Abstract. The aim of the paper is to construct projectable Bott linear connections in the lifted foliation on the transverse bundle of a foliation, using linear and nonlinear transverse connections. Considering a connection adapted to a Hamiltonian foliation, one lift it on the transverse bundle and one prove that the lifted foliation is a Riemannian one (as proved by one of authors, the last property is fulfilled automatically if the Hamiltonian is 2–homogeneous). The results extend similar ones of Miernowski and Mozgawa.

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1 Preliminaries

Let us consider $M$ an $(n + m)$-dimensional manifold which will be assumed to be connected and orientable.

A codimension $n$ foliation $\mathcal{F}$ on $M$ is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of $M$;

(ii) For every $i \in I$, $\varphi_i : U_i \to T$ are submersions, where $T$ is an $m$-dimensional manifold, called transversal manifold;

(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

\begin{equation}
\varphi_j = f_{i,j} \circ \varphi_i
\end{equation}

for every $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

Every fiber of $\varphi_i$ is called a plaque of the foliation. Condition (1.1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by $\varphi_i$ and $\varphi_j$ coincides. The
manifold $M$ is decomposed into a family of disjoint immersed connected submanifolds of dimension $m$; each of these submanifolds is called a leaf of $\mathcal{F}$.

We say that $\mathcal{F}$ is transversely orientable if on $T$ can be given an orientation which is preserved by all $f_{i,j}$. By $T\mathcal{F}$ we denote the tangent bundle to $\mathcal{F}$ and $\Gamma(\mathcal{F})$ is the space of its global sections i.e. vector fields tangent to $\mathcal{F}$.

In this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(x^u, x^v)$ $u = 1, \ldots, m$, $\bar{u} = 1, \ldots, n$ on an open subset $\bar{U}$ on which the foliation is trivial and defined by the equations $dx^v = 0$, $\bar{u} = 1, \ldots, n$.

We notice that the total spaces of the conormal bundle $Q^*\mathcal{F}$ of $\mathcal{F}$ carries a natural foliation $\tilde{\mathcal{F}}$ of codimension 2 such that the leaves of $\tilde{\mathcal{F}}$ are covering spaces of the leaves of $\mathcal{F}$, and it is called the natural lift of $\mathcal{F}$ to its conormal bundle $Q^*\mathcal{F}$.

If we denote by $\{dx^u\}$, $\bar{u} = 1, \ldots, n$ the corresponding local coframe on $Q^*\mathcal{F}$ then we can induce a chart $(x^u, p_\bar{u}, x^v)$ on $Q^*\mathcal{F}$ where $p_\bar{u}dx^u \in \Gamma(Q^*\mathcal{F})$, and the system of equations $x^\bar{u} = \text{const.}$, $p_\bar{u} = \text{const.}$ defines the foliation $\tilde{\mathcal{F}}$.

Let $Q\tilde{\mathcal{F}} = T(Q^*\mathcal{F})/T\mathcal{F}$ be the normal bundle of the foliated manifold $(Q^*\mathcal{F}, \tilde{\mathcal{F}})$. The vectors $\left\{ \frac{\partial}{\partial x^u}, \frac{\partial}{\partial p_\bar{u}} \right\}$, $\bar{u} = 1, \ldots, n$ form a natural frame of $Q\tilde{\mathcal{F}}$ at the point $(x^u, p_\bar{u}, x^v) \in Q^*\mathcal{F}$. The canonical projection $\pi: Q^*\mathcal{F} \to M$ given by $\pi(x^u, p_\bar{u}, x^v) = (x^u, x^v)$ induces another projection $\pi_*: T(Q^*\mathcal{F}) \to TM$ which maps the tangent vectors to $\tilde{\mathcal{F}}$ in the vectors tangent to $\mathcal{F}$. Thus $\pi_*$ induces a mapping $\tilde{\pi}_*: Q\tilde{\mathcal{F}} \to Q\mathcal{F}$ and denote by $V(Q^*\mathcal{F}) = \ker \tilde{\pi}_*$ which is a vertical bundle spanned by the vectors $\left\{ \frac{\partial}{\partial p_\bar{u}} \right\}$, $\bar{u} = 1, \ldots, n$.

