η-pseudolinearity on differentiable manifolds

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Abstract. After 1970, the classic convexity theory was generalized from the Euclidean to the Riemannian setting . The convex sets were defined by the property of containing the geodesic segments between any two points (in the same way that line segments behave, as Euclidean geodesics) and the convex (differentiable) functions by the positiveness of their Riemannian Hessian.

In some previous papers, we made a step forward, by extending the Riemannian convexity to *affine differential* convexity: the geodesic segments were replaced by segments of auto-parallel curves of some arbitrary linear connections and the Riemannian Hessian by the affine differential one.

In this paper, we generalize in a similar way the classic notion of η-pseudolinearity, jumping from the classic setting directly to the differential setting. An unexpected link between the η-pseudolinearity and submersions is also discovered.

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1 Introduction

When dealing with functions defined on (open) subsets in $\mathbb{R}^n$, many generalizations of *classical* convexity are known: pseudoconvexity, invexity, pseudolinearity, etc.(see Avriel & all [2], Craven [6], Giorgi [7], Mishra & Giorgi [10], Mititelu [11] for reviews). Each of them catches some features of convexity and leads to interesting results, extending the framework of (classic) Convex Analysis.

After 1970, several schools of optimization generalized the classic convexity theory from the Euclidean to the *Riemannian* setting (see Udriste [16], Rapcszak [15] for reviews). The convex sets were defined by the property of containing the geodesic segments between any two points (in the same way that line segments behave, as Euclidean geodesics) and the convex (differentiable) functions by the positiveness of their Riemannian Hessian. All the classical algorithms work well in this generalized framework.

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A step forward was made by our papers (see [14] for details), where the Riemannian convexity was extended to *affine differential* convexity: geodesic segments were replaced by segments of auto-parallel curves of arbitrary linear connections and the Riemannian Hessian by the differential affine one. Similar studies involved generalized invexity also ([13]).

A variation of the last setting is the following: we have to find other families of relevant curves, in order to connect points (replacing the geodesics or the auto-parallel curves), like a "distance action" from Theoretical Physics. The flow of some vector fields families can provide the answer. This theory depends only on some differentiable objects on the manifold and, in that sense, is less restrictive than the affine differential one.

In this paper, we extend the (classic notion of) η-pseudolinearity, jumping from the classic setting directly to the differentiable one, studied on arbitrary differentiable manifolds.

Consider a real-valued differentiable function f on an open subset D of $\mathbb{R}^n$ and a function η defined on $D \times D$ with values in $\mathbb{R}^n$. The function f is said to be (classic) η-pseudoconvex (Hanson [8], Kaul and Kaur [9]) if, for $x, y \in D$,

$$\nabla f(x)^T \eta(y,x) \geq 0 \quad \text{implies} \quad f(y) \geq f(x). \tag{1}$$

If f and $-f$ are simultaneously (classic) η-pseudoconvex, then f is called (classic) η-pseudolinear ([1]). The detailed study in Ansari & all. [1] for η-pseudolinear functions extended what was known about pseudolinear functions (see, for example, Avriel & all. [2], Cambini [4], Chew and Choo [5]).

Consider now M an n-dimensional differentiable manifold and TM the total space of its tangent bundle. We denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and by $\mathcal{X}(M)$ the $\mathcal{F}(M)$-module of (differentiable) vector fields.

Let $\eta : M \times M \to TM$ a function, such that $\eta(x,y)$ belongs to the tangent space of M at y, for every x, $y \in M$. We call η a *bundle function* on M. In particular, $\eta(x,.)$ may be a vector field in $\mathcal{X}(M)$, for every $x \in M$; in this case, the function η depends smoothly on the second variable.

