On a class of singular vector subbundles

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Abstract. The aim of the paper is to prove a Walczak formula for a couple of singular distributions and a suitable Riemannian metric which satisfies certain compatibility conditions.

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1 Introduction

A vector bundle is denoted by $\pi : E \to M$, or E for short, when no confusion is possible.

According to [3], a *generalized vector subbundle* (a g.v.s. for short) $\mathcal{D}$ of a vector bundle $E \to M$ is the assignment of a subspace $\mathcal{D}_x \subset E_x = \pi^{-1}(x) \subset E$ to every $x \in M$. A vector $X_x \in \mathcal{D}_x$ is *allowed* if there is a smooth section Y of $\mathcal{D}_{|U_x}$ on an open neighborhood U_x of x, such that $Y_x = X_x$. Denote by $\mathcal{A}(\mathcal{D})_x \subset \mathcal{D}_x$ the set of allowed vectors in x. The null vector $\bar{0}_x \in E_x$ is obviously allowed since $\bar{0}_x \in \mathcal{A}(\mathcal{D})_x$, thus $\mathcal{A}(\mathcal{D})_x$ is non-void. It is easy to see that $\mathcal{A}(\mathcal{D}) = \underset{x\in M}{\cup} \mathcal{A}(\mathcal{D})_x$ is a generalized vector subbundle. According to [3], $\mathcal{D}$ is *smooth* if $\mathcal{A}(\mathcal{D}) = \mathcal{D}$. In general, for an arbitrary $\mathcal{D}$, $\mathcal{A}(\mathcal{D})$ is smooth, according to its construction. More properties of g.v.s.s can be found in [1, 6]. Important tools in their study are based on [2, 4, 5, 7, 10].

Let us observe that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are smooth g.v.s. of E, then $\mathcal{D}_1 + \mathcal{D}_2 \subset E$ is also a smooth g.v.s.

Let E and E' be two vector bundles over the same base M. A morphism $f : \mathcal{D} \to \mathcal{D}'$ of two g.v.s. $\mathcal{D} \subset E$ and $\mathcal{D} \subset E$ is a collection of linear maps $f_x : \mathcal{D}_x \to \mathcal{D}'_x$, $(\forall) x \in M$, such that there is a linear morphism of vector bundles $F : E \to E'$ that restricts to f in every fiber. An isomorphism is obviously given by the existence of a couple of inverse morphisms $f : \mathcal{D} \to \mathcal{D}'$ and $f^{-1} : \mathcal{D}' \to \mathcal{D}$ that are mutually inverse. Notice that in this case the extending linear morphisms F of f and F' of f^{-1} may be not necessary isomorphisms, but an isomorphism of E and E' induces an isomorphism of g.v.s.

Let $\mathcal{D} \subset E$ be a g.v.s. of E. Consider a Riemannian metric g on the fibers of E and let $\mathcal{D}^{\perp_g} \subset E$ be the orthogonal g.v.s., i.e. $\mathcal{D}_x^{\perp_g} = (\mathcal{D}_x)^{\perp_g}$, $(\forall) x \in M$.

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Then $(\forall)x \in M$, the vector space $\mathcal{D}_x^{\perp_g}$ is canonically isomorphic with the annihilator $\mathcal{D}_x^\circ = \{\omega_x \in E_x^*, \omega_x(X_x) = 0, (\forall)X_x \in \mathcal{D}_x\}$, via the musical isomorphism $\cdot^\# : E \to E^*$ given by the metric g.

Let us observe that two orthogonal g.v.s., corresponding to two different Riemannian metrics, are both isomorphic to the annihilator, thus they are isomorphic.

A morphism $f : \mathcal{D} \to \mathcal{D}'$ of g.v.s., induced by $F : E \to E'$, restricts to a morphism $f_\mathcal{A} : \mathcal{A}(\mathcal{D}) \to \mathcal{A}(\mathcal{D}')$, induced by the same F.

Also, if $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic, then $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{D}')$ are isomorphic via the same linear morphisms. If $\mathcal{D} \subset E$ is a smooth g.v.s. and g is a Riemannian metric on the fibers of E, then we say that:

- $\mathcal{D}^{\perp_g}$ is a *co-smooth orthogonal* of $\mathcal{D}$;
- $\mathcal{D}^{\vdash_g} = \mathcal{A}(\mathcal{D}^{\perp_g}(\mathcal{D})) \subset E$ is a *smooth orthogonal* of $\mathcal{D}$ and
- $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ is a *smooth orthogonal completion* of $\mathcal{D}$.

