Integration of tensor fields

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Abstract. The aim of this paper is to introduce the idea of integration of tensor field as a reverse process to the Lie differentiation. The definitions of indefinite and definite integrals for tensor fields are similar to the analogous definitions for integrable functions in undergraduate differential calculus. In our definition the definite integral of a function is also a function, not a number, and the definite integral of a general tensor field is also a tensor field of the same type. A few geometrical examples included in the text clarify the topic being discussed.

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1 Introduction

One of the most significant tools in differential geometry and global analysis, in continuous environment mechanics and dynamical systems is the notion of vector field. Let us list some basic concepts related to vector fields which we use throughout this article: trajectories and flows, interaction of flows, phase portrait, dragging of tensor fields (including functions, vector fields and differential forms, [2]) along a flow, Lie derivatives, and integration of tensor fields. As it was mentioned in the Abstract, the main purpose of the present paper is to introduce the idea of integration of a tensor field as a reverse process to the Lie differentiation. In §2, we recall some well-known facts about Lie derivatives on manifold in the form convenient in what follows. It is known that the main property of the Lie derivative is its independence of change of coordinates. The Lie differentiation technique is developed in [1], where computation formulas are derived in nonholonomic basis. The nonholonomy object (J.A. Schouten, [3]) appearing in calculation formulas is a consequence of interacting of non-commuting basis operators, and allows to apply this technique to the theory of Lie groups. In particular, the structure constants are precisely the nonholonomy object of the left- or right-invariant basis in a Lie group. In §3 and §4, we define indefinite and definite integrals of a tensor field and explain the geometrical meaning of integrals by simple examples. In particular, in §4 are calculated the integrals of shift and rotation operators in the flows of rotations on the plane and in the space.

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2 Lie derivative on manifolds

Let M be a smooth n-dimensional manifold, and let X be a smooth vector field[1] on M. Therefore a flow $a_t = \exp tX$ as a one-parameter group of transformations of M is associated with X. Choosing local coordinates u^i, $i = 1, 2, \ldots, n$, on a neighborhood $U \subset M$, the flow a_t is determined by the system of first-order ordinary differential equations (ODEs)

$$(u^i)' = x^i(u), \tag{2.1}$$

where the prime denotes the differentiation with respect to a parameter t, and x^i are components of the vector field X at a point $u \in U$. More precisely, the flow a_t is a local pseudogroup of local transformations of M, because the theorem of uniqueness and existence of solutions of the system (2.1) has a local character. Such a relation between the local and the global should be kept in mind.

In the flow a_t points move along own trajectories, and functions are dragged according to the composition law:

$$u \rightsquigarrow u_t = a_t(u), \qquad f \rightsquigarrow f_t = f \circ a_t.$$

Moreover, each tensor field S of a general type (p, q) on M may be dragged along by the flow a_t for each value of t to define a one-parameter family of tensor fields, indicated by the abbreviation

$$S \quad \rightsquigarrow \quad S_t = \hat{a}_t S.$$

Then the derivative of this family with respect to t defines the *Lie derivative* of S by

$$S' = \mathcal{L}_X S = \lim_{t \to 0} \frac{S_t - S}{t}. \tag{2.2}$$

The tensor fields S and S' are of the same type. The Lie derivative of S with respect to X is usually denoted by the symbol $\mathcal{L}_X S$. But for the sake of convenience we use primes in order to denote the Lie derivative with respect to the *fixed* vector field X.

The Lie derivative along the vector field X of a function f on M is defined so that it is the ordinary derivative along X, i.e.,

$$f' = Xf = (f \circ a_t)'_{t=0} = \lim_{t \to 0} \frac{1}{t}(f_t - f). \tag{2.3}$$

The Lie derivative along the vector field X of another vector field Y on M is defined by the Lie bracket

$$\mathcal{L}_X Y = [X, Y]. \tag{2.4}$$

Given local coordinates u^i on some neighborhood $U \subset M$, the Lie derivative of the vector field $Y = \frac{\partial}{\partial u^i} y^i$ along $X = \frac{\partial}{\partial u^i} x^i$ is defined by (2.4) in coordinate notation:[2]

$$\mathcal{L}_X Y = [X, Y] = \frac{\partial}{\partial u^i}(X y^i - Y x^i). \tag{2.5}$$

[1] The smoothness of functions, vector fields and any tensor fields means that the relevant objects occurring will be assumed to be differentiable of sufficiently high class C^p or, if it is necessary, even C^∞ or C^ω. It is assumed that tensor fields are sufficiently smooth so that derivatives can be taken.

