Optimality conditions and duality results of the nonlinear programming problems under $(p, r)$-invexity on differentiable manifolds

Shreyasi Jana and Chandal Nahak

Abstract. The main purpose of this paper is to study a pair of optimization problems on differentiable manifolds under $(p, r)$-invexity assumptions. By using the $(p, r)$-invexity assumptions on the functions involved, optimality conditions and duality results (Mond-Weir, Wolfe and mixed type) are established on differentiable manifolds. We construct counterexample to justify that our investigations are more general than the existing work available in the literature.

M.S.C. 2010: 26B25, 58A05, 58B20, 90C26, 90C46.
Key words: manifold; generalized invexity; duality.

1 Introduction

The field of optimization is concerned with the study of maximization and minimization of mathematical functions. Convexity plays a vital role in optimization theory. In the recent years, several generalizations have been developed for the classical properties of convexity. In 1981, Hanson [8] introduced the concept of generalized convexity by generalizing the difference $(x - y)$ in the definition of convex function to any function $\eta(x, y)$. Craven [6] named them as invex functions. Hanson’s initial results inspired a great deal of subsequent work, which had greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Ben-Israel and Mond [5] introduced a new generalization of convex sets and convex functions. Jeyakumar [10] studied the properties of preinvex functions and their role in optimization and mathematical programming problems. Zalmai [21] introduced $\rho - (\eta, \theta)$-invexity and Antczak [3] introduced $(p, r)$-invex sets and functions. Mandal and Nahak [13] generalized the results of Zalmai [21] and Antczak [3] by introducing $(p, r) - \rho - (\eta, \theta)$-invexity.

The concept of convexity in linear topological spaces relies on the possibility of connecting any two points of the space by the line segment between them. But if we
are given an abstract manifold then we can not join two points by a line segment. In this case, when the linear space is replaced by a manifold, the line segment is replaced by a geodesic arc. Earlier Luenberger [12] studied gradient projection methods along geodesic. Rapcsak [18] introduced optimality conditions for a minimization problem taking the constraint set as a differentiable manifold. In year 1991, Rapcsak [19] proposed geodesic convexity by taking Riemannian manifold as a definition of domain. Udriste [20] introduced the concept of duality for a convex programming problem on Riemannian manifolds. The notion of invex functions on a manifold was introduced by Pini [16]. Mititelu [15] generalized the invexity notions by introducing \((\rho, \eta)\)-invex, \((\rho, \eta)\)-pseudoinvex and \((\rho, \eta)\)-quasivex functions, respectively. Mititelu [15], also had established the necessary and sufficient conditions of Karush-Kuhn-Tucker type for a vector programming problem defined on a differentiable manifold. Ferrara and Mititelu [7] developed the Mond-Weir type duality for vector programming problems on a differentiable manifold. Barani and Pouryayevali [4] introduced the geodesic invex sets, geodesic invex and geodesic preinvex functions on Riemannian manifolds. Ahmad et al. [2] extended these results by introducing geodesic \(\eta\)-pre-pseudo invex functions and geodesic \(\eta\)-pre-quasi invex functions. In year 2012, Iqbal et al. [9] introduced geodesic \(E\)-convex sets, geodesic \(E\)-convex functions. Agarwal et al. [1] introduced geodesic \(\alpha\)-invex sets, geodesic \(\alpha\)-invex and \(\alpha\)-preinvex functions.

In this paper we have generalized the work of Antczak [3] from functions defined on \(\mathbb{R}^n\) to functions defined on differentiable manifolds. We have defined \((p, r)\)-invex functions and studied optimality conditions and duality results (weak, strong, converse duality) on differentiable manifolds.

2 Preliminaries

In this section, we recall some definitions and known results from differentiable manifolds which can be found in ([11], [17]).

Definition 2.1. A differentiable manifold \(M\) of dimension \(n\) is a space which locally looks like \(\mathbb{R}^n\), i.e., every point of it has a neighborhood homeomorphic to an open set in \(\mathbb{R}^n\) and on which \(C^k\) functions for \(1 \leq k \leq \infty\) are defined.

Definition 2.2. A curve on a differentiable manifold \(M\) is a smooth (i.e., \(C^\infty\)) map \(\gamma\) from some interval \((-\delta, \delta)\) of the real line into \(M\).