Proposition 1.1. *For a smooth $\mathcal{D}$, the following properties hold:*

1. *– the smooth orthogonal of $\mathcal{D}^{\models_g}$ is null (i.e. $(\mathcal{D}^{\models_g})^{\vdash_g} = \bar{0}$) and consequently;*
2. *– the smooth orthogonal completion of $\mathcal{D}^{\models_g}$ is $\mathcal{D}^{\models_g}$ itself (i.e. $(\mathcal{D}^{\models_g})^{\models_g} = \mathcal{D}^{\models_g}$);*
3. *– the smooth orthogonal completion $\mathcal{D}^{\models_g} = \mathcal{D} + \mathcal{D}^{\vdash_g}$ has the property that its maximal dimension of the fibers is taken on an open dense of M;*
4. *– in the case when $\mathcal{D}$ has a regular dimension r, then $\mathcal{D}^{\perp_g} = \mathcal{D}^{\vdash_g}$ and $\mathcal{D}^{\models_g} = E$.*

In fact we can recognize $\mathcal{D}^{\vdash_g}$ and $\mathcal{D}^{\models_g}$ in some cases, according to the following simple result, that follows from definitions.

Proposition 1.2. *Let $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{D}_0 \subset E$, all smooth, such that $\mathcal{D}_0^{\vdash_g} = \{\bar{0}\}$ and $\mathcal{D}_{0,x}$ is an gorthogonal decomposition $\mathcal{D}_{1,x} \oplus \mathcal{D}_{2,x}$, $(\forall)x \in M$. Then $\mathcal{D}_2 = \mathcal{D}_1^{\vdash_g}$, $\mathcal{D}_1 = \mathcal{D}_2^{\vdash_g}$ and $\mathcal{D}_1^{\models_g} = \mathcal{D}_2^{\models_g} = \mathcal{D}_0$.*

Let $\mathcal{D}$ be a generalized vector subbundle. If $S \subset M$, denote by $\mathcal{D}_S = \bigcup_{x\in S} \mathcal{D}_x$ the restriction of $\mathcal{D}$ to S. For $x \in M$, denote $r(x) = \dim \mathcal{D}_x$. Consider

$$\mathcal{R} = \{r(x) : x \in M\} = \{r_i\}_{i=\overline{0,k}}, \tag{1.1}$$

where $r_{\min=}r_0 < r_1 < \cdots < r_k = r_{\max}$. For $r_i \in \mathcal{R}$, we denote by

$$\Sigma_{r_i} = \{x \in M : \dim \mathcal{D}_x = r_i\},\ \Sigma_{<r_i} = \{x \in M : \dim \mathcal{D}_x < r_i\},$$

$$\Sigma_{\le r_i} = \{x \in M : \dim \mathcal{D}_x \le r_i\} = \Sigma_{r_i} \cup \Sigma_{<r_i},\ \Sigma_{>r_i} = \{x \in M : \dim \mathcal{D}_x > r_i\},$$

$$\Sigma_{\ge r_i} = \{x \in M : \dim \mathcal{D}_x \ge r_i\} = \Sigma_{r_i} \cup \Sigma_{>r_i}.$$

We say that the subset $\Sigma_{r_{\min}}$ is the *minimal set* and $\Sigma_{r_{\max}}$ is the *maximal set.* The subsets $\Sigma_{<r_i}$ and $\Sigma_{\le r_i}$ are closed and their complements, the sets $\Sigma_{\ge r_i}$ and $\Sigma_{>r_i}$ are open in M. The subset $\Sigma_{r_i} \subset \Sigma_{\ge r_i}$ is the minimal subset of $\mathcal{D}_{|\Sigma_{\ge r_i}}$ and $\Sigma_{>r_i}$ is void if $i = k$ and is equal to $\Sigma_{\ge r_{i+1}}$ if $0 \le i < k$.

Let us denote by $\Sigma_i^\mathcal{D}$ the set Σ_i of $\mathcal{D}$, and $\mathcal{R}^\mathcal{D} = \{r^\mathcal{D}(x) : x \in M\} = \{r_i\}_{i=\overline{0,k}}$, where $k = \max \mathcal{R}^\mathcal{D}$. The set $\Sigma_k^\mathcal{D} \subset M$ is open; if it is also a dense set in M, then the set $\Sigma_{m-k}^{\mathcal{D}^{\vdash_g}}$ contains the set $\Sigma_k^\mathcal{D}$, thus it is also a dense set in M, thus $m - r_k$ is the maximal dimension of the g.v.s. $\mathcal{D}^{\vdash_g}$. For example, it is the particular case when $\mathcal{D}$ is tangent to the leaves of a singular Riemannian foliation.

2 Locally regular vector subbundles

A *regular family* of vector subbundles of the vector bundle ξ over the base M is an open cover $\{U_i\}_{i\in I}$ of M such that:

• there is a vector subbundle $\xi_i \subset \xi$ over the base U_i with the dimension of the fibers r_i and

• on an intersection $U_i \cap U_j \neq \varnothing$, $r_i \leq r_j$, the vector bundle $\xi_{i|U_i\cap U_j}$ is a vector subbundle of $\xi_{j|U_i\cap U_j}$ if $r_i < r_j$, or $\xi_{i|U_i\cap U_j} = \xi_{j|U_i\cap U_j}$ if $r_i = r_j$.