Lemma 1.1. Let $o: M \to Q^*\mathcal{F}$ be the zero section of the conormal bundle $Q^*\mathcal{F}$. Then the set $o(M)$ is saturated on $Q^*\mathcal{F}$ with leaves of the foliation $\tilde{\mathcal{F}}$.

## 2 Foliated vector bundles

Given a foliated manifold $(M, \mathcal{F})$, we say that a vector bundle $p: E \to M$ of rank $E = k$ is a foliated vector bundle if there is a foliated vector bundle atlas on $E$ (i.e. the transition matrices are basic functions as components). There is a foliation $\mathcal{F}_E$ on $E$ such that the canonical projection $\pi$ is a foliated map that induces a local diffeomorphism on leaves. For example, the transverse bundles, as well as the vertical transverse bundles are foliated vector bundles. The projection $p$ of the slashed bundle $p: \tilde{E} \to M$ is a foliated map.

A foliated vector bundle map $F: E' \to E$ over the foliated map $f: M' \to M$ is defined in a similar way, asking that $F$ be covered by foliated vector bundle maps in a foliated vector bundle atlas on $E$, $M$, $E'$ and $M'$ (i.e. the local matrices are basic functions as components). Analogous one can consider the definition of a foliated vector subbundle etc.

For a foliated vector bundle $p: E \to M$, considering the slashed bundle $p: \tilde{E} \to M$, where $\tilde{E} = E - \{\text{zero section}\}$, the differential map $p_* : T\tilde{E} \to TM$ induces a foliated vector bundle map $\tilde{p}_*: Q\mathcal{F}_{\tilde{E}} \to Q\mathcal{F}$ that is surjection on fibers. We denote by $V(Q\mathcal{F}_{\tilde{E}}) = \ker \tilde{p}_*$; it is a foliated vector subbundle of $Q\mathcal{F}_{\tilde{E}}$ (over the base $\tilde{E}$), we call it as the foliated vertical bundle of $E$, and we denote by $I : V(Q\mathcal{F}_{\tilde{E}}) \to Q\mathcal{F}_{\tilde{E}}$ the foliated inclusion. A transverse non-linear connection on $E$ is a foliated subbundle
\( H(QF_E) \subset QF_E \) such that

\[
(2.1) \quad QF_E = V(QF_E) \oplus H(QF_E).
\]

As in the non-foliated case, a transverse non-linear connection is equivalently defined by \( H(QF_E) = \ker \tilde{C} \), where \( \tilde{C} \) is a left splitting of the inclusion \( I : V(QF_E) \to QF_E \), i.e. a foliated epimorphism \( \tilde{C} : QF_E \to V(QF_E) \) such that \( \tilde{C} \circ I = 1_{V(QF_E)} \).

Using local coordinates
- \((x^u, x^a), u = 1, \ldots, m, \bar{u} = 1, \ldots, n \) on \( M \),
- \((x^u, x^a, y^a), u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k \) on \( E \),
- \((x^u, \bar{x}^a, \bar{x}^a, \bar{y}^a, \bar{X}^a, \bar{Y}^a), u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k \) on \( QF_E \),

then some coordinates \((x^u, \bar{x}^a, \bar{y}^a, \bar{Y}^a), u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k \) follow on \( V(QF_E) \) and the foliated epimorphism \( \tilde{C} \) has the local form

\[
(2.2) \quad (x^u, x^a, \bar{y}^a, \bar{X}^a, \bar{Y}^a) \xrightarrow{\tilde{C}} (x^u, x^a, \bar{y}^a, N^a_\bar{a}(x^u, \bar{y}^a)X^\bar{a} + \bar{Y}^a).
\]

Notice that the local correspondences

\[
(x^u, \bar{y}^a, \bar{X}^a, \bar{Y}^a) \xrightarrow{\tilde{C}} (x^u, \bar{y}^a, N^a_\bar{a}(x^u, \bar{y}^a)X^\bar{a} + \bar{Y}^a)
\]

are non-linear connections on the transverse model; we call them as the local projected (non-linear) connections.

