The existence and the number of critical points in an abstract perturbation problem

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Abstract. We use the relative homology groups of the sublevel sets associated to some functions which appear in an abstract perturbation problem in order to prove some criticality criteria. It is also proved the stability of critical groups. We give an estimation for the minimal number of critical points of a function using the category of the space with respect to the perturbed function.

Key words: critical point, Palais-Smale condition, deformation retract, relative homology, $f$-category, critical group.

1 Introduction

This paper is devoted to the study of the existence of critical points in an abstract perturbation problem under compactness assumptions of Palais-Smale type. A motivation is that perturbations arrive in many problems in Nonlinear Analysis; often it is difficult to study the function directly and we need to perturb it. Examples can be found in [1]. For Banach-Finsler manifolds, this kind of problems, in the presence of Palais-Smale type condition of [19]-[20], is studied in [21]. On the other hand, the use of relative homology in the study of the stability of critical values for two functions close enough, satisfying the Palais-Smale condition, goes back to [17]. In the last years, the classical critical point theory was extended to continuous functions on metric spaces; see [7]-[13]. The origin and evolution of the Palais-Smale condition in critical point theory is emphasized in [18], while [11] contains a purely topological approach of Morse theory.

It is well-known the fact that the minimum cardinality of the set of critical points of a function is difficult to compute. Estimating methods for this number go from the classical Lusternik-Schnirelmann category of the space, see [16], to $\varphi$-category, see [2]. Under compactness conditions, the number of critical points of a function can be estimated from below by the Lusternik-Schnirelmann category of the space; in [3]...
the authors introduced the category of a space with respect to a given function and it
was proved that, in suitable hypothesis, this number is a better lower bound for the
number of critical points of the function.

In this paper we use the relative homology groups of the sublevel sets associated
to some functions which appear in an abstract perturbation problem in order to
prove some criticality criteria. In Section 2 we review some of the standard facts
on critical point theory on metric spaces and we briefly introduce the (Lusternik-
Schnirelman) category depending on a functional. We also prove some technical
lemmas. Section 3 is devoted to the study of the existence and the number of critical
points in our abstract perturbation problem. More precisely, we are interested in
finding conditions such that a given interval is critical/non-critical for a function but
not for the associated perturbed function respectively conditions such that a given
interval is critical for both the original function and its perturbation. We also prove
the stability of critical groups under perturbations and we give an estimation for the
minimal number of critical points of the function using the category of the space with
respect to the perturbed function.

2 Preliminaries

2.1 Critical point theory on metric spaces

Let \( X \) be a metric space endowed with the metric \( d \). If \( x \in X \) and \( r > 0 \), then
\( B_r(x) \) denotes the open ball in \( X \) of center \( x \) and radius \( r \). Let \( f : X \to \mathbb{R} \) be a
continuous function. Recall from [7]-[13] basic notions and properties concerning the
critical point theory on metric spaces.

The weak slope of \( f \) at \( x \), denoted by \( |df|(x) \), is the supremum of all \( \sigma \in [0, \infty) \)
such that there exist \( \delta > 0 \) and a continuous map \( \mathcal{H} : B_\delta(x) \times [0, \delta] \to X \) which
satisfies the conditions

\[
d(\mathcal{H}(y,t),y) \leq t; \quad f(\mathcal{H}(y,t)) \leq f(y) - \sigma t,
\]

for all \( y \in B_\delta(x) \) and all \( t \in [0, \delta] \).

We call a point \( x \in X \) a critical point of \( f \) if \( |df|(x) = 0 \). A real number \( c \) is called
a critical value of \( f \) if there exists \( x \in X \) such that \( |df|(x) = 0 \) and \( f(x) = c \).

If \( X \) is a \( C^1 \)-Finsler manifold and \( f \in C^1(X,\mathbb{R}) \), then \( |df|(x) = \| df_x \| \), for any
\( x \in X \); in this case the notion of critical point agrees with the classical one, i.e. \( x \in X \)
a critical point of \( f \) if \( df_x = 0 \).