Definition 2.3. A tangent vector on a curve \(\gamma\) at a point \(p\) of \(M\) is defined as the map

\[ \dot{\gamma}_p : C^\infty(M) \rightarrow \mathbb{R}, \quad f \mapsto \dot{\gamma}_p(f) \equiv \frac{d}{dt}(f \circ \gamma) \big|_p. \]

Definition 2.4. The set of all tangent vectors at a point \(p\) of \(M\) is called the tangent space at \(p\) and is denoted by \(T_p M\).

Definition 2.5. A manifold whose tangent spaces are endowed with a smoothly varying inner product with respect to a point \(x \in M\) is called a Riemannian manifold. The smoothly varying inner product, denoted by \(\left< \xi_x, \zeta_x \right>\) for every two elements \(\xi_x\) and \(\zeta_x\) of \(T_x M\), is called a Riemannian metric. If \(M\) is a differentiable manifold, then there always exist Riemannian metrics on \(M\).
Definition 2.6. A geodesic is a curve whose tangents are parallel to it. A geodesic is a generalization of the notion of a straight line to curved spaces.

Definition 2.7. [19] Let $M$ be an $n$-dimensional Riemannian manifold. A set $A \subset M$ is called $g$-convex set (geodesic convex set) if any two points of $A$ are joined by a geodesic belonging to $A$, i.e., for all $x, y \in A$, there exists a geodesic $\gamma : [0, 1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 2.8. [19] Let $M$ be an $n$-dimensional Riemannian manifold and $A$ be a $g$-convex set in $M$. A function $f : A \to \mathbb{R}$ is said to be geodesic convex if for any two points $x, y \in A$ and geodesic $\gamma : [0, l] \to A$ with $\gamma(0) = x$ and $\gamma(l) = y$ and all $t \in [0, 1]$, we have,

$$f(\gamma(tl)) \leq tf(\gamma(0)) + (1 - t)f(\gamma(l)).$$

If $f$ is differentiable, we have

$$f(y) - f(x) \geq \nabla f(x) \dot{\gamma}(0) l,$$

where $\nabla f(x)$ and $\dot{\gamma}(0)$ mean, respectively, the gradient of $f$ at the point $x$ and the derivative of $\gamma(t)$ with respect to $t$ at the point $0$.

Let $M$ be an $n$-dimensional differentiable manifold and $T_p M$ be the tangent space to $M$ at $p$. Also assume that $TM = \bigcup_{p \in M} T_p M$ is the tangent bundle of $M$. Let $\alpha$ be a differentiable curve on $M$ with $\alpha(0) = p \in M$. Then the tangent vector to the curve $\alpha$ at $p$ is $\dot{\alpha}(0) \in T_{\alpha(0)} M = T_p M$. Assume that $N$ is another differentiable manifold and $\phi : M \to N$ is a differentiable map.

Definition 2.9. ([7]) The linear map $d\phi_p : T_p M \to T_{\phi(p)} N$ defined by $d\phi_p(v) = \phi'(p)v$ is called the differential of $\phi$ at the point $p$.

Let $F : M \to \mathbb{R}$ be a differentiable function. The differential of $F$ at $p$, namely $dF_p : T_p M \to T_{F(p)} \mathbb{R} \equiv \mathbb{R}$, is given by $dF_p(v) = dF(p)v$, $v \in T_p M$.

The length of a curve $\gamma : [a, b] \to M$ is defined by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt.$$ 

For any two points $p, q \in M$, we define $d(p, q) = \inf \{L(\gamma) : \gamma$ is curve joining $p$ to $q\}$. Then $d$ is a distance which induces the original topology on $M$. We consider now a map $\eta : M \times M \to TM$ such that $\eta(p, q) \in T_p M$ for every $p, q \in M$. For a differentiable function $f : M \to \mathbb{R}$, Pini [16] defined invexity in the following manner.

Definition 2.10. The differentiable function $f$ is said to be $\eta$-invex or invex on a differentiable manifold $M$ if for any $x, y \in M$,

$$f(x) - f(y) \geq d_\eta f(y)(\eta(x, y)).$$

Later on Mititelu [15] generalized the above definition as follows.