We say that a transverse non-linear connection is
- 1–homogeneous, if the local functions \((x^u, \bar{y}^a) \to N^a_\bar{a}(x^u, \bar{y}^a)\) are 1–homogeneous in the second group of variables, i.e.

\[
N^a_\bar{a}(x^u, \lambda \bar{y}^a) = \lambda N^a_\bar{a}(x^u, \bar{y}^a), \quad (\forall) \lambda > 0;
\]

- linear, if the local functions \((x^u, \bar{y}^a) \to N^a_\bar{a}(x^u, \bar{y}^a)\) are linear in the second group of variables, i.e.

\[
N^a_\bar{a}(x^u, \bar{y}^a) = \Gamma^a_{\bar{a} \bar{b}}(x^u)\bar{y}^\bar{b}.
\]

Considering a transverse non-linear connection, the local Berwald connections associated with the local projected non-linear connections glue together to a transverse linear connection on the foliated vector bundle \( V(QF_E) \), that we call as the transverse Berwald connection associated with the given transverse non-linear connection.

Using local coordinates as above, if a transverse non-linear connection that the left splitting \( \tilde{C} \) has the local form \((2.2)\), has its transverse Berwald linear connection \( \nabla \) given by

\[
(x^u, x^a, \bar{y}^a, Y^a, \bar{X}^a, \bar{Z}^a, W^a) \xrightarrow{\nabla} (x^u, x^a, \bar{y}^a, Y^a, \bar{Z}^a, \partial N^a_\bar{a}(x^u, \bar{y}^a)X^\bar{a}Y^\bar{a} + W^a).
\]

Let us denote by \( \Pi : T\tilde{E} \to QF_E \) the canonical projection on the normal bundle of \( F_E \); it induces also a vector bundle isomorphism \( \tilde{\Pi} : V(T\tilde{E}) \to V(QF_E) \), that corresponds canonically to the identity map of \( p^*_E E \). A non-linear connection \( \tilde{C} : \)
$T\tilde{E} \rightarrow V(T\tilde{E})$ is projectable if there is a foliated map $\tilde{C} : Q\mathcal{F}_E \rightarrow V(Q\mathcal{F}_E)$ such that the following diagram is commutative.

$$
\begin{align*}
T\tilde{E} & \xrightarrow{C} VT\tilde{E} \\
\Pi \downarrow & \downarrow \Pi \\
Q\mathcal{F}_E & \xrightarrow{\tilde{C}} V(Q\mathcal{F}_E)
\end{align*}
$$

(2.3)

It is easy to see that $\tilde{C}$ is unique and it is a left splitting of the inclusion $\tilde{I} : V(Q\mathcal{F}_E) \rightarrow T\tilde{E}$, thus a transverse non-linear connection on $\tilde{E}$.

We say that a non-linear connection $C : T\tilde{E} \rightarrow V(T\tilde{E})$ is of Bott type if $X^b \in \Gamma(T\mathcal{F}_E)$, $(\forall)X \in \Gamma(T\mathcal{F})$, where $X^b$ is the horizontal lift and $T\mathcal{F}$ is the tangent bundle to the leaves of $\mathcal{F}$.

**Proposition 2.1.** If $\tilde{C}$ is a transverse non-linear connection, then there is a unique Bott type nonlinear connection that is projectable on $\tilde{C}$.

We say that a Bott type nonlinear connection on $Q\mathcal{F}$ or on $Q^*\mathcal{F}$ is a projectable Bott nonlinear connection if it obtained as in the above Proposition 2.1.

Let us use some local coordinates $(x^u, x^a, y^a)$ on a foliated vector bundle $E$. If the transverse non-linear connection $\tilde{C} : Q\mathcal{F}_E \rightarrow V(Q\mathcal{F}_E)$ has the local form $(x^u, x^a, y^a, X^a, Y^a) \xrightarrow{\tilde{C}} (x^u, x^a, y^a, X^a\tilde{N}_a^b(x^u, y^b) + Y^a)$, then its Bott connection has the local form

$$
(x^u, x^a, y^a, X^a, Y^a) \xrightarrow{C} (x^u, x^a, y^a, X^a\tilde{N}_a^b(x^u, y^b) + Y^a).
$$

(2.4)

This is also the general form of a Bott non-linear connection. A simple characterization of a Bott connection is as follows.

**Proposition 2.2.** A projectable non-linear connection is a Bott connection iff $T\mathcal{F}_E \subset H(T\tilde{E})$.