We use the notation \( K[f] = \{ x \in X \mid |df|(x) = 0 \} \) for the critical set of \( f \). If \( c \)
is a real number, then \( K[c] = K[f] \cap f^{-1}(c) \) is the critical set of level \( c \) of \( f \) and
\( X_c(f) = \{ x \in X \mid f(x) < c \} \) denotes the set of sublevel \( c \) of \( f \).

We say that \( f \) satisfies the Palais-Smale condition, denoted by \((PS)\), if any se-
quence \( (x_n) \) in \( M \) such that \( \{ f(x_n) \} \) is bounded and \( |df(x_n)| \to 0 \) has a convergent
subsequence. We say that \( f \) satisfies the Palais-Smale condition at level \( c \), denoted by \((PS)_c\), if any sequence \( (x_n) \) in \( M \) such that \( f(x_n) \to c \) and \( |df(x_n)| \to 0 \) has a
convergent subsequence.

If \((PS)\) is satisfied, then the local condition \((PS)_c\) holds for all \( c \in \mathbb{R} \), while
the converse is not true. On the other hand, the function \( x \mapsto |df|(x) \) is lower
If we consider the exact sequence of the triple \((\minology \text{ is justified by the fact that if } \text{definition of } 2.2 \text{ The category depending on a functional })\)

\[ \text{Proof.} \]

\[ \alpha \text{ homomorphism } H \subset A \subset \text{for any } \text{with real coefficients. Recall that for a (weak) deformation retract } Y, \text{define } n \text{ for all } \text{and } \text{where } H \text{Y, X} \text{denote } \text{by taking } X \text{fcat} \text{(ε, f)} = 0 \text{for any } f \text{cat } (X) = \text{fcat}_X (X). \]

\[ 2.3 \text{ Tools from algebraic topology} \]

\[ \text{Denote by } H_n (Y, X) \text{ the } n^{th} \text{ relative singular homology group of the pair } (Y, X), X \subset Y, \text{with real coefficients. Recall that for a (weak) deformation retract } X \text{ of } Y \text{we have, for any } n, H_n (Y, X) = 0. \text{See [14]. Let } A, B, C \text{ and } D \text{ be topological spaces such that } A \subset B \subset C \subset D. \text{We prove the following auxiliary result:} \]

\[ \text{Lemma 2.1. If } H_n (C, A) = 0 \text{ for any } n, \text{then there exists, for any } n, \text{an injective homomorphism } H_n (D, A) \rightarrow H_n (D, B). \]

\[ \text{Proof. We consider the exact sequence of the triple } (D, C, A). \text{Then we have:} \]

\[ \cdots \rightarrow H_n (C, A) \rightarrow H_n (D, A) \rightarrow H_n (D, C) \rightarrow H_{n-1} (C, A) \rightarrow \cdots \]

\[ \text{Because } H_n (C, A) = 0 \text{ for any } n, \text{it follows that } i_n \text{ is an isomorphism. Denote } \alpha_n \text{ and } \beta_n \text{ the homomorphisms induced by the inclusions: } (D, A) \hookrightarrow (D, B) \text{ and } (D, B) \hookrightarrow (D, C) \text{ respectively. Then we have } \beta_n \circ \alpha_n = i_n, \text{which shows that } \alpha_n \text{ is injective.} \]
Lemma 2.2. If $H_n(D, B) = 0$ for any $n$, then there exists, for any $n$, an injective homomorphism $H_n(B, A) \rightarrow H_n(C, A)$.

Proof. The conclusion follows from the equality $\delta_n \circ \gamma_n = j_n$, where $\gamma_n, \delta_n$ are the homomorphisms induced by the inclusions $(B, A) \hookrightarrow (C, A)$ and $(C, A) \hookrightarrow (D, A)$ respectively and $j_n : H_n(B, A) \rightarrow H_n(D, A)$ is an isomorphism. □

Lemma 2.3. If $H_n(B, A) = 0$ and $H_n(D, C) = 0$ for any $n$, then we have $H_n(C, A) \cong H_n(C, B) \cong H_n(D, B)$.