Definition 1.1. (i) A function $f \in \mathcal{F}(M)$ is called *η-pseudoconvex on M* if, for $x, y \in M$,

$$df_x(\eta(y,x)) \geq 0 \quad \text{implies} \quad f(y) \geq f(x). \tag{2}$$

If f and $-f$ are simultaneously η-pseudoconvex on M, then f is called *η-pseudolinear* on M. (Similar definitions work for functions defined on open sets of M).

When M is (an open set in) $\mathbb{R}^n$, we recover the previous (classic) notions. This is not the only reason that (2) is indeed more general than (1): its main quality is that it does not depend on "metric" notions, like the gradient. By replacing the use of the gradient by the exterior differential (notions which are equivalent in the Euclidean setting) enables us to extend the formalism from open sets in $\mathbb{R}^n$ to arbitrary differentiable manifolds. This simple trick will provide in the sequel a differential machinery, able to replace the Riemannian (and in particular, the Euclidean) one.

(ii) A family of parametrized curves, indexed on $M \times M$, is called *η-compatible* if for every $x, y \in M$, $c_{x,y} : [0,1] \to M$, it satisfies the properties: $c_{x,y}(0) = x$, $c_{x,y}(1) = y$, $c'_{x,y}(t) = \eta(y, c_{x,y}(t))$, for $t \in [0,1]$.

(iii) A subset D of M will be called *η-convex* if there exists a family of η-compatible curves, indexed on $D \times D$ and whose images are contained in D, connecting x with y, for every $x, y \in D$. The set D will be called *strongly η-convex* if it is η-convex and if it contains all the families of η-compatible curves connecting any two of its points (the points may coincide); in particular, D must contain all the η-compatible loop curves through its points.

(iv) Let $D \subset M$ be an η-convex subset and c a family of η-compatible curves whose images are contained in D. A function $f : D \to \mathbb{R}$ is called *pre-invex with respect to η and c* if, for every $x, y \in D$ and $t \in [0, 1]$,

$$f(c_{x,y}(t)) \leq tf(y) + (1-t)f(x).$$

We say the function η *satisfies the property (C) with respect to c and f* if, for every $x, y \in D$ and $t \in [0, 1]$,

$$df_{c_{x,y}(t)}(\eta(x, c_{x,y}(t))) = -tdf_x(\eta(y, x)) \text{ and } df_{c_{x,y}(t)}(\eta(y, c_{x,y}(t))) = (1-t)df_x(\eta(y, x)).$$

(v) We say the function η *is skew-symmetric with respect to f* if, for every $x, y \in D$,

$$df_x(\eta(y, x)) = -df_y(\eta(x, y)).$$

For the sake of simplicity, we shall write $\eta(x, y) = -\eta(y, x)$. We note that the property (C) implies the skew-symmetry of η.

Remark 1.2. Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a family of vector fields defined by $\eta(y, x) = y - x$ (the "affine structure function" on $\mathbb{R}^n$) and $c : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ a family of parametrized curves, given by $c_{x,y}(t) = x + t\eta(y, x)$. Then c is η-compatible; this is the classic situation, considered, for example, in Mohan & Neogy ([12]).

2 Caracterizations of η-pseudolinearity on differentiable manifolds

In this section, we prove three generalizations of similar results from the classic setting ([1]). First, we have a characterization of η-pseudolinear functions.

Proposition 2.1. *Let M be a differentiable manifold, $f \in \mathcal{F}(M)$, $\eta : M \times M \to TM$ a bundle function on M and K an η-convex subset of M, such that η satisfies the property (C) on K, with respect to f. If f is η-pseudolinear on K then, for any two points $x, y \in M$, $\eta(y, x) \in ker(df_x)$ if and only if $f(x) = f(y)$.*

Proof. First, suppose that f is η-pseudolinear and consider $x, y \in K$, such that $df_x(\eta(y, x)) = 0$. Because f and $-f$ are η-pseudoconvex, it follows that $f(y) \geq f(x)$ and $-f(y) \geq -f(x)$ respectively, hence $f(y) = f(x)$.