Definition 2.11. The differentiable function $f$ is said to be $(\rho, \eta)$-invex at $y$ if there exist an $\eta : M \times M \to TM$ and $\rho \in \mathbb{R}$ such that

$$\forall x \in M : f(x) - f(y) \geq d_\eta f(y)(\eta(x, y)) + \rho d^2(x, y).$$

Definition 2.12. ([4]) A closed $\eta$-path joining the points $y$ and $u = \alpha_{x,y}(1)$ is a set of the form $P_{yu} = \{v : v = \alpha(t) : t \in [0, 1]\}$. 

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3 \textbf{(p,r)-Invexity} 

In the year 2001, Antczak [3] introduced \((p,r)\)-invex function over \(\mathbb{R}^n\) which generalizes the notion of invexity.

**Definition 3.1 (Antczak (2001)).** Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a differentiable function and \(p, r\) be arbitrary real numbers. The function \(f\) is said to be \((p,r)\)-invex with respect to \(\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) at \(u\), if any one of the following conditions holds

\[
\frac{1}{r} e^{rf(x)} \geq \frac{1}{r} e^{rf(u)} [1 + \frac{r}{p} \nabla f(u)(e^{p\eta(x,u)} - 1)], \text{ for } p \neq 0, r \neq 0, \tag{3.1}
\]

\[
\frac{1}{r} e^{rf(x)} \geq \frac{1}{r} e^{rf(u)} [1 + r \nabla f(u)\eta(x,u)], \text{ for } p = 0, r \neq 0, \tag{3.2}
\]

\[
f(x) - f(u) \geq \frac{1}{p} \nabla f(u)(e^{p\eta(x,u)} - 1), \text{ for } p \neq 0, r = 0, \tag{3.3}
\]

\[
f(x) - f(u) \geq \nabla f(u)\eta(x,u), \text{ for } p = 0, r = 0. \tag{3.4}
\]

We introduce the \((p,r)\)-invex function on a differentiable manifold \(M\). Using \((p,r)\)-invexity assumptions, we derive optimality conditions and duality results (weak, strong and converse duality theorems) for a pair of optimization problems.

**Definition 3.2.** Let \(M\) be an \(n\)-dimensional differentiable manifold and \(f : M \rightarrow \mathbb{R}\) be a differentiable function. Let \(\eta\) be a map \(\eta : M \times M \rightarrow TM\) such that \(\eta(x,u) \in T_uM\) for all \(x,u \in M\). The exponential map on \(M\) is a map \(\exp_u : T_uM \rightarrow M\) and the differential of the exponential map \((d\exp_u)_a : T_a(T_uM) \cong T_uM \rightarrow T_aM\), where \(a = t_0\eta(x,u), t_0 \in [0, 1]\), and \(c \in P_{xu}\) where \(P_{xu}\) is a closed path joining the point \(x\) and \(u\). Let \(p, r\) be arbitrary real numbers. If for all \(x \in M\), the relations

\[
\frac{1}{r} (e^{rf(x)} - f(u)) - 1 \geq \frac{1}{p} df_u([(d\exp_u)_a(p\eta(x,u))] - I), \text{ for } p \neq 0, r \neq 0, \tag{3.5}
\]

\[
\frac{1}{r} (e^{rf(x)} - f(u)) - 1 \geq df_u(\eta(x,u)), \text{ for } p = 0, r \neq 0, \tag{3.6}
\]

\[
f(x) - f(u) \geq \frac{1}{p} df_u([(d\exp_u)_a(p\eta(x,u))] - I), \text{ for } p \neq 0, r = 0, \tag{3.7}
\]

\[
f(x) - f(u) \geq df_u(\eta(x,u)), \text{ for } p = 0, r = 0, \tag{3.8}
\]

hold, then \(f\) is said to be \((p,r)\)-invex function at \(u\) on \(M\). Here \(I \in T_xM\) such that for a co-ordinate chart \(\phi\), \(\phi(I) = 1\), where \(1 = (1, 1, ..., 1) \in \mathbb{R}^n\).