Proof. We consider the exact sequences of the triple $(D, C, B)$:

$\ldots \rightarrow H_{n+1}(D, C) \rightarrow H_n(C, B) \xrightarrow{i_n} H_n(D, B) \rightarrow H_n(D, C) \rightarrow \ldots$

We have $H_{n+1}(D, C) = H_n(D, C) = 0$, which implies that $i_n$ is an isomorphism.

In the same way, by using the exact sequences of the triple $(C, B, A)$, we have:

$\ldots \rightarrow H_n(B, A) \rightarrow H_n(C, A) \xrightarrow{j_n} H_n(C, B) \rightarrow H_{n-1}(B, A) \rightarrow \ldots$

Because $H_n(B, A) = H_{n-1}(B, A) = 0$, we obtain that $j_n$ is an isomorphism. □

3 The existence and the number of critical points

3.1 The existence of critical points on metric spaces

Let $X$ be a complete metric space. Recall the second deformation theorem, which we need proving the main results of this paper:

Theorem 3.1. ([7]) Let $f : X \rightarrow \mathbb{R}$ be a continuous function, $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$. Assume that for any $u \in [a, b]$, $f$ satisfies the $(PS)$-condition on $f^{-1}([a, u])$, $f$ has no critical point $x$ with $a < f(x) < b$ and either $K_a[f] = \emptyset$ or the connected components of $K_a[f]$ are single points. Then $X_a(f) \cup K_a[f]$ is a weak deformation retract of $X_b(f)$.

We consider two continuous functions $f, g : X \rightarrow \mathbb{R}$ with $g > 0$. For any $\varepsilon > 0$ enough small, the perturbation of $f$ with the function $g$ is given by

$f^\varepsilon = f + \varepsilon g$.

In [5] we obtained, for Finsler manifolds, sufficient conditions for the existence of a critical value for the perturbed function $f^\varepsilon$ in a given interval, with suitable assumptions, including $(PS)$. In [6], a stability property of critical values for continuous functions on metric spaces was proved.

In this section we study the existence of critical points of $f$ and $f^\varepsilon$, assuming that the following condition is fulfilled:

$(3.1) \quad a + \varepsilon g(x) \leq b, \quad \forall \ x \in X.$

We can now formulate our main results.
Theorem 3.2. Let \([a, b]\) be a non-critical interval for \(f^\varepsilon\), i.e. \(f^\varepsilon\) has no critical values in \([a, b]\). Suppose that both \(f\) and \(f^\varepsilon\) satisfy the \((PS)_c\) condition for any \(c \in [a, b]\). If there exists \(m\) such that \(H_m(X_b(f), X_a(f^\varepsilon)) \neq 0\), then \(K[f] \cap f^{-1}([a, b]) \neq \emptyset\).

Proof. From (3.1) we obtain the next inclusions:

\[
X_a(f^\varepsilon) \subseteq X_a(f) \subseteq X_b(f^\varepsilon) \subseteq X_b(f).
\]
Because \([a, b]\) is a non-critical interval for \(f^\varepsilon\) and \(f^\varepsilon\) satisfies the \((PS)_c\) condition for any \(c \in [a, b]\), the Theorem 3.1 implies that \(X_a(f^\varepsilon)\) is a weak deformation retract of \(X_b(f^\varepsilon)\). This shows that for any \(n\) we have \(H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) = 0\). We can apply the Lemma 2.1 and we obtain that, for any \(n\), there exists an injective homomorphism
\[
H_n(X_b(f), X_a(f^\varepsilon)) \rightarrow H_n(X_b(f), X_a(f)).
\]
By contradiction, assume that \(K[f] \cap f^{-1}([a, b]) = \emptyset\). From Theorem 3.1 it follows that \(X_a(f)\) is a weak deformation retract of \(X_b(f)\); consequently \(H_n(X_b(f), X_a(f)) = 0\) for any \(n\). On the other hand, the homomorphism from (3.3) being injective, we obtain that \(H_n(X_b(f), X_a(f^\varepsilon)) = 0\) for any \(n\). But this contradicts the fact that there exists \(m\) such that \(H_m(X_b(f), X_a(f^\varepsilon)) \neq 0\). □