[2] We use the *Einstein summation*, i.e., the convention that repeated indices are implicitly summed over. Any index that is to be summed over we write in the upper position.

3 Integration of tensor fields

In the previous section we have defined the Lie derivative of tensor field along a flow $a_t = \exp tX$ of a vector field X. Analogously, one can speak about an *integration* of tensor fields. In particular, we need to recover a tensor field from its known Lie derivative with respect to the vector field X.

Definition 3.1. The *indefinite integral* of a function f'_t, see (2.3), with respect to the parameter t is defined as the set of all antiderivatives of f'_t along the flow a_t of X, symbolized by

$$\int f'_t dt = f_t + f_0, \tag{3.1}$$

where f_0 is an *invariant* of X, i.e., $Xf_0 = 0$.

Definition 3.2. The *definite integral* of f'_t on a closed interval $[a, b]$ is defined by the *Newton–Leibniz formula*

$$\int_a^b f'_t dt = f_t \Big|_a^b = f(b) - f(a). \tag{3.2}$$

If in (3.1) and (3.2) f is a tensor field, then along with the Lie differentiation one can speak about an *integration of tensor fields* along the flow of X.

Let S and Q be smooth tensor fields of the same type on M.

Definition 3.3. A tensor field Q is said to be an *antiderivative* of S along the flow a_t of X if

$$Q' = \mathcal{L}_X Q = S.$$

Let Q_1 and Q_2 be tensor fields of the same type and suppose one of them is an antiderivative of S. Then the second one is an antiderivative of S if and only if $Q_1 - Q_2 = Q_0$, where Q_0 is an invariant tensor field along the flow of X, i.e. $\mathcal{L}_X Q_0 = 0$.

Definition 3.4. The *indefinite integral* of the tensor field S with respect to t is defined as the set of all antiderivatives of S along the flow a_t of X, symbolized by

$$\int S_t dt = Q_t + Q_0, \tag{3.3}$$

where Q is an antiderivative of S and $\mathcal{L}_X Q_0 = 0$.

The next Proposition relates the integration and the Lie differentiation of tensor fields.

Proposition 3.1. *Let Q be an antiderivative of S along the flow a_t of X and suppose S is continuous on a closed interval $[a, b]$. Then the definite integral of S is defined by*

$$\int_a^b S_t dt = Q_b - Q_a. \tag{3.4}$$

Proof. Let the closed interval $[a, b]$ be partitioned by points

$$a = t_0 < t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_{n-1} < t_n = b.$$

Then the definite integral of S is defined by taking the limit of the sum

$$\int_a^b S_t dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^n S_{\xi_i} \Delta t_i,$$

where S_{ξ_i} is the value of S at an arbitrary point $\xi_i \in (t_{i-1}, t_i)$ and $\Delta t_i = t_i - t_{i-1}$ is the length of the subinterval, $i = 1, 2, \ldots, n$. According to the mean value theorem there is a one point ξ_i in each open interval (t_{i-1}, t_i) such that

$$S_{\xi_i} \Delta t_i = Q_{t_i} - Q_{t_{i-1}}.$$

We have

$$Q_b - Q_a = \sum_{i=1}^n (Q_{t_i} - Q_{t_{i-1}}),$$

which can be rewritten as

$$Q_b - Q_a = \sum_{i=1}^n S_{\xi_i} \Delta t_i. \tag{3.5}$$

Then taking the limit of the sum in the right-hand side of (3.5) as $n \to \infty$ we obtain (3.4). □

Let Y be a differentiable vector field on M. Then (2.4) and (3.4) yield

$$\int_a^b [X, Y]_t dt = Y_b - Y_a. \tag{3.6}$$

4 Geometrical examples

Example 4.1. Let us consider the linear vector field

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on the xy plane. The flow a_t of X is a uniform circular motion around the origin:

$$a_t : (x, y) \mapsto (x \cos t - y \sin t, y \cos t + x \sin t).$$

The indefinite integral of a function along the flow a_t is defined by (3.1), where $f_0 = f_0(I)$ is a function of the invariant $I = x^2 + y^2$ of X.