In the case when $\tilde{C}$ comes from a transverse linear connection on $E$, denoted by $\tilde{\nabla}$, then the Bott type (non-linear) connection $C$ comes from a linear connection $\nabla$ on $E$ and conditions on $\nabla$ reads:

- if $s(Y)$ is a locally transverse vector field to $\mathcal{F}_E$ and $A$ is a local foliated section in $\Gamma(E)$, then $\nabla_{s(Y)}A$ is a local foliated section in $\Gamma(E)$ (projectability condition) and
- if $X$ is tangent to $\mathcal{F}_E$ and $A \in \Gamma(E)$ is foliated, then $\nabla_X A = 0$ (Bott condition).

If $E = Q\mathcal{F}$ and $\Pi_0 : TM \rightarrow Q\mathcal{F}$ is the canonical projection, then the Bott condition becomes the classical one:

- if $X$ is tangent to $\mathcal{F}$ and $A = \Pi_0(Y) \in \Gamma(Q(\mathcal{F}))$, then $\nabla_X \Pi_0(Y) = \Pi_0[X, Y]$.

Using formula (2.4) in the case of a linear connection, we obtain that the following statement is true.

**Proposition 2.3.** If $\tilde{\nabla} : \Gamma(Q\mathcal{F}_E) \rightarrow \Gamma(Q^*\mathcal{F}_E \otimes Q\mathcal{F}_E)$ is a transverse linear connection on a foliated bundle $E$, then there is a unique Bott connection $\nabla : \Gamma(Q\mathcal{F}_E) \rightarrow \Gamma(T^*\tilde{E} \otimes Q\mathcal{F}_E)$ that locally projects on $\tilde{\nabla}$. 

In particular if $E = Q^*F$, we say that a Bott connection on $Q\tilde{F}$ is a projectable Bott connection if it obtained as in Proposition 2.3.

The link between the Bott condition for a non-linear connection and a linear connection is given by the Berwald linear connection.

**Proposition 2.4.** If $C$ is a Bott non-linear connection on a foliated vector bundle, then its Berwald linear connection is a Bott linear connection.

We can use in the proof that the Berwald linear connection $\nabla$ of $C$ has the property that $\nabla_X A = v [X^h, A]$, for any $X \in \mathcal{X}(M)$ and $A \in \Gamma(V(TE))$, where the projector $v$ and the lift $h$ are according to $C$.

Notice that, in general, the converse of Proposition 2.4 does not hold.

### 3 Foliations and connections

In this section we consider a connection adapted to a Hamilton foliation and we show also that the lifted foliation of a Hamilton foliation in the conormal bundle is a Riemannian foliation. Using [5], it follows that any Cartan foliation coming from a transverse Finsler metric is a Riemannian foliation.

We say that a foliation $\mathcal{F}$ is a transverse Hamiltonian one if there is a basic function $H : Q^*\mathcal{F} \to \mathbb{R}$ that has a non-degenerate vertical Hessian $h$, called a transverse Hamiltonian. For every $X \in \mathcal{X}(T(Q^*\mathcal{F}))$ we have $\Pi(X) = \tilde{X} \in \Gamma(Q\tilde{F})$, where $\Pi : T(Q^*\mathcal{F}) \to Q\tilde{F}$ is the canonical projection.

The inverse $h^{-1}$ of Hessian $h$ induces a vector bundle isomorphism $J_h : \pi^*_0 Q^*\tilde{F} \to \pi^*_0 Q\tilde{F} \cong V(Q^*\mathcal{F}) = V(Q\tilde{F})$, called the musical isomorphism. We can consider now the vector bundle map $J : \Gamma(Q\tilde{F}) \to \Gamma(V(Q^*\mathcal{F})) \subset \Gamma(Q\tilde{F})$, $J(X) = J_0(\tilde{\pi}_*(X))$, where $\tilde{\pi}_* : Q\tilde{F} \to Q\mathcal{F}$ is the canonical transverse projection. It is easy to see that $J \circ J = 0$ and $\text{Im } J = V(Q^*\mathcal{F})$, thus $J$ is a vector bundle epimorphism. A transverse non-linear connection can be given by an almost product endomorphism $P$ in the fibers of $Q\tilde{F}$ (i.e. $P^2 = 1_{Q\tilde{F}}$) such that the vectors in the fibers of $V(Q^*\mathcal{F})$ are exactly the eigenvectors corresponding to the eigenvalue $-1$ of $P$. The link between $P$ and the transverse map $C : Q\tilde{F} \to V(Q^*\mathcal{F})$ is $C = \frac{1}{2} (1_{Q\tilde{F}} - P)$. We denote by $\tilde{L}$ the transverse Lie derivation.