Conversely, let $x, y \in M$ such that $f(x) = f(y)$. Because K is an η-convex subset, there exists a family of η-compatible curves $c_{x,y} : [0, 1] \to M$, whose images are contained in K, such that: $c_{x,y}(0) = x$, $c_{x,y}(1) = y$, $c'_{x,y}(t) = \eta(y, c(t))$, for any $t \in [0, 1]$.

Consider a fixed $t \in (0, 1)$ and denote $z = c_{x,y}(t)$. Suppose ad absurdum that $f(z) > f(x)$. Due to the η-pseudoconvexity of f, it follows that $df_z(\eta(x, z)) < 0$.

From condition (C), it follows that

$$df_z(\eta(x,z)) = -t df_x(\eta(y,x)) = \frac{t}{t-1} df_z(\eta(y,z)).$$

Combined, the previous two inequalities lead to $df_z(\eta(y,z)) > 0$. Because f is η-pseudoconvex, we cannot have $f(y) < f(z)$, which contradicts the initial supposition.

This means that the claim $f(c_{x,y}(t)) > f(x)$ is false. In a similar way, by using the η-pseudoconvexity of $(-f)$, we derive that $f(c_{x,y}(t)) < f(x)$ is also false. So, we proved that $f(c_{x,y}(t)) = f(x)$, for every $t \in (0,1)$.

We calculate now

$$df_x(\eta(y,x)) = \lim_{t\to 0_+} df_{c(t)}(\eta(y,c(t))) = \lim_{t\to 0_+} df_{c_{x,y}(t)}(c'_{x,y}(t)) = \lim_{t\to 0_+} d(f \circ c_{x,y})(t)) = 0.$$

We proved $\eta(y,x) \in ker(df_x)$. □

Proposition 2.2. *Let M be a differentiable manifold, $f \in \mathcal{F}(M)$, $\eta : M \times M \to TM$ a bundle function on M and K an η-convex subset of M. Then, f is η-pseudolinear if and only if there exists a positive function $\varrho : K \times K \to \mathbb{R}$, such that, for every $x, y \in K$,*

$$f(y) - f(x) = \varrho(x,y) df_x(\eta(y,x)).$$

Proof. First, consider $f \in \mathcal{F}(M)$ an η-pseudolinear function on K. Let $x, y \in M$. If $df_x(\eta(y,x)) = 0$, we define $\varrho(x,y) = 1$. From Proposition 2.1, we know that $f(x) = f(y)$, so the required formula holds. If $df_x(\eta(y,x)) \neq 0$, we define

$$\varrho(x,y) = \frac{f(y) - f(x)}{df_x(\eta(y,x))}.$$

From the η-pseudolinearity of f it follows that $\varrho(x,y) > 0$.

Conversely, suppose that a function ϱ exists, satisfying the properties of the hypothesis. Let $x, y \in M$ such that $df_x(\eta(y,x)) \geq 0$. It follows that

$$f(y) - f(x) = \varrho(x,y) df_x(\eta(y,x)) \geq 0,$$

which implies $f(y) \geq f(x)$. We proved that f is η-pseudoconvex. A similar argument prove the η-pseudoconcavity of f. Hence f is η-pseudolinear on K. □

Proposition 2.3. *Let M be a differentiable manifold, $\eta : M \times M \to TM$ a bundle function on M, $f \in \mathcal{F}(M)$ an η-pseudolinear function, K an η-convex subset of M and $F : \mathbb{R} \to \mathbb{R}$ a differentiable function, regular or having only isolated critical points. Then, $F \circ f$ is also η-pseudolinear.*

Proof. A function is η-pseudolinear if and only if so is its opposite, thus we may restrict ourselves to the case $F' \geq 0$. Due to a similar argument, it suffices also to prove only the η-pseudoconvexity of $F \circ f$.