**Remark 3.3.** We denote the exponential map on the manifold by \(\exp(x)\) for \(x \in M\) and \(e^x\) for \(x \in \mathbb{R}\).
Example 3.4. The circle $S$ can be thought of as the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ of the Euclidean space $\mathbb{R}^2$. In the case of the circle $S$ the possible co-ordinate charts are

- $U_1 = \{(x, y) : x > 0\}$ \quad $\phi_1(x, y) = y$
- $U_2 = \{(x, y) : x < 0\}$ \quad $\phi_2(x, y) = y$
- $U_3 = \{(x, y) : y > 0\}$ \quad $\phi_3(x, y) = x$
- $U_4 = \{(x, y) : y < 0\}$ \quad $\phi_4(x, y) = x$.

We define a differentiable function on $S$. Let $x = (x_1, x_2) \in S$ and $f : S \rightarrow \mathbb{R}$ be defined by $f(x) = x_1 + \sin x_2$. Let $u = (u_1, u_2) \in S$.

The tangent space of $S$ at $u$ is the set $T_uS = \{v \in \mathbb{R}^2 : u \cdot v = 0\}$.

We choose $\eta : S \times S \rightarrow T_uS$ as $\eta(x, u) = (-u_2, u_1) \in T_uS$.

Let $a = \eta(x, u) = (-u_2, u_1)$.

We now find $d\eta(a)$. We take a chart $\phi_3(-u_2, u_1) = \phi(-u_2, u_1) = u_2$ at $a$ and the identity mapping as a chart $\psi$ at $f(a)$. Here both $S$ and $\mathbb{R}$ are of dimension 1. We now find the Jacobian matrix $\psi f \phi^{-1}$ at $\phi(a)$.

\[
d\eta(\frac{\partial}{\partial \phi})(\psi) = \frac{\partial}{\partial \phi}(\psi f) = \frac{\partial}{\partial \phi}(f(-u_2, u_1)) = \frac{\partial}{\partial u_2}(-u_2 + \sin u_1) = -1.
\]

i.e., $d\eta(\eta(x, u)) = -1$.

Now $e^{f(u)-f(u)} - 1 - d\eta(\eta(x, u)) = e^{f(x)-f(u)} - 1 + 1 = e^{f(x)-f(u)} \geq 0, \forall x, u \in S$.

Hence $f$ is $(0, 1)$-inex on $S$.

But if we take $x = (1/\sqrt{2}, -1/\sqrt{2}) \in S$, $u = (1/\sqrt{2}, 1/\sqrt{2}) \in S$.

Then $f(x) - f(u) = 1/\sqrt{2} - \sin 1/\sqrt{2} - 1/\sqrt{2} - \sin 1/\sqrt{2} = -1.299$ and $df_u(\eta(x, u)) = -1$. Hence $f(x) - f(u) < df_u(\eta(x, u))$, i.e., $f$ is not inex on $S$.

3.1 Sufficient Optimality Conditions

In recent years, many traditional optimization methods have been successfully generalized to minimize objective functions on manifolds. Mititelu [15] established necessary and sufficient conditions of Karush-Kuhn-Tucker (KKT) [14] type for a vector programming problem on differentiable manifolds. Consider the following primal optimization problem on a differentiable manifold $M$:

\[\text{(P)} \quad \text{Minimize } f(x)\]

subject to $g_i(x) \leq 0$, \quad $i = 1, ..., m$,

where $f : M \rightarrow \mathbb{R}$, $g_i : M \rightarrow \mathbb{R}$, \quad $i = 1, ..., m$, are differentiable functions. Let $D$ denote the set of all feasible solutions of (P).

Let $\bar{x} \in D$ be an optimal solution of (P) and we define the set $J^o = \{j \in 1, ..., m : g_j(\bar{x}) = 0\}$. Suppose that the domain $D$ satisfies the following constraint qualification at $\bar{x}$:

\[R(\bar{x}) : \exists v \in TM : d(g_{J^o})_{\bar{x}}(v) \leq 0.\]

Here $d(g_{J^o})_{\bar{x}}(v)$ is the vector components of $d(g_j)_{\bar{x}}(v)$, $\forall j \in J^o$, taken in increasing order of $j$.

