ..., \widehat{t^\beta}, ..., t^m)|_{t^\alpha=0} \tag{2.8}$$

$$Y_{1\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m)|_{t^\beta=1} = Y_{1\beta}(t^1, ..., \widehat{t^\beta}, ..., t^m)|_{t^\alpha=1} \tag{2.9}$$

$$Y_{0\alpha}(t^1, ..., \widehat{t^\alpha}, ..., t^m)|_{t^\beta=1} = Y_{1\beta}(t^1, ..., \widehat{t^\beta}, ..., t^m)|_{t^\alpha=0}. \tag{2.10}$$

In the paper [7] were determined the solutions of the recurrences of the form (2.6) and was obtained the following result

Theorem 2.1. *Let $m \geq 2$, $a, b \in \mathbb{R}$ and λ_1, λ_2 the roots of the polynomial $P(\lambda) = \lambda^2 + a\lambda + b$. Suppose that the $(m-1)$-sequences*

$$f_1, f_2, \dots, f_m \colon \mathbb{N}^{m-1} \to \mathbb{R},\ g_1, g_2, \dots, g_m \colon \mathbb{N}^{m-1} \to \mathbb{R},$$

satisfy, for any $\alpha, \beta \in \{1, 2, \dots, m\}$, the compatibility conditions

$$\begin{aligned} f_\alpha(t^1, \dots, \widehat{t^\alpha}, \dots, t^m)\Big|_{t^\beta=0} &= f_\beta(t^1, \dots, \widehat{t^\beta}, \dots, t^m)\Big|_{t^\alpha=0}, \\ g_\alpha(t^1, \dots, \widehat{t^\alpha}, \dots, t^m)\Big|_{t^\beta=1} &= g_\beta(t^1, \dots, \widehat{t^\beta}, \dots, t^m)\Big|_{t^\alpha=1}, \\ f_\alpha(t^1, \dots, \widehat{t^\alpha}, \dots, t^m)\Big|_{t^\beta=1} &= g_\beta(t^1, \dots, \widehat{t^\beta}, \dots, t^m)\Big|_{t^\alpha=0}, \\ &\forall t^1, \dots, t^{\alpha-1}, t^{\alpha+1}, \dots, t^{\beta-1}, t^{\beta+1}, \dots, t^m \in \mathbb{N}. \end{aligned}$$

Then the unique m-sequence $x \colon \mathbb{N}^m \to \mathbb{R}$ which verifies

$$\begin{aligned} & x(t + 2\cdot\mathbf{1}) + ax(t + \mathbf{1}) + bx(t) = 0, \quad \forall t \in \mathbb{N}^m, \\ & x(t)\Big|_{t^\gamma=0} = f_\gamma(t^1, \dots, \widehat{t^\gamma}, \dots, t^m), \quad \forall (t^1, \dots, \widehat{t^\gamma}, \dots, t^m) \in \mathbb{N}^{m-1}, \\ & x(t)\Big|_{t^\gamma=1} = g_\gamma(t^1, \dots, \widehat{t^\gamma}, \dots, t^m), \quad \forall (t^1, \dots, \widehat{t^\gamma}, \dots, t^m) \in \mathbb{N}^{m-1}, \\ & \qquad \forall \gamma \in \{1, 2, \dots, m\}, \end{aligned}$$

is defined by the following formulas:
i) If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then

$$\begin{aligned} x(t) = {} & \frac{\lambda_1^{t^\beta} - \lambda_2^{t^\beta}}{\lambda_1 - \lambda_2}\, g_\beta(t^1 - t^\beta + 1, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta + 1) \\ & - \frac{\lambda_2 \lambda_1^{t^\beta} - \lambda_1 \lambda_2^{t^\beta}}{\lambda_1 - \lambda_2}\, f_\beta(t^1 - t^\beta, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta), \ \textit{if}\ \ \mu(t) = t^\beta. \end{aligned}$$