A regular family of vector bundles gives rise to a smooth generalized vector subbundle and as well as to a cosmooth generalized vector subbundle, as follows.

For every $x \in M$, we denote by $r(x) = \max\limits_{x\in U_i} r_i$, $s(x) = \min\limits_{x\in U_i} r_i$ and $\mathcal{D}_x = (\xi_i)_x$, $\mathcal{E}_x = (\xi_j)_x$, where $r(x) = r_i$ and $s(x) = r_j$. Then $\mathcal{D} = \bigcup\limits_{x\in M} \mathcal{D}_x$ is a smooth generalized vector subbundle of ξ and $\mathcal{E} = \bigcup\limits_{x\in M} \mathcal{E}_x$ is a cosmooth generalized vector subbundle of ξ. Notice that $U_{r_k} = \Sigma_{r_k} = \Sigma_{\geq r_k}$.

We say that a smooth generalized vector subbundle $\mathcal{D}$ is *locally regular* if it is defined by a regular family of vector subbundles.

Thus for a smooth generalized vector subbundle $\mathcal{D}$ that is *locally regular* then every subset $\Sigma_i \subset M$ has an open neighborhood U_i such that:

• there is a vector bundle ξ_i over the base U_i with the dimension of the fibers r_i;

• the fibers of ξ_i are contained in the fibers of $\mathcal{D}_{U_i}$ and

• on an intersection $U_i \cap U_j$, $r_i < r_j$, the vector bundle $\xi_{i|U_i\cap U_j}$ is a vector subbundle of $\xi_{j|U_i\cap U_j}$.

Analogously, for a cosmooth generalized vector subbundle $\mathcal{D}$ is *locally regular* then every subset $\Sigma_i \subset M$ has an open neighborhood U_i such that:

• there is a vector bundle ξ_i over the base U_i with the dimension of the fibers r_i;

• the fibers of $\mathcal{D}_{U_i}$ are contained in the fibers of ξ_i and

• on an intersection $U_i \cap U_j$, $r_i < r_j$, the vector bundle $\xi_{i|U_i\cap U_j}$ is a vector subbundle of $\xi_{j|U_i\cap U_j}$.

For a smooth g.v.s. $\mathcal{D} \subset E$, a *natural anchor* is a vector bundle morphism $\rho : E \to \mathcal{D} \subset E$ that is a surjection on fibers. A (possibly local) vector field $X \in \Gamma(\mathcal{D})$ is an *eigenvector section* (e.s. for short) if there is an $u \in \mathcal{F}(M)$ such that $\rho(X) = u \cdot X$; we say that the eigenvector field is *regular* (r.e.f. for short) if $u(x) \neq 0$ for every x in the domain of X or if $u(x) \neq 0$ and $X_x \neq \bar{0}_x$, for every x the same.

A Riemannian metric g on the fibers of E can give rise to natural anchors on a locally regular g.v.s., as follows.

Consider j, $U_j \supset \Sigma_j$, and a test function φ_j corresponding to the closed set $M\backslash U_i$. We construct first the linear projections $\tilde{P}_i$ defined as the natural orthogonal projection $\tilde{P}_i : E_{U_i} \to \xi_i$ on U_i and the null map $E_{M\backslash U_i} \to E_{M\backslash U_i}$ on $M\backslash U_i$, then $P_i : E \to E$, $P_i = \varphi_i \tilde{P}_i$; let us notice that P_i is a smooth linear morphism and its values belong to the fibers of $\mathcal{D}$, without being, in general, a surjection. We define the linear morphism

$$P = \sum_{i=o}^{k} P_i.$$

Proposition 2.1. *The linear morphism P is a natural anchor.*

Using the proof of the above Proposition, the following result can be proved.

Proposition 2.2. *Let $\{e_1, \ldots, e_{r(x)}\} \subset \mathcal{D}_x$ be an orthonormal base. Then there are local (around x) eigenvector sections $\{E_1, \ldots, E_{r(x)}\}$ corresponding to P, such that $E_{i,x} = e_i$, $(\forall)i = \overline{1, r(x)}$.*

The following converse property holds.

Proposition 2.3. *Let $P \in End(E)$ such that for every $x \in M$ and every orthonormal base $\{e_1, \ldots, e_{r(x)}\} \subset \mathcal{D}_x$ then there are local (around x) eigenvector sections $\{E_1, \ldots, E_{r(x)}\}$ corresponding to P, such that $E_{i,x} = e_i$, $(\forall)i = \overline{1, r(x)}$. Then the smooth generalized vector subbundle given by the image of P is locally regular.*

3 Generalized vector subbundles

The results that follows asserts that every smooth g.v.s. has a natural anchor (see [8] for more details).

