On the group of isometries of the space

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Abstract. In this paper we consider the group of spatial rotations and translations in the universe, in case of a bounded universe and some other assumptions. This compact group is studied and assuming that the universe is simply connected, this group is shown to be isomorphic to Spin(4), i.e. $S^3 \times S^3$, while the observable universe is homeomorphic to $S^3$.

Key words: rotation; translation; Spin(4); isomorphism.

1 Introduction

In this paper we consider a group of all translations and rotations in the space or globally in the universe. It is well known that if the space is $\mathbb{R}^3$, this group is given by $\left\{ \begin{pmatrix} M & h^T \\ 0 & 1 \end{pmatrix} \right\}$, where $M \in SO(3, \mathbb{R}^3)$ and $h$ is the vector of translation. But this affine group must be changed, because the space is bounded, such that this 6-dimensional group of transformations is at least a compact Lie group. Each translation will be considered as a rotation for angle $\phi$, such that the corresponding translation should be $R\phi$, where $R$ is extremely large constant, which may be called radius of the universe. In this paper we give a description of this group.

In order the space and time to be of the same dimension, many authors ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12]) have considered the time as 3-dimensional. In order to preserve the Lorentz transformations (up to isomorphism) it is convenient to replace the group of Lorentz transformations $O^+_1(1,3)$ with its isomorphic Lie group $SO(3, \mathbb{C})$. In [9] the Lorentz transformation is converted into complex orthogonal transformation on $\mathbb{C}^3$ in the frame of the 3-dimensional temporal coordinates. While the Lie algebra of $SO(3, \mathbb{C})$ is given by $\begin{pmatrix} C & A \\ -A & C \end{pmatrix}$, where $A$ and $C$ are antisymmetric $3 \times 3$ matrices, the Lie algebra of the required group $G_{rt}$ of rotations and translations in [11] is introduced by $\begin{pmatrix} C & B \\ B & C \end{pmatrix}$.
where \( C \) and \( B \) are antisymmetric \( 3 \times 3 \) matrices. The matrix \( C \) is the Lie algebra which corresponds to the spatial rotations, i.e. to the Lie algebra of \( \text{SO}(3, \mathbb{R}) \) and the group of unit quaternions \( S^3 \), while \( B \) is an antisymmetric matrix which corresponds to the set of "translations". So, if we put

\[
C = \begin{bmatrix}
0 & -c_3 & c_2 \\
c_3 & 0 & -c_1 \\
-c_2 & c_1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & -b_3 & b_2 \\
b_3 & 0 & -b_1 \\
-b_2 & b_1 & 0
\end{bmatrix},
\]

\((c_1, c_2, c_3)\) determines the direction of 3-vector for a spatial rotation, while \((b_1, b_2, b_3)\) determines the direction of 3-vector of a spatial translation. If \((b_1, b_2, b_3) \sim 1/R\), the vector \( R(b_1, b_2, b_3) \) can be considered approximately as a vector of translation \( \vec{r} \), i.e. \((b_1, b_2, b_3) \approx \vec{r} R\). But the coefficient of proportionality \( 1/R \) is not necessary to be so small constant ([11]), and some important conclusions are given in [11]. For example, it leads to consideration of the motion of a spinning body on a horizontal plane in a gravitational field ([11]), which can be experimentally verified. Analogously as there is not a strict separation between the spatial coordinates and the 1-dimensional temporal coordinate with respect to the Special Relativity, there is not strict separation between the rotations and translations. Analogous to the invariant \( I = -ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 \) from the Spacial Relativity, according to the considered 3+3+3-model of space-time (3 classic spatial coordinates, 3 coordinates about the rotation of the body and 3-temporal coordinates) there appear 4 invariants ([11]):

\[
I_1 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\eta_1^2 + d\eta_2^2 + d\eta_3^2 - c^2 d\theta_1^2 - c^2 d\theta_2^2 - c^2 d\theta_3^2,
\]

\[
I_2 = d\xi_1 d\eta_1 + d\xi_2 d\eta_2 + d\xi_3 d\eta_3,
\]

\[
I_3 = d\xi_1 d\theta_1 + d\xi_2 d\theta_2 + d\xi_3 d\theta_3,
\]

\[
I_4 = d\eta_1 d\theta_1 + d\eta_2 d\theta_2 + d\eta_3 d\theta_3,
\]

where \( d\xi_1, d\xi_2, d\xi_3 \) correspond to the space displacement, \( d\eta_1, d\eta_2, d\eta_3 \) are induced by the rotation and \( d\theta_1, d\theta_2, d\theta_3 \) correspond to the temporal displacement. The last two invariants \( I_3 \) and \( I_4 \) are equal to 0. The geometrical/physical interpretations of these invariants are presented in [11]. While in the paper [11] it is shown that \( I_1, I_2, I_3 \) are invariants and some applications are given, in this paper we consider in more details the group \( G_{rt} \) and show that its universal covering is isomorphic to \( \text{Spin}(4) \).