Lemma 3.1. (Necessary Karush-Kuhn-Tucker (KKT) condition) ([15]) If a feasible point $\bar{x} \in M$ is an optimal solution of the problem (P) and satisfies the constraint qualification $R(\bar{x})$, then there exists multiplier $\xi = (\xi_1, ..., \xi_m)^T \in \mathbb{R}^m$, such that the
following conditions hold

(3.9) \[ df_x + \xi^T dg_x = 0, \]

(3.10) \[ \xi^T g(\bar{x}) = 0, \]

(3.11) \[ \xi \geq 0, \quad i = 1, 2, \ldots, m, \]

here \( g = (g_1, g_2, \ldots, g_m)^T. \)

**Theorem 3.2.** Assume that a point \( \bar{x} \in M \) is feasible for problem (P), and let the KKT conditions (3.9)-(3.11) be satisfied at \( (\bar{x}, \xi). \) If the objective function \( f \) and the function \( \xi^T g \) are \((p, r)\)-invex with respect to the same function \( \eta \) at \( \bar{x} \) on \( D, \) then \( \bar{x} \) is a global minimum point of the problem \( (P). \)

**Proof.** Let \( x \) be a feasible point for the problem \( (P). \) Since \( f \) and \( \xi^T g \) are \((p, r)\)-invex with respect to the same function \( \eta \) at \( \bar{x} \) on \( D, \) we have,

(3.12) \[ \frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{p} df_x(d(exp_\eta(x, \bar{x})) - I), \]

(3.13) \[ \frac{1}{r}(e^{r(\xi^T g(x)-\xi^T g(\bar{x}))} - 1) \geq \frac{\xi^T}{p} dg_x(d(exp_\eta(x, \bar{x})) - I). \]

Adding (3.12) and (3.13), we have

\[ \frac{1}{r}[(e^{r(f(x)-f(\bar{x}))} - 1 + e^{r(\xi^T g(x)-\xi^T g(\bar{x}))} - 1) \geq \frac{1}{p} (df_x + \xi^T dg_x)(d(exp_\eta(x, \bar{x})) - I), \]

and by KKT condition (3.9), we have

\[ \frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{r} (1 - e^{r(\xi^T g(x)-\xi^T g(\bar{x}))}), \]

or, by KKT condition (3.10)

\[ \frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{r} (1 - e^{r(\xi^T g(x))}). \]

Without loss of generality, let \( r > 0 \) (in the case when \( r < 0 \) the proof is analogous; one should change only the direction of some inequalities below to the opposite one).

Since \( x \) is a feasible point of \( (P), \) then \( g(x) \leq 0 \) and \( \xi \geq 0 \) imply that \( 1 - e^{r(\xi^T g(x))} \geq 0 \) and \( e^{r(f(x)-f(\bar{x}))} \geq 1. \)

Hence \( f(x) \geq f(\bar{x}). \) Therefore, \( \bar{x} \) is an optimal solution of the problem \( (P). \) \( \square \)
3.2 Mond-Weir Type Duality

The central part of the optimization problem is the duality theory. In several optimization problems evaluating the dual maximum is comparatively easier than solving a primal minimization problem. The concept of duality for a convex programming problem on a Riemannian manifold was first introduced by Udriste [20]. Ferrera and Mititelu [7] developed a duality of Mond-Weir type for a vector mathematical programming problem involving invex functions on a differentiable manifold. In our work, we establish the duality results for the problem \((P)\) involving \((p, r)\)-invex functions over a differentiable manifold.

For the optimization problem \((P)\), the Mond-Weir dual problem [7] \((MWD)\) is defined in the following form

\[
\begin{align*}
(MWD) \quad \text{Maximize } & f(u) \\
\text{subject to } & df_u + y^T dg_u = 0, \\
& y^T g(u) \geq 0, \quad y \in \mathbb{R}_+^m, \\
\end{align*}
\]

where \(f, g_i : M \to \mathbb{R}, \ i = 1, 2, ..., m\) are differentiable functions. Let \(W_1\) denote the set of all feasible solutions of \((MWD)\).

\[\text{Remark 3.5.} \] Throughout the remaining sections of this paper, without loss of generality, we assume \(r > 0\) (in the case when \(r < 0\) the proof is analogous; one should change only the direction of some inequalities to the opposite one but finally will get same results). The theorems will be proved only in the case when \(p \neq 0, r \neq 0\) (other cases can be dealt with likewise).