ii) If $\lambda_1 = \lambda_2$, then

$$\begin{aligned} x(t) = {} & t^\beta \lambda_1^{t^\beta - 1}\, g_\beta(t^1 - t^\beta + 1, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta + 1) \\ & - (t^\beta - 1)\lambda_1^{t^\beta}\, f_\beta(t^1 - t^\beta, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta), \ \textit{if}\ \ \mu(t) = t^\beta. \end{aligned}$$

iii) If $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1, \lambda_2 = r(\cos\theta \pm i \sin\theta)$, with $r > 0$, $\theta \in (0, 2\pi) \setminus \{\pi\}$, then

$$\begin{aligned} x(t) = {} & \frac{r^{t^\beta - 1} \sin t^\beta \theta}{\sin\theta}\, g_\beta(t^1 - t^\beta + 1, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta + 1) \\ & - \frac{r^{t^\beta} \sin(t^\beta - 1)\theta}{\sin\theta}\, f_\beta(t^1 - t^\beta, \dots, \widehat{t^\beta}, \dots, t^{m-1} - t^\beta), \ \textit{if}\ \ \mu(t) = t^\beta. \end{aligned}$$

2.1 Samuelson-Hicks discrete multitime recurrence equation with constant coefficients

2.1.1 Characteristic equation

Let us consider the second order homogeneous discrete multitime diagonal recurrence equation (2.6), with constant coefficients γ and α. Looking for a solution of the form

$$Y(t) = v\lambda^{<\epsilon, t>}, \ <\epsilon, \mathbf{1}>= 1,$$

it appears the characteristic equation

$$\lambda^2 - (\gamma + \alpha)\lambda + \alpha = 0.$$

The discriminant of this equation is $\delta = (\gamma+\alpha)^2 - 4\alpha$. If $\delta > 0$, then we have real roots $\lambda_1 > 0, \lambda_2 > 0$; if $\delta = 0$, then $\lambda_1 = \lambda_2 > 0$; if $\delta < 0$, then the roots λ_1, λ_2 are complex conjugated with modulus $\sqrt{\alpha}$. The stability requires $|\lambda_i| < 1$, $i = 1, 2$.

2.1.2 Generating function

First computation variant We start with the recurrence

$$Y_{m+2\,n+2} - (\gamma + \alpha)Y_{m+1\,n+1} + \alpha Y_{mn} = 0,\ m \geq 0,\ n \geq 0.$$

Our aim is to determine a closed form for the corresponding generating function

$$F(x, y) = \sum_{m=0,n=0} Y_{mn} x^m y^n$$

(analytic in a neighborhood of the origin; it can often be extended by analytic continuation) in terms of the given initial conditions and the recurrence relation, and then to derive a direct access formula for the coefficients Y_{mn} of the development of $F(x, y)$ in Taylor series around origin.

Multiplying with $x^{m+2}y^{n+2}$ and summing, it follows

$$\sum_{m,n\geq 0} Y_{m+2\,n+2} x^{m+2} y^{n+2} - (\gamma + \alpha) \sum_{m,n\geq 0} Y_{m+1\,n+1} x^{m+2} y^{n+2}$$

$$+\alpha \sum_{m,n\geq 0} Y_{mn} x^{m+2} y^{n+2} = 0.$$

We replace each double infinite sum in this equation by an algebraic expression involving the generating function, respectively

$$\sum_{m,n\geq 0} Y_{m+2\,n+2} x^{m+2} y^{n+2} = F(x, y) - \sum_{m\geq 0} Y_{m0} x^m - \sum_{n\geq 0} Y_{0n} y^n + Y_{00}$$

$$-y \sum_{m\geq 1} Y_{m1} x^m - x \sum_{n\geq 1} Y_{1n} y^n + xyY_{11},$$

$$\sum_{m,n\geq 0} Y_{m+1\,n+1} x^{m+2} y^{n+2} = xy \left(F(x, y) + Y_{00} - \sum_{m\geq 0} Y_{m0} x^m - \sum_{n\geq 0} Y_{0n} y^n \right)$$

$$\sum_{m,n\geq 0} Y_{mn} x^{m+2} y^{n+2} = x^2 y^2 F(x, y).$$

We denote

$$\phi_0(x) = \sum_{m\geq 0} Y_{m0} x^m,\ \psi_0(y) = \sum_{n\geq 0} Y_{0n} y^n,$$

$$\phi_1(x) = \sum_{m\geq 1} Y_{m1} x^m,\ \psi_1(y) = \sum_{n\geq 1} Y_{1n} y^n.$$