Theorem 3.1. *If M is a connect manifold, E is a vector bundle over M and $\mathcal{D}$ is a smooth generalized subbundle of E, then there is a smooth endomorphism Φ on the fibers of E such that its images are the fibers of $\mathcal{D}$.*

Lemma 3.2. *Let $W_1, \ldots, W_n \subset W_0 \subset V$ be vector subspaces and g be a (positive definite) scalar product on V. Let us denote by $\Pi_i : V \to V_i$, $i = \overline{0, n}$, the orthogonal projection and let $\alpha_0, \alpha_1, \ldots \alpha_n$ be some strict positive real scalars. Then the linear map*

$$\Pi : V \to W_0, \Pi(\bar{x}) = \alpha_0 \Pi_0(\bar{x}) + \cdots + \alpha_n \Pi_n(\bar{x})$$

is a surjection and the restriction of Π to W_0 is an authomorphism, more specifically a symmetric endomorphism according to g, having strict positive eigenvalues.

4 Fundamental forms of singular distributions

If g is a Riemannian metric on the fibers of E, then it induces natural anchors $P_1 = \Pi : E \to \mathcal{D}$, $P_2 = \Pi^{\vdash_g} : E \to \mathcal{D}^{\vdash_g}$ and $P = P_1 + P_2 = \Pi^{\vDash_g} : E \to \mathcal{D}^{\vDash_g}$.

In spite that Π induces an automorphism $\Pi_{x|\mathcal{D}_x} : \mathcal{D}_x \to \mathcal{D}_x$, $(\forall)x \in M$, it does not induces always an isomorphism of $\Gamma_1 = \Pi(\Gamma(E))$ and $\Gamma(\mathcal{D})$, since Γ_0 is finite generated and, in general, $\Gamma(\mathcal{D})$ is not.

We particularize now to the case $E = TM$. We call a generalized vector subbundle $\mathcal{D} \subset TM$ as a *singular distribution.*

Let g be a Riemannian metric on M and ∇ be its Levi-Civita connection. Consider $\Gamma_1 = P_1(\mathcal{X}(M)) \subset \Gamma(\mathcal{D})$, i.e. the sections $X' \in \Gamma(\mathcal{D})$ of the form $X' = \Pi(X)$, $X \in \mathcal{X}(M)$. Then Γ_1 is an $\mathcal{F}(M)$–submodule of $\Gamma(\mathcal{D})$. Analogously, we denote $\Gamma_2 = P_2(\mathcal{X}(M)) \subset \Gamma(\mathcal{D}^{\vdash_g})$.

Let us consider the maps $\tilde{B}_1, \tilde{B}_2 : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, defined by

$$\tilde{B}_1(Y, X) = \nabla_{P_1 X} P_2 Y, \tilde{B}_2(X, Y) = \nabla_{P_2 Y} P_1 X.$$

The *structural tensors* of $\mathcal{D} \subset TM$ are the bilinear maps $B_i : \mathcal{X}(M) \times \mathcal{X}(M) \to \Gamma_i$ given by

$$\begin{aligned} B_1(Y, X) &= P_1(\tilde{B}_1(Y, Y)) = P_1 \nabla_{P_1(X)} P_2(Y), \\ B_2(X, Y) &= P_2(\tilde{B}_2(X, Y)) = P_2 \nabla_{P_2(Y)} P_1(X), (\forall) X, Y \in \mathcal{X}(M). \end{aligned}$$

They restrict to bilinear forms $B_1' : \Gamma(\mathcal{D}^{\vdash_g}) \times \Gamma(\mathcal{D}) \to \Gamma_1$ and $B_2' : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}^{\vdash_g}) \to \Gamma_2$, having also the properties that if $X \in \Gamma(\mathcal{D}^{\vdash_g})$ and $Y \in \mathcal{X}(M)$, or if $X \in \mathcal{X}(M)$ and $Y \in \Gamma(\mathcal{D})$, then $B_1(X, Y) = B_2(Y, X) = 0$.

In general, P_1 and P_2 are not projectors, unless the case when $\mathcal{D}$ is regular. Let us consider the bilinear forms

$$B_j^{(i)} : \mathcal{X}(M) \times \mathcal{X}(M) \to \Gamma_2, i, j = \overline{1, 2},$$

$$B_1^{(1)}(X_1, X_2) = P_2^2 \nabla_{P_1 X_1} P_1 X_2 - P_2 \nabla_{P_1 X_1} P_1^2 X_2;$$

$$B_1^{(2)}(X_1, X_2) = P_2^2 \nabla_{P_1 X_1} P_1 X_2 - P_2 \nabla_{P_1^2 X_1} P_1 X_2;$$