Remark 3.1. Let \(f\) be as above, satisfying the \((PS)_c\) condition for any \(c \in [a, b]\) and let us consider the family of perturbations of \(f\),

\[
\left\{ f^\varepsilon = f + \varepsilon g \mid 0 < \varepsilon, \ 0 < g < \frac{b - a}{\varepsilon}, \ f^\varepsilon \ satisfies \ (PS)_c, \ \forall \ c \in [a, b] \right\}.
\]
Theorem 3.2 shows that if we can find a function \(f^\varepsilon\) in this family and an integer \(m\) such that \(f^\varepsilon\) has no critical values in \([a, b]\) and \(H_m(X_b(f), X_a(f^\varepsilon)) \neq 0\), then \(f\) has at least a critical value between \(a\) and \(b\).

In the above theorem, we obtained sufficient conditions such that the interval \([a, b]\) is non-critical for the perturbed function \(f^\varepsilon\), but it is critical for \(f\). The next theorem gives sufficient conditions such that the interval \([a, b]\) is non-critical for \(f\), but it is critical for the perturbed function \(f^\varepsilon\):

Theorem 3.3. Let \([a, b]\) be a non-critical interval for \(f\). Assume that both \(f\) and \(f^\varepsilon\) satisfy the \((PS)_c\) condition for any \(c \in [a, b]\) and there exists \(m\) such that \(H_m(X_b(f), X_a(f^\varepsilon)) \neq 0\). Then \(K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset\).

Proof. If \([a, b]\) is a non-critical interval for \(f\) and \(f\) satisfies the \((PS)_c\) condition for any \(c \in [a, b]\), then the Theorem 3.1 implies that \(H_n(X_b(f), X_a(f)) = 0\) for any \(n\). From (3.2) and Lemma 2.2, there exists, for any \(n\), an injective homomorphism

\[
H_n(X_a(f), X_a(f^\varepsilon)) \rightarrow H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)).
\]
Suppose that \(K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) = \emptyset\). Therefore by Theorem 3.1 we get that \(X_a(f^\varepsilon)\) is a weak deformation retract of \(X_b(f^\varepsilon)\). Then \(H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) = 0\), for any \(n\). But the homomorphism from (3.4) is injective; this contradicts the fact that there exists \(m\) such that \(H_m(X_a(f), X_a(f^\varepsilon)) \neq 0\). □
Remark 3.2. The above theorem is a generalization of [5, Theorem 3.1.(i)].

Now we establish sufficient conditions such that the interval \([a, b]\) is non-critical for both \(f\) and \(f^\varepsilon\).

Theorem 3.4. Let \(f\) and \(f^\varepsilon\) be as above satisfying the \((PS)_c\) condition for any \(c \in [a, b]\). Assume that \(H_n(X_a(f), X_a(f^\varepsilon)) = 0\) and \(H_n(X_b(f), X_b(f^\varepsilon)) = 0\) for any \(n\). If there exists \(m\) such that \(H_m(X_b(f^\varepsilon), X_a(f^\varepsilon)) \neq 0\), then \(K[f] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset\) and \(K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset\).

Proof. We have from Lemma 2.3 and (3.2) that

\[ H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) \cong H_n(X_b(f^\varepsilon), X_a(f)) \cong H_n(X_b(f), X_a(f)). \]

The conclusion follows from Theorem 3.1, by contradiction.

We obtain now sufficient conditions for the existence of critical values in the given interval \([a, b]\) for \(f\) and for the perturbed function \(f^\varepsilon\) respectively, by using the relative homology involving the level set \((f^\varepsilon)^{-1}(a)\).

Theorem 3.5. Let \(f\) and \(f^\varepsilon\) be as above satisfying the \((PS)_c\) condition for any \(c \in [a, b]\).