It follows the functional equation $Q(x,y)F(x,y) = G(x,y)$, which define the generating function $F(x,y) = \frac{G(x,y)}{Q(x,y)}$, where

$$Q(x,y) = 1 - (\gamma + \alpha)xy + \alpha x^2 y^2,$$

$$G(x,y) = -(\gamma + \alpha)xy(\phi_0(x) + \psi_0(y) - Y_{00}) - xyY_{11}$$
$$+y\phi_1(x) + x\psi_1(y) + (\phi_0(x) + \psi_0(y) - Y_{00}).$$

The polynomial $Q(x,y)$ is the transform in $\frac{1}{\lambda}$ of the characteristic polynomial of the recurrence.

We write $G(x,y) = K(x,y) - U(x,y)$, with

$$K(x,y) = -(\gamma + \alpha)xy(\phi_0(x) + \psi_0(y) - Y_{00}) - xyY_{11}$$

and

$$U(x,y) = -y\phi_1(x) - x\psi_1(y) - (\phi_0(x) + \psi_0(y) - Y_{00}).$$

Then

$$K(x,y) = F(x,y)G(x,y) + U(x,y),$$

where $K(x,y)$ is a polynomial in xy, like Q, whose coefficients are functions depending on the initial conditions. The function $U(x,y)$ is affine, with coefficients depending on the initial conditions.

Using the formal power series expansion

$$F(x,y) = G(x,y) \sum_{k=0}^{\infty} ((\gamma + \alpha) - \alpha xy)^k \; x^k y^k,$$

we find the general term Y_{mn} of the bivariate economic sequence.

Particular solution If we use the following data

$$\begin{array}{rclcr}
\phi_0(x) = Y_{00} & \Longleftrightarrow & Y_{m0} = 0 & \text{for} & m \geq 1 \\
\psi_0(y) = Y_{00} & \Longleftrightarrow & Y_{0n} = 0 & \text{for} & n \geq 1 \\
\phi_1(x) = x & \Longleftrightarrow & Y_{11} = 1, Y_{m1} = 0 & \text{for} & m \geq 2 \\
\psi(y) = y & \Longleftrightarrow & Y_{11} = 1, Y_{1n} = 0 & \text{for} & n \geq 2,
\end{array}$$

we obtain

$$F(x,y) = \frac{xy(1 - (\gamma + \alpha)) + Y_{00}}{1 - (\gamma + \alpha)xy + \alpha x^2 y^2}.$$

Denoting by r_1, r_2 the real roots of the polynomial $Q(x,y)$, it follows the bivariate sequence

$$Y_{nn} = \frac{1}{\alpha}\left(\frac{A}{r_1^{n+1}} + \frac{B}{r_2^{n+1}}\right) \text{ and } Y_{mn} = 0, \text{ for } m \neq n,$$

depending on the parameters α and γ.

Second computation variant Let us start with the univariate Samuelson-Hicks recurrence

$$a_{n+2} - (\alpha + \gamma)a_{n+1} + \alpha a_n = 0.$$

Using a method similar to the previous one, we obtain the generating function

$$F(x) = \frac{a_0 + (a_1 - (\alpha+\gamma)a_0)x}{1 - (\alpha+\gamma)x + \alpha x^2},$$

where a_0, a_1 are fixed by the initial conditions.