Let $x, y \in M$ such that

$$0 \leq d(F \circ f)_x(\eta(y,x)) = F'(f(x)) df_x(\eta(y,x)).$$

From continuity properties, it follows that $df_x(\eta(y,x)) \geq 0$. The η-pseudoconvexity of f and the monotonicity of F ensures that $f(y) \geq f(x)$. □

3 η-pseudolinear programs

We generalize some results concerning the η-pseudolinear programs, first proved in the classical setting by Ansari, Schaible and Yao [1].

Let M be a differentiable manifold, $f \in \mathcal{F}(M)$, $\eta : M \times M \to TM$ a bundle function on M and K an η-convex subset of M with respect to a family of curves c. We consider the problem of minimizing the function f on K and denote by S the set of all the solutions for this problem. In what follows, we suppose $S \neq \varnothing$.

Proposition 3.1. *If f is pre-invex on K, with respect to η and c, then the solution set S is η-convex.*

Proof. Let $x, y \in S$. Then, for any $z \in K$, we have $f(x) \le f(z)$ and $f(y) \le f(z)$. Since K is an η-convex subset with respect the family of η-compatible curves $c_{x,y} : [0,1] \to M$, whose image is contained in K, we have: $c_{x,y}(0) = x$, $c_{x,y}(1) = y$, $c'_{x,y}(t) = \eta(y, c(t))$, for any $t \in [0,1]$.

From the pre-invexity of f, it follows that, for every $t \in [0,1]$ and $z \in K$, we obtain

$$f(c_{x,y}(t)) \le tf(y) + (1-t)f(x) \le tf(z) + (1-t)f(z) = f(z)$$

It follows that $c_{x,y}(t) \in S$, so S is η-convex. □

Theorem 3.2. *Let M be a differentiable manifold, $f \in \mathcal{F}(M)$, $\eta : M \times M \to TM$ a bundle function on M and K an η-convex subset of M. Suppose f is η-pseudolinear on K and η satisfies the condition C with respect to f. Let x_0 be a fixed element of the solution set S for the η-program associated to f. Then S coincides with each of the following sets:*

$$S_1 = \{x \in K \mid df_x(\eta(x_0, x)) = 0\} \;\; ; \;\; S_2 = \{x \in K \mid df_{x_0}(\eta(x, x_0)) = 0\}$$

$$S_3 = \{x \in K \mid df_x(\eta(x_0, x)) \ge 0\} \;\; ; \;\; S_4 = \{x \in K \mid df_{x_0}(\eta(x, x_0)) \ge 0\}$$

$$S_5 = \{x \in K \mid df_{x_0}(\eta(x, x_0)) = df_x(\eta(x_0, x))\}$$

$$S_6 = \{x \in K \mid df_{x_0}(\eta(x, x_0)) \ge df_x(\eta(x_0, x))\}$$

Proof. Step I. A point $x \in M$ belongs to S if and only if $f(x) = f(x_0)$; that is (via the Proposition 1) if and only if $df_x(\eta(x_0, x)) = 0$. It follows that $S = S_1$.

Step II. In a similar way, $x \in S$ is equivalent successively to $f(x) = f(x_0)$, to $df_x(\eta(x_0, x)) = 0$ and to $df_{x_0}(\eta(x, x_0)) = 0$, due to the fact that $\eta(x_0, x) = -\eta(x, x_0)$. It follows that $S = S_2$.

Step III. From $S = S_1$, it is obvious that $S \subset S_3$. If $x \in S_3$, from the Proposition 2.2 we know there exists a function $\varrho : K \times K \to \mathbb{R}$ such that $\varrho(x, x_0) > 0$ and

$$f(x_0) = f(x) + \varrho(x, x_0) df_x(\eta(x_0, x)) \ge f(x).$$

It results that $f(x) = f(x_0)$, so $x \in S$. We got $S = S_3$.

Step IV. Combining ideas from Step II and III we can easily prove that $S = S_4$.

Step V. The inclusion $S_5 \subset S_6$ follows directly from their definitions.