$$B_2^{(1)}(Y, Z) = P_1^2 \nabla_{P_2 Y} P_2 Z - P_1 \nabla_{P_2 Y} P_2^2 Z;$$

$$B_2^{(2)}(Y, Z) = P_1^2 \nabla_{P_2 Y} P_2 Z - P_1 \nabla_{P_2^2 Y} P_2 Z.$$

We say that the singular distribution $\mathcal{D}$ is *normally allowed* or ∇ is *adapted* if

$$B_j^{(i)} = 0, i, j = \overline{1, 2}. \tag{4.1}$$

Proposition 4.1. *If $\mathcal{D}$ is* normally allowed *then for all $X_1, X_2, X, Y, Z \in \mathcal{X}(M)$, the following relations hold true:*

$$P_2^2[P_1 X_1, P_1 X_2] = P_2[P_1^2 X_1, P_1 X_2] = P_2[P_1 X_1, P_1^2 X_2],$$

$$P_1^2[P_2 Y, P_2 Z] = P_1[P_2^2 Y, P_2 Z] = P_1[P_2 Y, P_2^2 Z],$$

$$P_2^2 \nabla_{P_2 Y} P_1 X = P_2 \nabla_{P_2^2 Y} P_1 X = P_2 \nabla_{P_2 Y} P_1^2 X;$$

$$P_1^2 \nabla_{P_1 X} P_2 Y = P_1 \nabla_{P_1^2 X} P_2 Y = P_1 \nabla_{P_1 X} P_2^2 Y.$$

Proposition 4.2. *Given $\mathcal{D}$, there is a linear connection ∇ that satisfies (4.1).*

Notice that an adapted connection is the Levi-Civita connection of the metric g that gives P_1 and P_2, satisfying the condition (4.1).

Let us consider

$$\begin{aligned} T_1(Y, X_1, X_2, Z) &= (\left(\nabla_{P_1 X_1} \tilde{B}_2\right)(X_2, Y), P_2 Z) \\ &= (P_2 \nabla_{P_1 X_1} B_2(X_2, Y) - B_2(\nabla_{P_1 X_1} P_1 X_2, Y) \\ &\quad - B_2(X_2, \nabla_{P_1 X_1} P_2 Y), Z) = (P_2 \nabla_{P_1 X_1} P_2 \nabla_{P_2 Y} P_1 X_2 \\ &\quad - P_2 \nabla_{P_2 Y} P_1 \nabla_{P_1 X_1} P_1 X_2 - P_2 \nabla_{P_2 \nabla_{P_1 X_1} P_2 Y} P_1 X_2, Z); \\ T_2(Y, X_1, X_2, Z) &= (\left(\nabla_{P_2 Y} \tilde{B}_1\right)(Z, X_1), P_1 X_2) \\ &= (P_1 \nabla_{P_2 Y} B_1(Z, X_1) - B_1(\nabla_{P_2 Y} P_2 Z, X_1) \\ &\quad - B_1(Z, \nabla_{P_2 Y} P_1 X_1), X_2)(P_1 \nabla_{P_2 Y} P_1 \nabla_{P_1 X_1} P_2 Z \\ &\quad - P_1 \nabla_{P_1 X_1} P_2 \nabla_{P_2 Y} P_2 Z - P_1 \nabla_{P_1 \nabla_{P_2 Y} P_1 X_1} P_2 Z, X_2), \end{aligned}$$

and

$$R(Y, X_1, X_2, Z) = (\nabla_{P_2 Y} P \nabla_{P_1 X_1} P_1 X_2 - \nabla_{P_1 X_1} P \nabla_{P_2 Y} P_1 X_2 \\ - \nabla_{P[P_2 Y, P_1 X_1]} P_1 X_2, P_2 Z).$$

Consider $f \in \mathcal{F}(M)$. Then

$$\begin{aligned} &T_1(Y, fX_1, X_2, Z) = T_1(Y, X_1, X_2, fZ) = fT_1(Y, X_1, X_2, Z), \\ &T_1(Y, X_1, fX_2, Z) = fT_1(Y, X_1, X_2, Z) + P_1 X_1(f)(P_2^2(\nabla_{P_2 Y} P_1 X_2) - P_2 \nabla_{P_2 Y} P_1^2 X_2), \\ &T_1(fY, X_1, X_2, Z) = fT_1(Y, X_1, X_2, Z) + P_1 X_1(f)(P_2^2(\nabla_{P_2 Y} P_1 X_2) - P_2 \nabla_{P_2^2 Y} P_1 X_2). \end{aligned}$$

As well, we get

$$\begin{aligned} &T_2(Y, X_1, fX_2, Z) = T_2(fY, X_1, X_2, Z) = fT_2(Y, X_1, X_2, Z), \\ &T_2(Y, fX_1, X_2, Z) = fT_2(Y, X_1, X_2, Z) + P_2 Y(f)(P_1^2(\nabla_{P_1 X_1} P_2 Z) - P_1 \nabla_{P_1 X_1} P_2^2 Z), \\ &T_2(Y, X_1, X_2, fZ) = fT_2(Y, X_1, X_2, Z) + P_2 Y(f)(P_1^2(\nabla_{P_1 X_1} P_2 Z) - P_1 \nabla_{P_1^2 X_1} P_2 Z). \end{aligned}$$