We have established the following duality results between \((P)\) and \((MWD)\).

**Theorem 3.3.** (Weak Duality) Let \(x\) and \((u, y)\) be the feasible solutions of \((P)\) and \((MWD)\), respectively. Moreover, assume that \(f\) and \(y^T g\) are \((p, r)\)-invex at \(u\) on \(M\) with respect to the same \(\eta\), then \(\inf(P) \geq \sup(MWD)\).

**Proof.** Since \(f\) and \(y^T g\) are \((p, r)\)-invex with respect to the same \(\eta\),

\[
\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p} df_u (d(exp_u(p\eta(x, u))) - I),
\]

\[
\frac{1}{r}(e^{y^T g(x)} - y^T g(u)) - 1) \geq \frac{y^T}{p} dg_u (d(exp_u(p\eta(x, u))) - I) \forall x \in D.
\]

Adding (3.14) and (3.15), we get

\[
\frac{1}{r}[e^{r(f(x)-f(u))} - 1 + e^{y^T g(x)} - y^T g(u) - 1] \geq \frac{1}{p} (df_u + y^T dg_u (d(exp_u(p\eta(x, u))) - I),
\]

or,

\[
\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{r} (1 - e^{y^T g(x)}).
\]

Since \(x\) is a feasible solution of \((P)\) and \(y^T g \geq 0\), then we have \(1 - e^{y^T g(x)} \geq 0\), i.e., \(e^{r(f(x)-f(u))} \geq 1\), or, \(f(x) \geq f(u)\) holds for \(\forall x \in D\) and \(u \in W_1\). Therefore \(\inf(P) \geq \sup(MWD)\).
Theorem 3.4. (Strong Duality) Let \( \bar{x} \) be an optimal solution of the problem (P) at which a constraint qualification \( R(\bar{x}) \) is satisfied. Then there exists \( \xi \in \mathbb{R}^n \) such that \( (\bar{x}, \xi) \) is a feasible solution of (MWD). Suppose that the hypotheses of the Weak Duality Theorem 3.3 hold, then \( (\bar{x}, \xi) \) is an optimal solution of the dual programming problem (MWD), and the objective values of (P) and (MWD) are equal.

Proof. Since a constraint qualification \( R(\bar{x}) \) is satisfied at \( \bar{x} \), then from the KKT necessary conditions (3.9)-(3.11), there exists \( \xi \) such that \( (\bar{x}, \xi) \) is a feasible solution of (MWD). Since the conditions of the Weak Duality Theorem 3.3 hold, then \( (\bar{x}, \xi) \) is an optimal solution of the dual problem (MWD), and the objective values of (P) and (MWD) are equal. \( \square \)

Theorem 3.5. (Converse Duality) Let \( (\bar{u}, y) \) be an optimal solution of the dual problem (MWD) such that \( \bar{u} \in D \). If \( f \) and \( y^T g \) are \( (p, r) \)-invex with respect to the same \( \eta \) at \( \bar{u} \) on \( M \). Then \( \bar{u} \) is an optimal solution of (P).

Proof. Since \( f \) and \( y^T g \) are \( (p, r) \)-invex with respect to the same \( \eta \) at \( \bar{u} \), we have

\[
(3.16) \quad \frac{1}{r}(e^{r(f(x)−f(\bar{u}))} − 1) \geq \frac{1}{p} d_f u (d(exp_{\nu}(pu(x, \bar{u}))) − I),
\]

\[
(3.17) \quad \frac{1}{r}(e^{r(y^T g(x)−g(\bar{u}))} − 1) \geq \frac{y^T}{p} d_g u (d(exp_{\nu}(pu(x, \bar{u}))) − I), \quad \forall x \in D.
\]

Adding (3.16) and (3.17), we get

\[
\frac{1}{r}[e^{r(f(x)−f(\bar{u}))} − 1 + e^{r(y^T g(x)−g(\bar{u}))} − 1)] \geq \frac{1}{p} (d_f u + y^T d_g u)(d(exp_{\nu}(pu(x, \bar{u}))) − I).
\]

Since \( (\bar{u}, y) \) is a feasible solution of (MWD), we have \( d_f u + y^T d_g u = 0 \), hence

\[
\frac{1}{r}(e^{r(f(x)−f(\bar{u}))} − 1) \geq \frac{1}{r}(1 − e^{r y^T g(x)}).
\]

Since \( x \in D \) and \( y \geq 0 \), we have, \( 1 − e^{r y^T g(x)} \geq 0 \Rightarrow e^{r f(x)−f(\bar{u})} \geq 1 \Rightarrow f(x) \geq f(\bar{u}) \).