## 2 On the compact group of all translations and rotations

After the previous introduction we study now the mentioned group \( G_{rt} \) of rotations and translations. According to the Lie algebra of \( G_{rt} \), the 6-dimensional Lie group of rotations and translations \( G_{rt} \) locally is generated by the following 6 subgroups of matrices:

\[
R_{x, \alpha} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha & 0 & 0 & 0 \\
0 & \sin \alpha & \cos \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \alpha & -\sin \alpha \\
0 & 0 & 0 & 0 & \sin \alpha & \cos \alpha
\end{bmatrix},
\]
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\[
R_{y,\beta} = \begin{bmatrix}
\cos \beta & 0 & \sin \beta & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-\sin \beta & 0 & \cos \beta & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \beta & 0 & \sin \beta \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\sin \beta & 0 & \cos \beta
\end{bmatrix},
\]

\[
R_{z,\gamma} = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 & 0 & 0 & 0 \\
\sin \gamma & \cos \gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \cos \gamma & -\sin \gamma & 0 \\
0 & 0 & \sin \gamma & \cos \gamma & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
T_{x,\alpha} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \alpha & 0 & 0 & 0 & \sin \alpha \\
0 & 0 & \cos \alpha & 0 & -\sin \alpha & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \sin \alpha & 0 & \cos \alpha & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
T_{y,\beta} = \begin{bmatrix}
\cos \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \beta & \sin \beta & 0 & 0 \\
0 & 0 & -\sin \beta & \cos \beta & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\sin \beta & 0 & 0 & 0 & 0 & \cos \beta
\end{bmatrix},
\]

\[
T_{z,\gamma} = \begin{bmatrix}
\cos \gamma & 0 & 0 & 0 & \sin \gamma & 0 \\
0 & \cos \gamma & 0 & -\sin \gamma & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \sin \gamma & 0 & \cos \gamma & 0 & 0 \\
-\sin \gamma & 0 & 0 & \cos \gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

where the first three are rotations around \(x\), \(y\) and \(z\) axes for angles \(\alpha\), \(\beta\) and \(\gamma\) respectively, while the last three are "translations" along the \(x\), \(y\) and \(z\) axes for lengths \(R_\alpha\), \(R_\beta\) and \(R_\gamma\). All these matrices are orthogonal \(6 \times 6\) matrices of type \[
\begin{bmatrix}
A & B \\
B & A
\end{bmatrix},
\]
and if \(1/R^2\) is neglected, then this group becomes the classical group of rotations and translations, analogously as the Special Relativity reduces to the classical Newtonian mechanics if we neglect \(c^{-2}\).

**Theorem 2.1.** (a) The mappings

\[
R_{x,\alpha} \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}, \quad R_{y,\beta} \mapsto \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix},
\]

\[
R_{z,\gamma} \mapsto \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad T_{x,\alpha} \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{bmatrix},
\]
generate locally an epimorphism of groups \( \varphi : G_{rt} \to SO(3, \mathbb{R}) \);

(b) The mappings
\[
R_{x,\alpha} \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix},
\quad R_{y,\beta} \mapsto \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix},
\quad R_{z,\gamma} \mapsto \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix},
\quad T_{x,\alpha} \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix},
\quad T_{y,\beta} \mapsto \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix},
\quad T_{z,\gamma} \mapsto \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

generate locally an epimorphism of groups \( \psi : G_{rt} \to SO(3, \mathbb{R}) \);

(c) There is a local isomorphism \( \theta : G_{rt} \to SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \), which is defined by \( \theta(M) = \varphi(M) \times \psi(M) \).

Proof. Let us consider the matrices of type \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \) which belong to \( SO(6, \mathbb{R}) \) and which are path connected with \( \text{diag}(1,1,1,1,1,1) \). This implies

\[
\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} A^T & B^T \\ B^T & A^T \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}
\]

and consequently

\begin{equation}
(2.1) \quad AA^T + BB^T = I \quad \text{and} \quad AB^T + BA^T = O.
\end{equation}

It is easy to verify that these matrices give a connected subgroup \( G \) of \( SO(6, \mathbb{R}) \). Its Lie algebra coincides with the Lie algebra of \( G_{rt} \), which was previously described. So, locally the Lie group \( G_{rt} \) can be considered that is given by the matrices \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \), where \( A \) and \( B \) are \( 3 \times 3 \) matrices which satisfy the conditions (2.1).