Analogously, we infer:

$$\begin{aligned} &R(Y, X_1, X_2, fZ) = fR(Y, X_1, X_2, Z), \\ &R(fY, X_1, X_2, Z) = fR(Y, X_1, X_2, Z) - P_1 X_1(f)(P_2^2(\nabla_{P_2 Y} P_1 X_2) - P_2 \nabla_{P_2^2 Y} P_1 X_2, Z), \\ &R(Y, fX_1, X_2, Z) = fT(Y, X_1, X_2, Z) - P_2 Y(f)(P_2 \nabla_{P_1^2 X_1} P_1 X_2 - P_2^2 \nabla_{P_1 X_1} P_1 X_2, Z), \\ &R(Y, X_1, fX_2, Z) = fR(Y, X_1, X_2, Z) - P_2 Y(f)(P_2 \nabla_{P_1 X_1} P_1^2 X_2 - P_2^2 \nabla_{P_1 X_1} P_1 X_2, Z) \\ &- P_1 X_1(f)(P_2^2(\nabla_{P_2 Y} P_1 X_2) - P_2 \nabla_{P_2 Y} P_1^2 X_2, Z). \end{aligned}$$

We denote also

$$S_1(Y, X_1, X_2, Z) = (\tilde{B}_2(X_2, \tilde{B}_2(X_1, Y)), P_2 Z),$$
$$S_2(Y, X_1, X_2, Z) = (\tilde{B}_1(Z, \tilde{B}_1(Y, X_1), P_1 X_2).$$

Proposition 4.3. *Let us assume that* $B_j^{(i)} = 0$, $i, j = \overline{1, 2}$. *Then*

- $R + S_1 + T_1 + S_2 + T_2 = 0$ *and*
- $R, S_i, T_i : \mathcal{X}(M)^4 \to \mathcal{F}(M)$, $i = \overline{1, 2}$, *are* $\mathcal{F}(M)$*–linear maps.*

An important example of a singular distribution $\mathcal{D}$ that is normally allowed is when $P = f \cdot I$, where $P = P_1 + P_2$, I is the identity endomorphism of TM and f is a real function in $\mathcal{F}(M)$ such that its support is the whole M, i.e. its non-zero set is dense in M; for example f has an at most countable set of zeros. Examples of singular distributions of this type, even integrable, are given below.

It is easy to see that the smooth orthogonal completion of a distribution $\mathcal{D}$ defined as the image of $P_1 = fI$, with f as above, is $\mathcal{D}$ itself, the smooth orthogonal $\mathcal{D}^{\vdash_g}$ is null and its orthogonal completion is $\mathcal{D}$.

On $\mathbb{R}^2$, consider the singular distribution $\mathcal{D}$ spanned by the vector field $\bar{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = (x, y, y, -x)$. The g-orthogonal distribution $\mathcal{D}^{\vdash_g}$, also singular, is spanned by the vector field $\bar{Y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = (x, y, x, y)$. The endomorphisms of $T\mathbb{R}^2$, given by

$$P_1(x, y, X, Y) = (x, y, -(-yX + xY)y, (-yX + xY)x) \text{ and}$$
$$P_2(x, y, X, Y) = (x, y, (xX + yY)x, (xX + yY)y)$$

are orthogonal projectors on $\mathcal{D}$ and $\mathcal{D}^{\vdash_g}$ respectively. Their sum is $P = P_1 + P_2 = (x^2 + y^2)I_2$. It is easy to see that $P_1(x, y, 1, 0)$ and $P_1(x, y, 0, 1)$ generate $\mathcal{D}$, while $P_2(x, y, 1, 0)$ and $P_2(x, y, 0, 1)$ generate $\mathcal{D}^{\vdash_g}$. This example can be easy extended to a couple of endomorphisms on $\mathbb{R}^n$ having as images the singular distributions $\mathcal{D}$, tangent to the spheres centered in the origin and the origin itself as a singular point, and $\mathcal{D}^{\vdash_g}$, generated by the position vector field. For $n = 3$, $\mathcal{D}$ is generated by the vector fields

$$\begin{cases} \bar{X}_1 = z\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial z} = (x, y, z, 0, z, -y), \\ \bar{X}_2 = x\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial x} = (x, y, z, -z, 0, x), \\ \bar{X}_3 = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y} = (x, y, z, y, -x, 0), \end{cases}$$

while $\mathcal{D}^{\vdash_g}$ is generated by $X_0 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$. Consider