**Proposition 3.1.** Let $H : Q^*\mathcal{F} \to \mathbb{R}$ be a transverse Hamiltonian, $X$ and $J_0$ be a transverse vector field for $\tilde{F}$ and the musical isomorphism, respectively. Then $P = -L_X J : \Gamma(Q\tilde{F}) \to \Gamma(Q\tilde{F})$ is an almost product endomorphism giving a transverse non-linear connection.

Let now $\nabla^v : \Gamma(V(Q^*\mathcal{F})) \to \Gamma\left(Q^*\tilde{F} \otimes V(Q^*\mathcal{F})\right)$ be a transverse linear connection, that we call a transverse vertical connection.

In the sequel we will use the basis $\left\{ \frac{\partial}{\partial x^a}, \frac{\partial}{\partial p_a} \right\}$, called adapted, as well as its dual $\{dx^a, dp_a = dp_a + N_{aw} dx^w\}$, accordingly to the decomposition (2.1). Using this coframe we can define the local connection forms by

$$\nabla^v \frac{\partial}{\partial p_a} = \omega^v_a \otimes \frac{\partial}{\partial p_a},$$

(3.1)
where

\begin{equation}
\omega^\alpha_a = \Gamma^\alpha_{\gamma a} dx^\gamma + \Gamma^\alpha_\delta dp_\delta = \left( \Gamma^\alpha_\gamma - \Gamma^\alpha_\delta N_\delta \right) dx^\gamma + \Gamma^\alpha_\delta dp_\delta = H^\alpha_\gamma dx^\gamma + \Gamma^\alpha_\delta dp_\delta.
\end{equation}

But \( V(Q^*F) \) and \( H(Q^*F) \) are dual vector bundles, thus the linear connection \( \nabla^v \) on \( V(Q^*F) \) give rise to a dual linear connection \( \nabla^h \) on \( H(Q^*F) \). Thus we can construct a linear connection \( \nabla \) in \( Q\tilde{F} \)

\begin{equation}
\nabla_X Y = \nabla_X^v (v(Y)) + \nabla_X^h (h(Y)),
\end{equation}

where \( Y \in \Gamma(Q\tilde{F}) \), \( X \in \Gamma(T(Q^*F)) \) and \( v : Q\tilde{F} \to V(Q^*F) \) and \( h : Q\tilde{F} \to H(Q^*F) \) are the vertical and horizontal projector respectively from decomposition (2.1). In particular we have

\begin{equation}
\nabla \frac{\delta}{\delta x^a} = -\omega^a_\delta \odot \frac{\delta}{\delta x^a},
\end{equation}

where \( \omega^a_\delta \) is given in (3.2).

Let us remark that we can consider a transverse linear connection \( \nabla^h : \Gamma (H(Q^*F)) \to \Gamma \left( Q^*\tilde{F} \otimes H(Q^*F) \right) \), that we call a transverse horizontal connection and then associate a dual transverse linear connection \( \nabla^v \), that is a vertical connection. The constructions of \( \nabla^h \) from \( \nabla^v \) and of \( \nabla^v \) from \( \nabla^h \) are mutually inverse, giving rise to a same transverse linear connection \( \nabla \).

If \( \varphi \in \Gamma \left( Q^*\tilde{F} \otimes Q\tilde{F} \right) \) is an 1–form with values in \( Q\tilde{F} \) locally given by

\begin{equation}
\varphi = \varphi^a \odot \frac{\delta}{\delta x^a} + \varphi_\delta \odot \frac{\partial}{\partial p_\delta},
\end{equation}

then following [1], [2], we can define an exterior differential \( D\varphi \) putting

\begin{equation}
D\varphi = (d\varphi^a + \varphi^\delta \wedge \omega^a_\delta) \odot \frac{\delta}{\delta x^a} + (d\varphi_\delta - \varphi^a \wedge \omega^a_\delta) \odot \frac{\partial}{\partial p_\delta}.
\end{equation}

A straightforward calculus show that the above formula is well-defined.