So \( \bar{u} \) is an optimal solution of (P). \( \square \)

3.3 Wolfe Type Duality

Motivated by the classical Wolfe type duality [14], for the optimization problem (P), we define the Wolfe type dual (WD) in the following form

\[
(WD) \quad \text{Maximize} \quad f(u) + \sum_{i=1}^m \xi_i g_i(u)
\]

subject to \( d_f u + \sum_{i=1}^m \xi_i d_g u_i = 0, \)

\( \xi_i \geq 0, \quad i = 1, 2, ..., m, \)

where \( f, g_i : M \to \mathbb{R}, \quad i = 1, 2, ..., m \) are differentiable functions. Let \( W_2 \) be the set of all feasible solutions of (WD).

We have proved the following duality results between (P) and (WD).
Theorem 3.6. (Weak Duality) Let \(x\) and \((u, \xi)\) be feasible solutions for (P) and (WD), respectively. Moreover, assume that \(f\) and \(\sum_{i=1}^{m} \xi_i g_i\) are \((p, r)\)-invex and \((p, -r)\)-invex respectively, at \(u\) on \(M\) with respect to the same \(\eta\), then \(\inf (P) \geq \sup (WD)\).

Proof. Since \(f\) and \(\sum_{i=1}^{m} \xi_i g_i\) are \((p, r)\)-invex and \((p, -r)\)-invex respectively, with respect to the same \(\eta\), then \(\forall x \in D\), we have

\[
\frac{1}{r}(e^{r(f(x) - f(u))} - 1) \geq \frac{1}{p}d_f u(d(exp(p\eta(x, u))) - I),
\]

\[
-\frac{1}{r}(e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} - 1) \geq \frac{1}{p} \sum_{i=1}^{m} \xi_i d g_i u(d(exp(p\eta(x, u))) - I).
\]

Adding (3.18) and (3.19), we get

\[
\frac{1}{r}[e^{r(f(x) - f(u))} - 1 - e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} + 1] \geq \frac{1}{p} \left(d_f u + \sum_{i=1}^{m} \xi_i d g_i u\right)(d(exp(p\eta(x, u))) - I).
\]

Since \((u, \xi)\) is a feasible solution of (WD), we have

\[
d_f u + \sum_{i=1}^{m} \xi_i d g_i u = 0.
\]

Hence

\[
e^{r(f(x) - f(u))} \geq e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u))}.
\]

Since \(x\) is a feasible solution of (P) and \(\xi_i \geq 0\), we have \(\sum_{i=1}^{m} \xi_i g_i(x) \leq 0\). Hence

\[
e^{r(f(x) - f(u))} \geq e^{r(\sum_{i=1}^{m} \xi_i g_i(u))}
\]

\[
\Rightarrow f(x) - f(u) \geq \sum_{i=1}^{m} \xi_i g_i(u),
\]

\[
\Rightarrow f(x) \geq f(u) + \sum_{i=1}^{m} \xi_i g_i(u),
\]

holds for \(\forall x \in D\) and \(u \in W_2\). Therefore \(\inf (P) \geq \sup (WD)\). \(\square\)

Theorem 3.7. (Strong Duality) Let \(\bar{x}\) be an optimal solution of the problem (P) at which a constraint qualification \(R(\bar{x})\) be satisfied. Then there exists \(\xi \in \mathbb{R}^m\) such that \((\bar{x}, \xi)\) is a feasible solution of (WD). Suppose that the hypotheses of the Weak Duality Theorem 3.6 hold, then \((\bar{x}, \xi)\) is an optimal solution of the dual programming problem (WD), and the objective values of (P) and (WD) are equal.