For the bivariate recurrence, the general solution, via characteristic equation, is

$$Y_{mn} = c_1 \lambda_1^{\min\{m,n\}} + c_2 \lambda_2^{\min\{m,n\}},$$

where the coefficients c_1, c_2 are fixed by initial conditions. Introducing the subsets $S_1 = \{m \geq 0,\, n \geq 0,\; m \leq n\}$, $S_2 = \{m \geq 0,\, n \geq 0,\; m > n\}$ of $\mathbb{N}^2$, the generating function becomes

$$F(x,y) = \sum_{S_1} (c_1\lambda_1^m + c_2\lambda_2^m)\,(xy)^m y^{n-m} + \sum_{S_2} (c_1\lambda_1^n + c_2\lambda_2^n)\,(xy)^n x^{m-n}$$

$$= \sum_{k\geq 0} \left[\sum_{m\geq 0} (c_1\lambda_1^m + c_2\lambda_2^m)\,(xy)^m \right] y^k + \sum_{k\geq 1} \left[\sum_{n\geq 0} (c_1\lambda_1^n + c_2\lambda_2^n)\,(xy)^n \right] x^k.$$

Using the univariate generating function, we find $F(x,y) = \frac{G(x,y)}{Q(x,y)}$, where

$$Q(x,y) = 1 - (\alpha+\gamma)xy + \alpha x^2 y^2,$$

$$G(x,y) = \sum_{k\geq 0} \left[Y_{0k} + (Y_{1\,k+1} - (\alpha+\gamma)Y_{0k})xy\right] y^k$$

$$+ \sum_{k\geq 1} \left[Y_{k0} + (Y_{k+1\,1} - (\alpha+\gamma)Y_{k0})xy\right] x^k.$$

3 Discrete multitime periodic coefficients Samuelson-Hicks model

We have in mind the single-time case [12] and similar ingredients with those in Section 9. Instead of constant parameters γ and α, we use variable parameters $\gamma(t)$ and $\alpha(t)$, with $\gamma(t) \in (0,1)$, $\alpha(t) > 0$, which are supposed to be T-diagonal-periodic.

A *discrete multitime periodic coefficients Samuelson-Hicks model* can be either the first order diagonal recurrence system

$$\begin{pmatrix} Y(t+\mathbf{1}) \\ C(t+\mathbf{1}) \end{pmatrix} = \begin{pmatrix} \gamma(t)+\alpha(t) & -\frac{\alpha(t)}{\gamma(t-\mathbf{1})} \\ \gamma(t) & 0 \end{pmatrix} \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix},\ t \in \mathbb{N}^m,$$

subject to the (initial and compatibility) conditions (2.3), (2.4), (2.5), or the second order homogeneous diagonal recurrence equation

$$Y(t+\mathbf{2}) - (\gamma(t+\mathbf{1}) + \alpha(t+\mathbf{1}))Y(t+\mathbf{1}) + \alpha(t+\mathbf{1})Y(t) = 0,\ t \in \mathbb{N}^m,$$

subject to the (initial and compatibility) conditions (2.7), (2.8), (2.9), (2.10).

A prominent role in the analysis of the foregoing recurrences is played by the so-called *Floquet multipliers* of the second order homogeneous diagonal recurrence equation, if its coefficients are T-diagonal-periodic. We reconvert this recurrence into an equivalent diagonal recurrence system

$$\begin{pmatrix} Y(t+\mathbf{1}) \\ Z(t+\mathbf{1}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha(t+\mathbf{1}) & \gamma(t+\mathbf{1})+\alpha(t+\mathbf{1}) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix},\ t \in \mathbb{N}^m.$$

The matrix of this diagonal recurrence system

$$A(t+\mathbf{1}) = \begin{pmatrix} 0 & 1 \\ -\alpha(t+\mathbf{1}) & \gamma(t+\mathbf{1})+\alpha(t+\mathbf{1}) \end{pmatrix}$$

is T-diagonal-periodic. A Floquet multiplier is nothing else than an eigenvalue of the matrix

$$D(t) = \prod_{k=0}^{T-1} A\big(t + (T-k-\mu(t))\cdot \mathbf{1}\big).$$

This matrix is called *monodromy matrix* of the foregoing first order recurrence system. It is a 2×2- matrix with determinant

$$\Delta(t) = \det\, D(t) = \alpha\big(t+(T-\mu(t))\cdot\mathbf{1}\big)\cdot \ldots \cdot \alpha\big(t+(1-\mu(t))\cdot\mathbf{1}\big).$$

Theorem 3.1. *The Floquet multipliers of the second order homogeneous diagonal recurrence equation, with diagonal-periodic coefficients, are the two roots of the quadratic equation $z^2 - z(Tr\ D) + \Delta = 0$. These multipliers depend on $t^1-\mu(t),\ldots,t^m-\mu(t)$.*

For the constant coefficients recurrence equation, the foregoing equation is reduced to $z^2-(\gamma+\alpha)z+\alpha=0$ and the Floquet multipliers are constants.