The bundle \( Q^*\tilde{F} \otimes Q\tilde{F} \) admits a natural section \( \eta \) given by

\begin{equation}
\eta = dx^a \otimes \frac{\partial}{\partial x^a} + dp_\delta \otimes \frac{\partial}{\partial p_\delta} = dx^a \otimes \frac{\delta}{\delta x^a} + \delta p_\delta \otimes \frac{\partial}{\partial p_\delta}.
\end{equation}

It is clear that the form \( \eta \) is well-defined.

The form \( \theta = D\eta \) is called the torsion form of the connection \( \nabla^h \) or its dual \( \nabla^v \).

Locally the form \( \theta \) can be expressed as follows:

\begin{equation}
D\eta = (dx^a \wedge \omega^a_\gamma) \odot \frac{\delta}{\delta x^a} + (d(p_\delta) - \delta p_\gamma \wedge \omega^a_\gamma) \odot \frac{\partial}{\partial p_\delta} = \theta^a \odot \frac{\delta}{\delta x^a} + \theta_\delta \odot \frac{\partial}{\partial p_\delta},
\end{equation}

where

\begin{equation}
\theta^a = \frac{1}{2} \left( H^a_\gamma^\delta - H^a_\delta^\gamma \right) dx^a \wedge dx^\delta - \Gamma^a_\delta dx^a \wedge \delta p_\delta,
\end{equation}
\((3.10)\quad \theta_v = -dN_v \wedge dx^\gamma - H^{\alpha}_e \delta p_\alpha \wedge dx^\delta - \frac{1}{2} \left( \Gamma^{\alpha \delta}_{\gamma e} - \Gamma^{\delta \alpha}_{\gamma e} \right) \delta p_\alpha \wedge \delta p_\delta.\)

The first term and the last one in formulas (3.9) and (3.10) respectively give two global transverse tensors that we call **horizontal torsion** and **vertical torsion** respectively.

Using formulas (3.9) and (3.10), it is easy to check that

a) the horizontal torsion vanishes if \( \theta (V, W) = 0, \) for all \((V, W) \in \Gamma (H(Q^*F))\) and

b) the vertical torsion vanishes if \( \theta (V, W) = 0, \) for all \((V, W) \in \Gamma (V(Q^*F))\).

If \( \nabla^h \) and \( \nabla^v \) are dual and a horizontal and a vertical transverse connection respectively, then, according to Proposition 2.3, they project to two projectable Bott connections \( \nabla^h \) and \( \nabla^v \) respectively.

We say that the **horizontal and vertical torsions** of \( \nabla^h \) and \( \nabla^v \) are just the horizontal and vertical torsions of \( \nabla^h \) and \( \nabla^v \) respectively.

**Proposition 3.2.** If \( \bar{g} \) is a non-degenerated and symmetric transverse bilinear form in the fibers of \( H(Q^*F) \) and \( \bar{N} \) is a Bott type nonlinear connection of \( Q\bar{F} \), then there is a unique projectable Bott linear connection \( \nabla^h \), in the horizontal bundle \( H(Q^*F) \), such that

1) \( \nabla^h \) has null horizontal and vertical torsions and

2) \( \bar{g} \) is parallel with respect to \( \nabla^h \), i.e. \( \nabla^h _X \bar{g} = 0, \) for all \( X \in \mathcal{X}(Q\bar{F}) \).

**Proposition 3.3.** If \( g \) is a non-degenerated and symmetric transverse bilinear form in the fibers of \( V(Q^*F) \) and \( N \) is a Bott type nonlinear connection of \( QF \), then there is a unique projectable Bott linear connection \( \nabla^v \), in the vertical bundle \( V(Q^*F) \), such that

1) \( \nabla^v \) has null horizontal and vertical torsions and

2) \( g \) is parallel with respect to \( \nabla^v \), i.e. \( \nabla^v _X g = 0, \) for all \( X \in \mathcal{X}(Q\bar{F}) \).