(i) Suppose that \(H_n(X_a(f), (f^\varepsilon)^{-1}(a)) = 0\) for any \(n\) and that there exists some \(m\) such that \(H_m(X_b(f^\varepsilon), (f^\varepsilon)^{-1}(a)) \neq 0\). Then \(K[f] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset\).

(ii) Suppose that \(H_n(X_b(f^\varepsilon), (f^\varepsilon)^{-1}(a)) = 0\) for any \(n\) and that there exists some \(m\) such that \(H_m(X_a(f), (f^\varepsilon)^{-1}(a)) \neq 0\). Then \(K[f] \cap f^{-1}([a, b]) \neq \emptyset\).

Proof. From (3.1) we obtain the inclusions

\[ (f^\varepsilon)^{-1}(a) \subseteq X_a(f^\varepsilon) \subseteq X_a(f) \subseteq X_b(f^\varepsilon); \]

\[ (f^\varepsilon)^{-1}(a) \subseteq X_a(f) \subseteq X_b(f^\varepsilon) \subseteq X_b(f). \]

Then we use Lemma 2.1.

As a limit case, we can look at the whole space \(X\) as a sublevel set of a function; for example, we can write \(X = X_{\infty}(f^\varepsilon)\). Then we get the following result:

Theorem 3.6. Let \(f\) and \(f^\varepsilon\) be as above satisfying the \((PS)_c\) condition for any \(c \in [a, b]\). Suppose that \(H_n(X_b(f^\varepsilon)) = 0\), for any \(n\) and there exists some \(m\) such that \(H_m(X_a(f), X_a(f^\varepsilon)) \neq 0\). Then \(K[f] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset\).

Proof. The conclusion follows from Lemma 2.2 and \(X_a(f^\varepsilon) \subseteq X_b(f^\varepsilon) \subseteq X_b(f) \subseteq X\).

3.2 The case of Banach-Finsler manifolds

Let \(M\) be a \(C^1\)-Finsler manifold and let \(f : M \to \mathbb{R}\) be of \(C^1\) class. If \(K[f]\) denotes the critical set of \(f\) and \(c \in \mathbb{R}\) a critical value of \(f\), the critical set of level \(c\) of \(f\) is \(K_c[f] = K[f] \cap f^{-1}(c)\) and \(M_c(f)\) is the set of sublevel \(c\) of \(f\). The second deformation theorem has the following formulation:
The assumptions of Theorem 3.9 imply that $f$ satisfies the $(PS)_c$ condition for any $c \in [a, b]$ and $K[f] \cap f^{-1}([a, b]) = \emptyset$. Then $M_a(f)$ is a strong deformation retract of $M_b(f)$.

Let $M$ be a $C^2$-Finsler manifold and let $f$ and $g$ be real functions defined on $M$, of $C^1$ class, such that $g > 0$. Denote $f^\varepsilon = f + \varepsilon g$, where $\varepsilon > 0$ is enough small and the condition (3.1) is fulfilled. It is obvious that Theorems 3.2-3.6 remains true in this context.

The problem of finding information about the critical set of the perturbed function $f^\varepsilon$, where $f$ is bounded from below and $dg$ is bounded on sets on which $g$ is bounded, is studied in [21]. We need the following result:

**Theorem 3.8.** ([21]) Let $f, g$ and $f^\varepsilon$ be as above. Assume that $f$ is bounded from below, $dg$ is bounded on sets on which $g$ is bounded, $f^\varepsilon$ satisfies the $(PS)$ condition and $K_c[f^\varepsilon] = \emptyset$ for any $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0$ is enough small and $c$ is a given real number. Then $M_c(f^\varepsilon)$ is a strong deformation retract of $M_c(f)$.

We obtain now sufficient condition such that the interval $[a, b]$ is critical for $f$ and also for the perturbed function $f^\varepsilon$.

**Theorem 3.9.** Let $f, g$ and $f^\varepsilon$ be as above. Assume that $f$ is bounded from below, $dg$ is bounded on sets on which $g$ is bounded, $f$ and $f^\varepsilon$ satisfy the $(PS)$ condition and $f^\varepsilon$ has no critical points at level $a$ and $b$, for any $\varepsilon \in (0, \varepsilon_0]$. If there exists $m$ such that $H_m(M_b(f^\varepsilon), M_a(f)) \neq 0$, then $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \emptyset$ and $K[f] \cap f^{-1}([a, b]) \neq \emptyset$.