From (3.3) it follows that the indefinite integral of a vector field $[X, Y]_t$ is of the form

$$\int [X, Y]_t dt = Y_t + Y_0,$$

where Y is a differentiable vector field on the xy plane, and Y_0 is of the form

$$Y_0 = \zeta(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

According to the condition

$$[X, Y_0] = (X\zeta + \eta)\frac{\partial}{\partial x} + (X\eta - \zeta)\frac{\partial}{\partial y} = 0$$

the functions ζ and η must satisfy the system of linear ODEs

$$\begin{cases} \zeta'' + \zeta = 0 \\ \eta'' + \eta = 0, \end{cases}$$

where prime denotes the derivative with respect to X.

Suppose two functions $f = x$ and $g = y$ be given on the xy plane. The draggings of these functions and the function $f + g = x + y$ along the flow of X are described by $f_t = x_t$, $g_t = y_t$ and $(f+g)_t = (x+y)_t$, respectively. Let us calculate the corresponding definite integrals on the closed interval $[a,b] = \left[0, \frac{\pi}{2}\right]$. By (3.2) we have

$$\int_0^{\frac{\pi}{2}} f_t' dt = \int_0^{\frac{\pi}{2}} x_t' dt = x_t\Big|_0^{\frac{\pi}{2}} = -x - y,$$
$$\int_0^{\frac{\pi}{2}} g_t' dt = \int_0^{\frac{\pi}{2}} y_t' dt = y_t\Big|_0^{\frac{\pi}{2}} = x - y,$$
$$\int_0^{\frac{\pi}{2}} (f+g)_t' dt = \int_0^{\frac{\pi}{2}} (x_t + y_t)' dt = (x_t + y_t)\Big|_0^{\frac{\pi}{2}} = -2y.$$

Consider the vector field $Y = \dfrac{\partial}{\partial y}$. The Lie derivatives of Y with respect to X are

$$Y' = [X, Y] = \frac{\partial}{\partial x}, \qquad Y'' = [X, Y'] = -\frac{\partial}{\partial y} = -Y.$$

Thus, we have $Y'' + Y = 0$ and the dragging of Y along the flow of X is described by the vector-function

$$Y_t = Ta_t Y = \sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y}.$$

Then using (3.6) we obtain the definite integral of the field $Y' = \dfrac{\partial}{\partial x}$ on the closed interval $\left[0, \frac{\pi}{2}\right]$:

$$\int_0^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x}\right)_t dt = Y_t\Big|_0^{\frac{\pi}{2}} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad \text{where} \left(\frac{\partial}{\partial x}\right)_t = \cos t\frac{\partial}{\partial x} - \sin t\frac{\partial}{\partial y}.$$

The Figures 1–3 illustrate the meaning of the definite integral of a vector field on the 1st quarter of the xy plane.

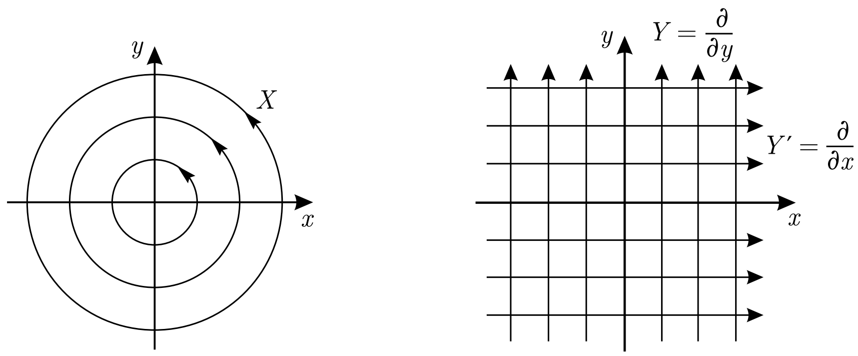

Figure 1: The flow of X is the uniform circular motion around the origin in the counter-clockwise direction. The Lie derivative of $Y = \dfrac{\partial}{\partial y}$ (*south wind*) with respect to X is the field $Y' = \dfrac{\partial}{\partial x}$ (*west wind*).