**Proof.** It can be easily inferred that the hypothesis above imply that all the hypothesis of Proposition 3.2 are in fact fulfilled for \( \bar{g} \); thus, using its conclusion by duality, the final conclusions of our statement follow for \( g \). \( \square \)

The Propositions 3.2 and 3.3 have special forms in the case of a regular transverse Hamiltonian \( H \) or a Cartan metric \( K^2 \).

Let us suppose that the foliation \( \mathcal{F} \) has a regular transverse Hamiltonian \( H : Q^*\mathcal{F} \rightarrow R \); it reads that \( H \) is a basic function for \( \mathcal{F} \) and its transverse Hessian \( h \) is a transverse non-degenerated bilinear form in the fibers of \( V(Q^*F) \). The Hessian of \( H \) on \( H(Q^*F) \) is, by its definition, the inverse \( h^{-1} \) of the Hessian \( h \) on \( V(Q^*F) \).

Then, according to Proposition 3.1, \( H \) gives rise to a transverse nonlinear connection \( \bar{N} \) in \( Q^*F \).

The Propositions 3.2 and 3.3 become in this case as follows, improving [2, Theorem 3.1].

**Theorem 3.4.** If \( H \) is a regular transverse Hamiltonian, then there are unique projectable Bott linear connections \( \nabla^h \) and \( \nabla^v \), in the horizontal bundle \( H(Q^*F) \) and the vertical bundle \( V(Q^*F) \) respectively, such that

1) \( \nabla^v \) and \( \nabla^h \) have null horizontal and vertical torsions and
2) the Hessians of $H$ are parallel with respect to $\nabla^h$ and $\nabla^v$, i.e. $\nabla^h_X h^{-1} = 0$ and $\nabla^v_X h = 0$, $(\forall) X \in \mathcal{X}(Q\tilde{F})$, in the fibers of $H(Q^*F)$ and $V(Q^*F)$, respectively.

If the vertical hessian $h$ is positively defined on $Q^*F$, and $H$ is differentiable on $Q^*F$ or on the slashed $Q^*F = Q^*F\setminus \partial(M)$, we obtain a transverse Riemannian metric for the foliation $\tilde{F}$ on the manifold $Q^*F$ or for the foliation $\tilde{F}_*$ on the manifold $Q^*F$ respectively.

**Proposition 3.5.** If $H$ is differentiable on $Q^*F$ or on the slashed $Q^*F$ and the vertical hessian $h$ is positively defined, then the foliated manifold $(Q^*F, \tilde{F})$ or $(Q^*F, \tilde{F}_*)$ respectively is Riemannian.

According to [8], we say that $H$ is allowed if:
1) $H$ is continuous on $Q^*F$, differentiable on the slashed $Q^*F$, positively defined (i.e. its vertical hessian is positively defined) and $H(x, p) \geq 0 = H(x, 0)$, $(\forall) x \in M$ and $p \in Q^*_x F$;
2) $H$ is locally projectable on a transverse Hamiltonian;
3) there is a basic function $\varphi : M \to (0, \infty)$, such that for every $x \in M$ there is $p \in Q^*_x F$ such that $H(x, p) = \varphi(x)$.

If a positively transverse Hamiltonian $H$ is 2–homogeneous (i.e. $H(x, \lambda p) = \lambda^2 H(x, p)$, $(\forall) \lambda > 0$), then $H$ is called a transverse Cartan form; it is also a positively admissible Hamiltonian, taking $\varphi \equiv 1$, or any positive constant.

Using the results in [8] we have that the following statement is true.

**Proposition 3.6.** If there is an allowed $H : Q^*F \to \mathbb{R}$ (in particular a transverse Cartan form), then the foliation $F$ is Riemannian.

Notice that the lagrangian version of the above result was proved in [5], improving [2, Theorem 3.2]. Proposition 3.6 follows by duality from the lagrangian form only in the case when the dual lagrangian of $H$ is also allowed; for example, in the case of a transverse Cartan form, when its dual is a Finslerian. In the general case, the dual Hamiltonian (lagrangian) of an allowed lagrangian (Hamiltonian) does not follows to be allowed; we have not yet a example to prove this statement, so we leave it as an open question.

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