**Proof.** It is obvious that

$$
M_a(f^\varepsilon) \subseteq M_a(f) \subseteq M_b(f^\varepsilon) \subseteq M_b(f).
$$

We apply the Theorem 3.8 twice:

- if $K_a[f^\varepsilon] = \emptyset$ for any $\varepsilon \in (0, \varepsilon_0]$, then $H_n(M_b(f), M_a(f^\varepsilon)) = 0$;

- if $K_b[f^\varepsilon] = \emptyset$ for any $\varepsilon \in (0, \varepsilon_0]$, then $H_n(M_b(f), M_b(f^\varepsilon)) = 0$.

Now we use the Lemma 2.3 and, for any $n$, we have:

$$
H_n(M_b(f^\varepsilon), M_a(f^\varepsilon)) \equiv H_n(M_b(f^\varepsilon), M_a(f)) \equiv H_n(M_b(f), M_a(f)).
$$

The conclusion follows from Theorem 3.7 by contradiction. \qed

**Remark 3.3.** The assumptions of Theorem 3.9 imply that

$$
H_n(M_a(f), M_a(f^\varepsilon)) = H_n(M_b(f), M_b(f^\varepsilon)) = 0
$$

for any $n$. We conclude that the Theorem 3.9 is a particular case of Theorem 3.4.

### 3.3 The stability of critical groups

Let $X$ be a complete metric space. Given a function $f : X \to \mathbb{R}$, we set:

$$
\|f\|_\infty = \sup_X |f|; \quad \text{Lip} \ (f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)};
$$

where $d(x, y)$ is the distance between $x$ and $y$ in $X$. 

The problem of finding information about the critical set of the perturbed function $f^\varepsilon$, where $f$ is bounded from below and $dg$ is bounded on sets on which $g$ is bounded, is studied in [21]. We need the following result:
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\[ \|f\|_{1,\infty} = \max\{\|f\|_{\infty}, \text{Lip}\ (f)\} . \]

It is obvious that \( \|f\|_{1,\infty} < \infty \) only if \( f \) is bounded and Lipschitz.

The homotopical stability of a minimum is studied in [15]; on the other hand, in [10] the authors extend the study to isolated critical points of continuous functions. We need the following result:

**Theorem 3.10.** ([10]) Let \( U \) be an open subset of \( X \) and \( x \in U \) the only critical point of \( f \) in \( \overline{U} \). Assume that \( f \) satisfy the \((PS)\) condition in \( U \). Then there exists \( \delta > 0 \) such that for any \( \tilde{f} : X \to \mathbb{R} \) continuous, having an unique critical point \( \tilde{x} \in U \), satisfying the \((PS)\) condition in \( \overline{U} \) and \( \|\tilde{f} - f\|_{1,\infty} \leq \delta \), we have \( C_\ast(\tilde{f}, \tilde{x}) = C_\ast(f, x) \).

Let \( f, g : X \to \mathbb{R} \) be continuous, \( g > 0 \); define \( f^\varepsilon = f + \varepsilon g \), \( \varepsilon > 0 \) small; we do not need the assumption (3.1). We have the following stability property of critical groups.

**Theorem 3.11.** Let \( U \) be an open subset of \( X \) and \( x \in U \) the only critical point of \( f \) in \( \overline{U} \). Assume that \( f \) satisfy the \((PS)\) condition in \( \overline{U} \). Then there exists \( \delta > 0 \) with the following property: for any \( \varepsilon > 0 \) such that \( f^\varepsilon \) has an unique critical point \( x_\varepsilon \) in \( \overline{U} \), satisfies the \((PS)\) condition in \( \overline{U} \) and \( \varepsilon\|g\|_{1,\infty} \leq \delta \), we have \( C_\ast(f^\varepsilon, x_\varepsilon) = C_\ast(f, x) \), i.e. the critical group does not depend on \( \varepsilon \).