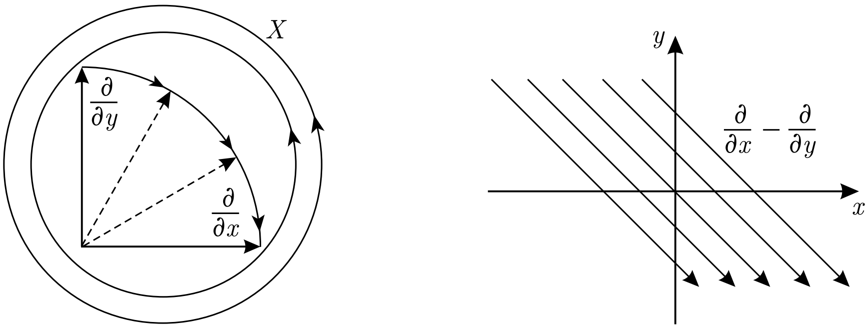

Figure 2: The field Y is rotated in moving frame according to the law $Y_t = Y\cos t + Y'\sin t$ (the wind changes own direction rotating clockwise). The calculating of definite integral $\displaystyle\int_0^{\frac{\pi}{2}} [X,Y]_t dt$ yields the field $\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y}$ (*north-west wind*).

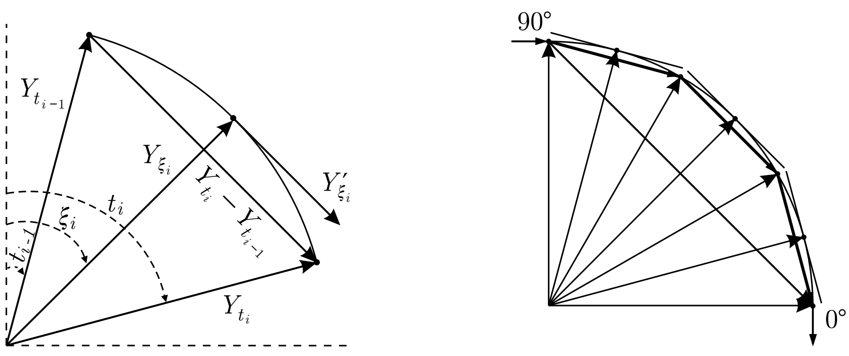

Figure 3: The summands for the integral sum are defined by the mean value theorem. Taking the limit of the integral sum we obtain the closing line to the hodograph of Y_t.

The *hodograph* is the plot of the velocity as function of time. The hodograph of the vector-function Y_t has the same trajectory as X but with opposite direction. The integral sum $\sum Y'_{\xi_i}\Delta t_i$ is a broken line to the hodograph and the integral $\dfrac{\partial}{\partial x}-\dfrac{\partial}{\partial y}$ is a straight line closing this broken line, see Figure 3.

Remark 4.2. Note that the formulas in the Example 4.1 can be obtained as follows. Consider the dragging of the natural frame $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$ and coframe (dx,dy) along the flow of $X=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$. The derivation formulas of the frame and coframe are

$$\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}'=\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad \begin{pmatrix}dx\\ dy\end{pmatrix}'=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}dx\\ dy\end{pmatrix},\tag{4.1}$$

where $C=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ is the Jacobi matrix of the components of X. The matrix C in the flow of X remains constant, and in the corresponding tangent space the flow Ta_t is determined by the exponential of the matrix Ct:

$$\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}_t=\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},\qquad \begin{pmatrix}dx\\ dy\end{pmatrix}_t=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\cdot\begin{pmatrix}dx\\ dy\end{pmatrix}.\tag{4.2}$$

The equalities (4.1) can be considered as ODEs in matrix notations and the corresponding solutions are presented by (4.2). Thus, the definite integrals on the closed interval $\left[0,\frac{\pi}{2}\right]$ of the corresponding vector fields and 1-forms can be calculated as follows:

$$\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}_0^{\frac{\pi}{2}}=\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}_{\frac{\pi}{2}}-\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}_0=\begin{pmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix},$$

$$\begin{pmatrix}dx\\ dy\end{pmatrix}_0^{\frac{\pi}{2}}=\begin{pmatrix}dx\\ dy\end{pmatrix}_{\frac{\pi}{2}}-\begin{pmatrix}dx\\ dy\end{pmatrix}_0=\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}\cdot\begin{pmatrix}dx\\ dy\end{pmatrix}.$$

Example 4.3. Let three vector fields

$$X=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z},\qquad Y=-z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z},\qquad Z=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$$

be given in the space xyz. The flows of X, Y and Z are rotations about three axes x, y and z, respectively.