(a) It is sufficient to prove that the given 6 mappings generate an epimorphism of groups \( \varphi : G \to SO(3, \mathbb{R}) \). If there exists such a homomorphism \( \varphi : G \to SO(3, \mathbb{R}) \) with the required conditions, then it is unique. So we will prove that such a homomorphism exists and it is given by

\begin{equation}
(2.2) \quad \varphi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = A + B.
\end{equation}

Using that the conditions (2.1) are satisfied, we obtain

\[
(A + B)(A + B)^T = (A + B)(A^T + B^T) =
\]
\[
= (AA^T + BB^T) + (AB^T + BA^T) = I + O = I,
\]

so

\[
\varphi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = A + B.
\]
and so \( A + B \in O(3, \mathbb{R}) \). Moreover, since \( \varphi : G \to O(3, \mathbb{R}) \) is a continuous mapping, \( G \) is a connected manifold and \( \varphi \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) = I \in SO(3, \mathbb{R}) \), we notice that \( \varphi : G \to SO(3, \mathbb{R}) \). This mapping is homomorphism because

\[
\varphi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} C & D \\ D & C \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} AC + BD & AD + BC \\ AD + BC & AC + BD \end{bmatrix} \right) = AC + BD + AD + BC = (A + B)(C + D) = \varphi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \cdot \varphi \left( \begin{bmatrix} C & D \\ D & C \end{bmatrix} \right).
\]

This mapping is also surjective, because for any matrix \( A \in SO(3, \mathbb{R}) \), \( A = \varphi \left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) \) and the matrix \( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \) is obviously path connected with \( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \). Finally, the required 6 conditions are trivially satisfied.

(b) In this case instead of (2.2) we have

\[
(2.3) \quad \psi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = A - B.
\]

and then proof is analogous to the proof in case (a).

(c) Firstly it is sufficient to prove that \( \theta : G \to \varphi(G) \times \psi(G) \) defined by \( \theta(M) = \varphi(M) \times \psi(M) \) is isomorphism.

\( \theta \) is injection: Assume that

\[
\varphi \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} C & D \\ D & C \end{bmatrix} \right), \quad \text{i.e.} \quad (A + B, A - B) = (C + D, C - D).
\]

Hence \( A = C \) and \( B = D \) and thus \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} C & D \\ D & C \end{bmatrix} \).

\( \theta \) is surjection: Assume that \((A, B) \in SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \). Then

\[
\theta \left( \begin{bmatrix} (A + B)/2 & (A - B)/2 \\ (A - B)/2 & (A + B)/2 \end{bmatrix} \right) = (A, B).
\]

Now, using that \( AA^T = BB^T = I \), it is easy to prove that

\[
\frac{A + B}{2} \left( \frac{A + B}{2} \right)^T + \frac{A - B}{2} \left( \frac{A - B}{2} \right)^T = I
\]

and

\[
\frac{A + B}{2} \left( \frac{A - B}{2} \right)^T + \frac{A - B}{2} \left( \frac{A + B}{2} \right)^T = 0.
\]

Moreover, since the mappings (2.2) and (2.3) are homomorphisms, we obtain that \( \theta \) is isomorphism.

We should only to prove that \( \begin{bmatrix} (A + B)/2 & (A - B)/2 \\ (A - B)/2 & (A + B)/2 \end{bmatrix} \) is path connected to

\[
\begin{bmatrix} I & O \\ O & I \end{bmatrix}
\]

in \( G \). This is obvious because \( \begin{bmatrix} (A + B)/2 & (A - B)/2 \\ (A - B)/2 & (A + B)/2 \end{bmatrix} \) is path connected to

\[
\begin{bmatrix} (A + I)/2 & (A - I)/2 \\ (A - I)/2 & (A + I)/2 \end{bmatrix}
\]

in \( G \) and
\[ \begin{bmatrix} \frac{A+I}{2} & \frac{A-I}{2} \\ \frac{A-I}{2} & \frac{A+I}{2} \end{bmatrix} \text{ is path connected to } \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ in } G. \]

Since the group \( G_{rt} \) was defined via its Lie algebra, it is determined only locally. If \( G_{rt} = G \), then \( G_{rt} \) is isomorphic to \( SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \). But if we require for the universe to be simply connected, and also for the group of rotations to be simply connected because of symmetry between them, then \( G_{rt} \) is also simply connected Lie group. In this case \( G_{rt} \) is homeomorphic to \( S^3 \times S^3 \) as universal covering of \( SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \).