$$P_1(\bar{X}) = (\bar{X} \cdot \bar{X}_1)\bar{X}_1 + (\bar{X} \cdot \bar{X}_2)\bar{X}_2 + (\bar{X} \cdot \bar{X}_3)\bar{X}_3,$$

which can be expressed as

$$\begin{aligned} P_1(x, y, z, X, Y, Z) &= (x, y, z, (zY - yZ)(0, z, -y) + (-zX + xZ)(-z, 0, x) \\ &\quad +(yX - xY)(y, x, 0)) = (x, y, z, (y^2 + z^2)X - xyY - xzZ, \\ &\quad -xyX + (x^2 + z^2)Y - yzZ, -xzX - yzY + (x^2 + y^2)Z), \end{aligned}$$

and

$$P_2(\bar{X}) = (\bar{X}\cdot\bar{X}_0)\bar{X}_0 = (x, y, z, x^2X + xyY + xzZ, xyX + y^2Y + yzZ + xzX + yzY + z^2Z).$$

Then $P = P_1 + P_2$ has the form $P(\bar{X}) = (x^2 + y^2 + z^2)\bar{X}$.

This example can be extended to $\mathbb{R}^n$, considering the singular distribution $\mathcal{D}$ on U spanned by the vector fields

$$\bar{X}_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad 1 \le i < j \le n + 1.$$

Then $\mathcal{D}^{\vdash_g}$ is generated by $X_0 = x^i\dfrac{\partial}{\partial x^i}$.

An other example is given considerering an open subset $U \subset \mathbb{R}^4$, with coordinates (x, y, z, t) and the distribution $\mathcal{D}$ on U generated by the vector fields $X_1 = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial t}$ and $X_2 = u\frac{\partial}{\partial y} + v\frac{\partial}{\partial z}$, where $u, v : U \to \mathbb{R}$ are two real functions such that the set of their common zeros is an open dense subset of U; for example, $u = xy$ and $v = zt$. If the set of common zeros of u and v is void, then $\mathcal{D}$ is regular, otherwise $\mathcal{D}$ is a singular distribution. The distribution $\mathcal{D}^{\vdash_g}$ contains the distribution $\mathcal{D}'$ generated by the vector fields $Y_1 = v\frac{\partial}{\partial x} - u\frac{\partial}{\partial t}$ and $Y_2 = v\frac{\partial}{\partial y} - u\frac{\partial}{\partial z}$. In fact, according to Proposition 1.2 we have $\mathcal{D}^{\vdash_g} = \mathcal{D}'$.

5 The mean curvature vector fields and the divergence formulas

Let us consider a local orthonormal base $\{e_i\}_{i=\overline{1,m}} \subset \Gamma(TM_U) = \mathcal{X}(U)$ of sections and denote

$$\Pi(e_i) = \Pi_i^j e_j, \quad \Pi^{\vdash_g}(e_i) = \bar{\Pi}_i^j e_j, \quad b(e_i, e_j) = b_{ij}^k e_k, \quad \nabla_{e_i} e_j = \Gamma_{ij}^k e_k.$$

Then

$$b(e_i, e_j) = -\Pi_i^l \left(e_l \left(\Pi_j^s \right) + \Gamma_{lt}^s \Pi_j^t \right) e_s.$$

$$\begin{aligned}&T_1(Y, X_1, X_2, Z) = \\ &(P_2 \nabla_{P_1 X_1} B_2(X_2, Y) - B_2(\nabla_{P_1 X_1} P_1 X_2, Y) - B_2(X_2, \nabla_{P_1 X_1} P_2 Y), Z) = \\ &(\nabla_{P_1 X_1} P_2 \nabla_{P_2 Y} P_1 X_2 - \nabla_{P_2 Y} P_1 \nabla_{P_1 X_1} P_1 X_2 - \nabla_{P_2 \nabla_{P_1 X_1} P_2 Y} P_1 X_2, P_2 Z).\end{aligned}$$

We have:

$$\begin{aligned}(\nabla_{P_1 X_1} P_2 \nabla_{P_2 Y} P_1 X_2, P_2 Z) = {} & P_1 X_1 (P_2 \nabla_{P_2 Y} P_1 X_2, P_2 Z) \\ & -(P_2 \nabla_{P_2 Y} P_1 X_2, \nabla_{P_1 X_1} P_2 Z) \\ (\nabla_{P_2 Y} P_1 \nabla_{P_1 X_1} P_1 X_2, P_2 Z) = {} & -(P_1 \nabla_{P_1 X_1} P_1 X_2, \nabla_{P_2 Y} P_2 Z).\end{aligned}$$

Let us consider $s_{mix} = \sum\limits_{s,t=1}^{m} R(P_2 e_t, P_1 e_s, P_1 e_s, P_2 e_t)$ and

$$\begin{aligned}\operatorname{div}_P X &= Trace(Y \to P \nabla_{PY} X) = \sum_{s=1}^{m} (P \nabla_{P e_s} X, e_s) \\ &= \sum_{s=1}^{m} (\nabla_{P e_s} X, P e_s) = \sum_{s=1}^{m} (P e_s (X, P e_s) - (X, \nabla_{P e_s} P e_s)).\end{aligned}$$