Let $x \in S$. From Steps I and II, we derive $df_x(\eta(x_0, x)) = 0 = df_{x_0}(\eta(x, x_0))$, hence $x \in S_5$; it follows that $S \subset S_5$.

Consider now $x \in S_6$; by definition, $df_{x_0}(\eta(x, x_0)) \geq df_x(\eta(x_0, x))$. Suppose ad absurdum that $x \notin S$. It follows that $f(x) > f(x_0)$ and $df_{x_0}(\eta(x, x_0)) > 0$, due to the η-pseudoconvexity of $(-f)$. From $\eta(x, x_0) = -\eta(x_o, x)$ we obtain successively $0 > df_{x_0}(\eta(x, x_0)) \geq df_x(\eta(x_0, x)$ and $df_x(\eta(x, x_0)) > 0$. From Proposition 2.2, we know there exists a function $\varrho : K \times K \to \mathbb{R}$ such that $\varrho(x, x_0) > 0$ and $f(x_0) = f(x) + \varrho(x, x_0) df_x(\eta(x_0, x))$. Then $f(x_0) > f(x)$, which contradicts the hypothesis. Hence $x \in S$.

The three previously proved inclusions show that $S = S_5 = S_6$. □

4 New examples of η-pseudolinear functions

In this section, we provide a large family of (new) examples of η-pseudolinear functions.

Theorem 4.1. *Let M be an n-dimensional differentiable manifold and $f \in \mathcal{F}(M)$ a differentiable submersion. Then there exists a function $\eta : M \times M \to TM$, such that f is η-pseudolinear.*

Proof. Let $p \in M$ and $(x^1, ..., x^n)$ local coordinates in a chart $U^{(p)}$ around p . On $U^{(p)}$, we write $df = \frac{\partial f}{\partial x^i} dx^i$. By hypothesis, f is a submersion, so there exists $i_0 \in \{1, 2, ..., n\}$ such that $\frac{\partial f}{\partial x^{i_0}}(p) \neq 0$. Without restraining the generality, we may suppose $\frac{\partial f}{\partial x^{i_0}} \neq 0$ on $U^{(p)}$.

We define $\eta^{(p)} : M \times U^{(p)} \to TM$, $\eta^{(p)}(y, x) = (f(y) - f(x)) \frac{\partial f}{\partial x^{i_0}}(x) \frac{\partial}{\partial x^{i_0}} \mid_x$. It follows that, for every $y \in M$ and $x \in U^{(p)}$, we have

$$df_x(\eta^{(p)}(y, x)) = \left(\frac{\partial f}{\partial x^{i_0}}(x) \right)^2 (f(y) - f(x)).$$

Consider now an locally finite open covering, subordonated to the covering $\{U^{(p)} \mid p \in M\}$ and a differentiable partition of unity associated to it. A standard argument allows the construction of a function $\eta : M \times M \to TM$, such that $df_x(\eta(y, x))$ and $(f(y) - f(x))$ differ modulo a positive function. This fact implies that f and $-f$ are η-pseudoconvex, hence f is η-pseudolinear. □

Remark 4.1. (i) A converse of Theorem 4.1 is also true: consider M an n-dimensional differentiable manifold, $\eta : M \times M \to TM$ and $f \in \mathcal{F}(M)$ an η-pseudolinear function. Suppose ad absurdum there exists a point $x_0 \in M$ such that $df_{x_0} = 0$. It follows that, for every $y \in M$, $df_{x_0}(\eta(y, x_0)) = 0$. From Proposition 2.2, we derive $f(y) = f(x_0)$, for every $y \in M$. So, the η-pseudolinearity of f implies f is a submersion.

(ii) The connection between submersions and η-pseudolinearity appears to be quite natural if we take into account: 1) the obvious covariance of both notions and 2) the fact that a submersion is, locally (modulo some diffeomorphism), a projection (hence *linear* !).

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