**Proof.** We can apply the Theorem 3.10 for \( f \) and \( f^\varepsilon \) because the equalities

\[ \|f^\varepsilon|_U - f|_U\|_{1,\infty} = \|\varepsilon g|_U\|_{1,\infty} = \sup_X (\varepsilon g|_U) = \varepsilon \sup_X g|_U = \varepsilon\|g\|_{1,\infty} \]

and

\[ \text{Lip} (f^\varepsilon|_U - f|_U) = \text{Lip} (\varepsilon g|_U) = \varepsilon \text{Lip} (g|_U) \]

proves the fact that

\[ \|f^\varepsilon|_U - f|_U\|_{1,\infty} = \max\{\varepsilon\|g\|_{1,\infty}, \varepsilon \text{Lip} (g)\} = \varepsilon\|g|_U\|_{1,\infty} . \]

\[ \square \]

### 3.4 The number of critical points

Let \( X \) be a topological space and \( B \subseteq X \). We have the following property:

**Proposition 3.12.** ([3]) Let \( f_1, f_2 : X \to \mathbb{R} \) be two continuous functions. For any \( k \in \mathbb{N} \) such that \( f_1\text{cat}_X(B) \geq k \), there exists \( \delta > 0 \) such that if \( \sup_{x \in B} |f_1(x) - f_2(x)| \leq \delta \), then \( f_1\text{cat}_X(B) \leq f_2\text{cat}_X(B) \).

From [3, Theorem 5.8] and comments from [3, Section 5.4], we have:

**Theorem 3.13.** ([3]) Let \( X \) be a locally contractible metric space and let \( f : X \to \mathbb{R} \) satisfying the \((PS)\) condition. If \( f \) is bounded from below, then \( f \) has at least \( f\text{cat}(X) \) critical points.

Let \( f, g : X \to \mathbb{R} \) be continuous functions with \( g > 0 \). For any \( \varepsilon > 0 \) enough small we consider the perturbation \( f^\varepsilon = f + \varepsilon g \) such that the condition (3.1) of Section 3.1 is fulfilled.

We estimate the minimal number of critical points of \( f \) by using the general category of the space. We prove first the following property.
Proposition 3.14. Let $k \in \mathbb{N}$ such that $\text{fcat}_{X}(B) \geq k$. Then there exists a perturbation $f^\varepsilon$ of $f$ such that $\text{fcat}_{X}(B) \geq k$.

Proof. Proposition 3.12 shows that there exists $\delta > 0$ such that if $\sup_{x \in B} |f^\varepsilon(x) - f(x)| \leq \delta$, then $\text{fcat}_{X}(B) \leq \text{fcat}_{X}(B)$.

The hypothesis (3.1) implies, in fact, that $\varepsilon g$ is bounded on $X$; then we can choose, from the family of perturbations of $f$, a function $f^\varepsilon$ such that $\sup_{x \in B} |f^\varepsilon(x) - f(x)| = \varepsilon \sup_{x \in B} |g(x)| \leq \delta$, i.e. we can choose $\varepsilon > 0$ enough small such that $\varepsilon \sup_{x \in B} |g(x)| \leq \delta$. □

Proposition 3.15. Let $k \in \mathbb{N}$ be such that $\text{fcat}_{X}(B) \geq k$, for any $\varepsilon > 0$. Then $\text{fcat}_{X}(B) \geq k$.

Proof. We apply the Proposition 3.12. □

We can now formulate the main result of this section.

Theorem 3.16. Let $X$ be a locally contractible metric space and let $k \in \mathbb{N}$ be such that $\text{fcat}_{X}(B) \geq k$, for any $\varepsilon > 0$. If $f$ is bounded from below and satisfies the (PS) condition, then $f$ has at least $k$ critical points.

Proof. From Proposition 3.15 we obtain that $\text{fcat}_{X}(B) \geq k$. Then we apply the Theorem 3.13. □

References

The existence and the number of critical points


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