Then

$$\begin{aligned}(H_2, X) &= -Trace(Y \to B_2(X, Y)) = -\sum_{s=1}^{m} \left(B_2 \left(X, e_s \right), e_s \right) \\ &= -\sum_{s=1}^{m} \left(P_2 \nabla_{P_2 e_s} P_1 X, e_s \right) = \sum_{s=1}^{m} \left(P_1 \nabla_{P_2 e_s} P_2 e_s, X \right),\end{aligned}$$

and hence $H_2 = \sum\limits_{s=1}^{m} P_1 \nabla_{P_2 e_s} P_2 e_s$ and we infer

$$\begin{aligned}\operatorname{div}_P H_2 &= \sum_{s,t=1}^{m} (\nabla_{P e_s} P_1 \nabla_{P_2 e_t} P_2 e_t, P e_s) \\ &= \sum_{s,t=1}^{m} (P e_t \left(P_1 \nabla_{P_2 e_s} P_2 e_s, P e_t \right) - (P_1 \nabla_{P_2 e_s} P_2 e_s, \nabla_{P e_t} P e_t)).\end{aligned}$$

Consider further $|H_2|^2_{P_1} = \sum\limits_{s,t=1}^{m} (P_1 \nabla_{P_2 e_s} P_2 e_s, \nabla_{P_2 e_t} P_2 e_t)$ and

$$\begin{aligned}\operatorname{div}_{P_1} X &= Trace(Y \to P_1 \nabla_{P_1 Y} X) = \sum_{s=1}^{m} (P_1 \nabla_{P_1 e_s} X, e_s) \\ &= \sum_{s=1}^{m} (\nabla_{P_1 e_s} X, P_1 e_s) = \sum_{s=1}^{m} (P_1 e_s\,(X, P_1 e_s) - (X, \nabla_{P_1 e_s} P_1 e_s)).\end{aligned}$$

Then we have

$$\begin{aligned}\operatorname{div}_{P_1} H_2 &= \sum_{s,t=1}^{m} (\nabla_{P_1 e_t} P_1 \nabla_{P_2 e_s} P_2 e_s, P_1 e_t) \\ &= \sum_{s,t=1}^{m} (P_1 e_t\,(P_1 \nabla_{P_2 e_s} P_2 e_s, P_1 e_t) - (P_1 \nabla_{P_2 e_s} P_2 e_s, \nabla_{P_1 e_t} P_1 e_t)),\end{aligned}$$

and thus $\operatorname{div}_{P_1} H_2 = \operatorname{div}_P H_2 + |H_2|^2_{P_1}$.

The *second fundamental forms* of the distributions are

$$h_1 : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}^{\vdash_g}) \qquad \text{and} \qquad h_2 : \Gamma(\mathcal{D}^{\vdash_g}) \times \Gamma(\mathcal{D}^{\vdash_g}) \to \Gamma(\mathcal{D}),$$

given by

$$\begin{cases} h_1(X_1, X_2) = P_2\,(\nabla_{P_1 X_1} P_1 X_2 + \nabla_{P_1 X_2} P_1 X_1) \\ h_2(Y, Z) = P_1\,(\nabla_{P_2 Y} P_2 Z + \nabla_{P_2 Z} P_2 Y)\,, \end{cases}$$

and the *integrability tensors* are

$$A_1 : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}^{\vdash_g}) \qquad \text{and} \qquad A_2 : \Gamma(\mathcal{D}^{\vdash_g}) \times \Gamma(\mathcal{D}^{\vdash_g}) \to \Gamma(\mathcal{D}),$$

given by

$$\begin{cases} A_1(X_1, X_2) = P_2\,(\nabla_{P_1 X_1} P_1 X_2 - \nabla_{P_1 X_2} P_1 X_1) \\ A_2(Y, Z) = P_1\,(\nabla_{P_2 Y} P_2 Z - \nabla_{P_2 Z} P_2 Y)\,. \end{cases}$$

One can easily remark that the maps h_2 and A_2 are similarly defined by symmetric formulas. These tensors are involved in the generalized Walczak formula described below.

Proposition 5.1. *We have the generalized Walczak formula*

$$\operatorname*{div}_P (H_1 + H_2) = s_{mix} + \|h_1\|^2 + \|h_2\|^2 - \|A_1\|^2 - \|A_2\|^2 - |H_1|^2_{P_2} - |H_2|^2_{P_1}\,.$$

We note that since in the general case the integral of the P-divergence does not vanish (because of P), the integral over the right side of the above formula (for M compact), does not vanish as well (see [11, 9] for the classical formulas).

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