## 3 Connection with the group \( Spin(4) \)

Finally we come to the following question: What is the shape of the universe? First we state the following proposition, which is a consequence of the second equality in (2.1).

**Proposition 3.1.** If \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \) belongs to the group \( G \) and \( A \) is non-singular, then \( A^{-1}B \) is an antisymmetric matrix.

If we neglect the rotations in the Lie group \( G_{rt} \), we obtain a subset \( U \) of \( G_{rt} \) which is the spatial part of the universe, but it is not a subgroup of \( G_{rt} \), similarly as the set of Lorentz boosts is not a subgroup of the Lorentz group of transformations. The products of two matrices in \( U \) as well as two Lorentz boosts, contain small spatial rotations. The set \( U \) of the spatial part of the universe locally is given by the 3-dimensional antisymmetric matrix \( A^{-1}B \) from the Proposition 3.1. Using the representations of \( T_{x,\alpha}, T_{y,\beta} \) and \( T_{z,\gamma} \), the corresponding antisymmetric matrices \( A^{-1}B \) are

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \tan \alpha \\
0 & -\tan \alpha & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & -\tan \beta \\
0 & 0 & 0 \\
\tan \beta & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & \tan \gamma & 0 \\
-\tan \gamma & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Hence, if we consider the group \( G_{rt} \) to be isomorphic to \( SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \), then the subset \( U \) is homeomorphic to \( SO(3, \mathbb{R}) \), i.e. \( RP^3 \) and then the function \( \tan \varphi \) from the previous matrices is considered as a function with period \( \pi \). But it is more convenient to consider the group \( G_{rt} \) to be isomorphic to \( S^3 \times S^3 \) and then the subset \( U \) is homeomorphic to \( S^3 \). In this case the function \( \tan \varphi \) from the previous matrices is considered as a function with period \( 2\pi \), because for a given angle \( \varphi \) and given direction correspond two different points from \( U \) which lead to the same matrix representation \( A^{-1}B \).

Now if we consider the group of isometries of the sphere \( S^3 \) which preserve orientation, we obtain again the same group \( S^3 \times S^3 \). Indeed, this group can be obtained if we consider \( S^3 \) embedded into \( \mathbb{R}^4 \). Locally this group is \( SO(4, \mathbb{R}) \), but its universal covering group \( \text{Spin}(4) \) is isomorphic to \( S^3 \times S^3 \). We skip the proof of the following theorem, because by straight verification it can be proved.

**Theorem 3.2.** The mapping

\[
\begin{bmatrix} A & b^T \\ -b & 0 \end{bmatrix} \mapsto \begin{bmatrix} A & B \\ B & A \end{bmatrix},
\]
where $B$ is antisymmetric matrix such that $B_{12} = b_3$, $B_{23} = b_1$, $B_{31} = b_2$, defines an isomorphism between the Lie algebras of $SO(4, \mathbb{R})$ and $G_{rt}$.

According to Theorem 3.2 we obtain the isomorphism between the universal coverings of the considered two groups, which is indeed $\text{Spin}(4)$. This alternative representation of the group $G_{rt}$ as isometries (which preserve orientation) of $S^3$ embedded into $\mathbb{R}^4$ leads to the following invariant element. Assuming that $d\theta_1 = d\theta_2 = d\theta_3 = 0$, then the invariant $I_1$ in 4 dimensions reduces to $I_1 = d\eta_{12} + d\eta_{23} + d\eta_{31} + d\xi_{2} = dx^2 + dy^2 + dz^2 + d\xi_2$, where $d\xi_2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2$, but we do not know the direction of $(d\xi_1, d\xi_2, d\xi_3)$. This invariant is analogous to the well know invariant from the Special Relativity $-ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$, which is also a special case of the invariant $I_1$.

4 Conclusion

Although the subset of translations, which is homeomorphic to the visible part of the universe from a chosen point, is homeomorphic to $S^3$, it is not convenient to consider it as embedded in $\mathbb{R}^4$. Also, it is not convenient to endow it with linear connection as a submanifold of $\mathbb{R}^4$. It is much more convenient to endow the total space $G_{rt}$ with a flat linear connection and non-zero torsion tensor arising from the Lie group structure, such that it admits 6 parallel vector fields. The Lie group $G_{rt}$ of isometries acts on itself, but not on a different space. Analogously the group $SO(3, \mathbb{C})$ acts on itself, instead of the Lorentz group which acts on the 4-dimensional space-time points.

References

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