4 Samuelson-Hicks model as two-time recurrence of way required

Let $t \in \mathbb{N}^2$. A first order linear bi-recurrence of the form

$$x(t^1+1,0) = F_1((t^1,0),x(t^1,0)),\ x(t^1,t^2+1) = F_2((t^1,t^2),x(t^1,t^2)),$$

with initial condition $x(0,0)=x_0$, is called *two-time recurrence of way required.* Of course, similar to this recurrence, we can consider *discrete multitime recurrences of way required* or *discrete multitime nonholonomic recurrences.*

Suppose we have linear two-time recurrence of way required, with constant coefficients,

$$x(t^1+1,0) = A_1 x(t^1,0),\ x(t^1,t^2+1) = A_2 x(t^1,t^2).$$

Theorem 4.1. *The solution (imposed by the path) of a linear two-time recurrence of way required, with constant coefficients, is*

$$x(t^1,t^2) = A_2^{t^2}\, A_1^{t^1}\, x_0.$$

We propose a *first order discrete two-time constant coefficients Samuelson-Hicks model* as a *two-time recurrence of way required*

$$\begin{pmatrix} Y(t^1+1,0) \\ C(t^1+1,0) \end{pmatrix} = \begin{pmatrix} \gamma_1+\alpha_1 & -\frac{\alpha_1}{\gamma_1} \\ \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} Y(t^1,0) \\ C(t^1,0) \end{pmatrix}, \ t \in \mathbb{N}^2,$$

$$\begin{pmatrix} Y(t^1+1,t^2+1) \\ C(t^1+1,t^2+1) \end{pmatrix} = \begin{pmatrix} \gamma_2+\alpha_2 & -\frac{\alpha_2}{\gamma_2} \\ \gamma_2 & 0 \end{pmatrix} \begin{pmatrix} Y(t^1,t^2) \\ C(t^1,t^2) \end{pmatrix}, \ t \in \mathbb{N}^2,$$

with initial condition $Y(0,0) = Y_0, C(0,0) = C_0$.

Solving this recurrence is evident, but the economic sense of the solution will be discussed in another paper.

5 Conclusions

This paper presents original results regarding the multivariate recurrence equations having as main sources the papers [4]-[8]. Our approach to multivariate recurrence equations is advantageous for practical problems. The original results have a great potential to solve problems in various areas such as ecosystem dynamics, financial modeling, and economics.

Though the techniques for deriving multivariate generating functions are sometimes paralleling the univariate theory, they achieve surprising depth in many problems. Analytic methods for recovering coefficients of generating functions once the functions have been derived have, however, been sorely lacking.

The authors lay no claims to the paper's being a complete presentation of all possibilities to introduce and study multitime Samuelson-Hicks models. The material presented here is a reflection of our scientific interests regarding the interaction between mathematics and economics.

Samuelson-Hicks Model of Business Cycles describes in fact the interaction between *multiplier* and *accelerator*. However, the adequate explanation of the multitime business cycles would require the reasons why the system starts moving in a diagonal direction or in the reverse direction or in path imposed direction, say, after striking the ceiling.

We hope to recruit further researchers into economic multivariate recurrences, which still have many interesting challenges to offer, and this explains the rather comprehensive nature of the paper.

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Authors' address:

University Politehnica of Bucharest
Faculty of Applied Sciences
Splaiul Independentei 313
Bucharest 060042, Romania.

Cristian Ghiu
Department of Mathematical Methods and Models
E-mail: crisghiu@yahoo.com

Raluca Tuligă (Coadă), Constantin Udrişte, Ionel Ţevy
Department of Mathematics and Informatics
E-mail: ralucacoada@yahoo.com , udriste@mathem.pub.ro , vascatevy@yahoo.fr