Proof. Since a constraint qualification \(R(\bar{x})\) is satisfied at \(\bar{x}\), then from the KKT necessary conditions (3.9)-(3.11), there exists \(\xi\) such that \((\bar{x}, \xi)\) is a feasible solution of (WD). Since the conditions of the Weak Duality Theorem 3.6 hold, then \((\bar{x}, \xi)\) is an optimal solution of the dual problem (WD) and the objective values of (P) and (WD) are equal. \(\square\)
Theorem 3.8. *(Converse Duality)* Let \((u, \xi)\) be an optimal solution of the dual problem \((WD)\) such that \(u \in D\). If \(f\) is \((p, r)\)-invex and \(\sum_{i=1}^{m} \xi_i g_i\) is \((p, -r)\)-invex with respect to the same \(\eta\) at \(u\) on \(M\). Then \(u\) is an optimal solution of \((P)\).

**Proof.** We prove it by contradiction. Let \(u\) is not an optimal solution of \((P)\). Hence \(\exists x \in D \ni f(x) < f(u)\). Since \((u, \xi)\) is an optimal solution of \((WD)\), we have

\[
(3.21) \quad f(u) + \sum_{i=1}^{m} \xi_i g_i(u) > f(x) + \sum_{i=1}^{m} \xi_i g_i(x),
\]
or,

\[
(3.22) \quad f(x) - f(u) < -\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u).
\]

Since \(f\) is \((p, r)\)-invex and \(\sum_{i=1}^{m} \xi_i g_i\) is \((p, -r)\)-invex we have from (3.20)

\[
e^r(f(x) - f(u)) \geq e^r(-\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u)),
\]

(3.23) \[\Rightarrow f(x) - f(u) \geq -\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u),\]

which is a contradiction to (3.22). Hence \(u\) is an optimal solution to \((P)\). \(\square\)

### 3.4 Mixed Type Duality

For the problem \((P)\), we consider the Mixed type dual problem \((MDP)\) in the following form

\[(MDP) \quad \text{Maximize} \quad f(u) + \sum_{i=1}^{m} \xi_i g_i(u)\]

subject to

\[
df + \sum_{i=1}^{m} \xi_idg_i = 0,
\]

\[
\sum_{i=1}^{m} \xi_i g_i(u) \geq 0, \quad \xi_i \geq 0, \quad i = 1, 2, ..., m.
\]

Let \(W_3\) be the set of all feasible solutions of \((MDP)\).

We have established the following duality results between \((P)\) and \((MDP)\), whose proofs are omitted as they are very similar to Theorem 3.6 to Theorem 3.8.

**Theorem 3.9.** *(Weak Duality)*. Let \(x\) and \((u, \xi)\) be feasible solutions for \((P)\) and \((MDP)\), respectively. Moreover, we assume that \(f\) and \(\sum_{i=1}^{m} \xi_i g_i\) are \((p, r)\)-invex and \((p, -r)\)-invex, respectively at \(u\) on \(M\) with respect to the same \(\eta\), then \(\inf (P) \geq \sup (MDP)\).
Theorem 3.10. (Strong Duality) Let $\bar{x}$ be an optimal solution of the problem (P) at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi \in \mathbb{R}^m_+$, such that $(\bar{x},\xi)$ is a feasible solution of (MDP). Suppose that the hypotheses of the Weak Duality Theorem 3.9 hold, then $(\bar{x},\xi)$ is an optimal solution of the dual programming problem (MDP), and the objective values of (P) and (MDP) are equal.

Theorem 3.11. (Converse Duality) Let $(u, \xi)$ be an optimal solution of the dual problem (MDP) such that $u \in D$. If $f$ is $(p, r)$-invex and $\sum_{i=1}^m \xi_i g_i$ is $(p, -r)$-invex with respect to the same $\eta$ at $u$ on $M$. Then $u$ is an optimal solution of (P).

4 Conclusions

In this paper, we introduce the notion of $(p, r)$-invex functions on differentiable manifolds. We establish optimality conditions and duality results under $(p, r)$-invexity assumptions for a general nonlinear programming problem that is built upon on differentiable manifolds. Variational problems and control problems on differentiable manifolds under generalized invexity will orient future research of the authors.

Acknowledgements. The authors wish to thank the referees for their valuable suggestions which improved the presentation of the paper.

References

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