Let us consider the dragging of Y along the flow of X, the dragging of Z along the flow of Y and the dragging of X along the flow of Z:

$$\begin{aligned}
Y'&=[X,Y]=Z, & Y''+Y&=0 &&\Longrightarrow & Y_t&=Y\cos t+Z\sin t,\\
Z'&=[Y,Z]=X, & Z''+Z&=0 &&\Longrightarrow & Z_t&=Z\cos t+X\sin t,\\
X'&=[Z,Y]=Y, & X''+X&=0 &&\Longrightarrow & X_t&=X\cos t+Y\sin t.
\end{aligned}$$

Let us calculate the integrals of X, Y and Z on a closed interval $[a,b]$:

$$\begin{aligned}\int_a^b Z_t dt &= \int_a^b [X,Y]_t dt = Y_b - Y_a = \\ &= 2\sin\frac{a-b}{2}\left(Y\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right),\\ \int_a^b X_t dt &= \int_a^b [Y,Z]_t dt = Z_b - Z_a = \\ &= 2\sin\frac{a-b}{2}\left(Z\sin\frac{a+b}{2} - X\cos\frac{a+b}{2}\right),\\ \int_a^b Y_t dt &= \int_a^b [Z,X]_t dt = X_b - X_a = \\ &= 2\sin\frac{a-b}{2}\left(X\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right).\end{aligned}$$

Taking $a=0$ and $b=\frac{\pi}{2}$ we obtain three vector fields

$$\int_0^{\frac{\pi}{2}} Z_t dt = (y+z)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \tag{4.3}$$

$$\int_0^{\frac{\pi}{2}} X_t dt = -y\frac{\partial}{\partial x} + (x+z)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \tag{4.4}$$

$$\int_0^{\frac{\pi}{2}} Y_t dt = y\frac{\partial}{\partial x} - (x+z)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}. \tag{4.5}$$

The flow of the field (4.3) is

$$(x,y,z) \mapsto \begin{cases} x_t = x\cos\sqrt{2}t + (y+z)\dfrac{\sin\sqrt{2}t}{\sqrt{2}} \\ y_t = y - x\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (y+z)\dfrac{1-\cos\sqrt{2}t}{2} \\ z_t = z - x\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (y+z)\dfrac{1-\cos\sqrt{2}t}{2} \end{cases}.$$

From the equalities $y_t - z_t = y - z$ and $2x_t^2 + (y_t+z_t)^2 = 2x^2 + (y+z)^2$ we obtain two invariants

$$I_1 = 2x^2 + (y+z)^2, \qquad I_2 = y - z.$$

It means that the level surfaces of the trajectories of the field (4.3) are elliptic cylinders with axis of rotation $\begin{cases} y+z=0 \\ x=0 \end{cases}$. The trajectories are ellipses on the intersections of the cylinders $I_1 = c > 0$ with planes $I_2 = c \geq 0$ perpendicular to the axis of rotation.

The flow of the field (4.4) is

$$(x,y,z)\mapsto\begin{cases} x_t = x - y\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (x+z)\dfrac{1-\cos\sqrt{2}t}{2} \\ y_t = y\cos\sqrt{2}t + (x+z)\dfrac{\sin\sqrt{2}t}{\sqrt{2}} \\ z_t = z - y\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (x+z)\dfrac{1-\cos\sqrt{2}t}{2} \end{cases},$$

and the invariants are

$$I_1 = 2y^2 + (x+z)^2, \qquad I_2 = x - z.$$

The level surfaces of the trajectories of the field (4.4) are elliptic cylinders with axis of rotation $\begin{cases} x+z=0 \\ y=0 \end{cases}$. The trajectories are ellipses on the intersections of the cylinders $I_1 = c > 0$ with planes $I_2 = c \geq 0$ perpendicular to the axis of rotation.

From $\int_0^{\frac{\pi}{2}} Y_t dt = -\int_0^{\frac{\pi}{2}} X_t dt$ it follows that the flows and invariants of the fields (4.4) and (4.5) are the same, but the trajectories of these fields are opposite directed.

In the Example 4.1 the definite integral on the xy plane is a straight line that closing the integral sum $\sum Y'_{\xi_i}\Delta t_i$, see Figure 3. In the xyz space we have analogous situation, but the closing line is a part of an ellipse.

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