On control affine systems

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Abstract. In this paper we continue the study of drift less control affine systems (distributional systems) considering a system with no constant rank of distribution and with positive homogeneous cost of Randers type. We will use the Pontryagin Maximum Principle in order to find the general solution. We have to remark that the optimal solutions of the control system are the geodesics in the framework of sub-Riemannian geometry.

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1 Introduction

The paper continues the study of drift less control affine systems (distributional systems) started by the author in [3], [5], [6], considering a system with no constant rank of distribution. It is well known that the optimal solution of a control system (see [1]) is provided by Pontryagin's Maximum Principle: that is the curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton's equations. We have to remark that the optimal solutions of our distributional system are the geodesics in the so-called sub-Riemannian geometry (see [2]). We are in the case of strong bracket generating distribution (i.e. the vector fields of the distribution and the first iterated Lie brackets generate the entire tangent space) with no constant rank. The well-known Chow's theorem guarantees that the system is controllable, that is the system can be brought from any state x_1 to other state x_2.

2 Control systems

Let M be a smooth n-dimensional manifold. We consider the control system

$$\frac{dx^i}{dt} = f^i(x, u),$$

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where $x \in M$ and the control u takes values in an open subset Ω of R^m. Let x_0 and x_1 be two points of M. An optimal control problem consists of finding the trajectories of our control system which connects x_0 and x_1 and minimizing the cost

$$\min \int_0^T L(x(t), u(t))dt, \quad x(0) = x_0, \ x(T) = x_1,$$

where L is the *Lagrangian* or *running cost*.
Necessary conditions for a trajectory to be an extreme are given by Pontryagin's maximum principle. The Hamiltonian reads as

$$H(x, p, u) = \langle p, f(x, u)\rangle - L(x, u), \quad p \in T^*M,$$

while the maximization condition with respect to the control variables u, namely

$$H(x(t), p(t), u(t)) = \max_v H(x(t), p(t), v),$$

leads to

$$\frac{\partial H}{\partial u} = 0.$$

The extreme trajectories satisfy the Hamilton equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{2.1}$$

2.1 Control affine systems

In the following we consider the drift less control-affine systems (distributional systems) given by

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x(t)), \tag{2.2}$$

where $x \in M$, $X_1, X_2, ..., X_m$ are smooth vector fields on M and the control $u = (u_1, u_2, ..., u_m)$ takes values in an open subset Ω of R^m.
The vector fields $X_i, i = \overline{1, m}$, generate a distribution $D \subset TM$ such that the rank of D is not necessary constant. Let x_0 and x_1 be two points of M. An optimal control problem consists of finding those trajectories of the distributional system which connect x_0 and x_1, while minimizing the cost

$$\min_{u(\cdot)} \int_I F(u(t))dt,$$

where F is a Minkowski norm (positive homogeneous) on D. The Lagrangian has the form $L = \frac{1}{2}F^2$.

2.1.1 Application

Let us consider in the three dimensional space R^3 the drift less control affine system

$$\dot{x}(t) = u^1 X_1 + u^2 X_2 + u^3 X_3, \tag{2.3}$$

with

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix},$$

and minimizing the cost

$$\min_{u(.)} \int F(u(t))dt,$$

where $F = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2} + \varepsilon u^1$, $0 \leq \varepsilon < 1$ is the positive homogeneous cost (Randers metric). We are looking for the solution of the above distributional system. The distribution D is generated by the vectors X_1 , X_2, X_3 and

$$rank D = \begin{cases} 3 & if \quad x \neq 0 \\ 2 & if \quad x = 0 \end{cases}$$

In the canonical base $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ of R^3 we have

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial z},$$

and the Lie brackets are given by

$$[X_1, X_2] = 0, \quad [X_1, X_3] = \frac{\partial}{\partial z} = X_4 \notin D, \quad [X_2, X_3] = 0.$$

It results that the distribution is nonholonomic, but is strong bracket generating, because the vector fields $\{X_1, X_2, X_3, X_4 = [X_1, X_3]\}$ generate the entire space R^3. From (2.3) we obtain

$$\frac{dx}{dt} = u^1 \overset{not}{=} s^1, \quad \frac{dy}{dt} = u^2 \overset{not}{=} s^2, \quad \frac{dz}{dt} = u^3 x \overset{not}{=} s^3.$$

The cost function can be written in the form ($x \neq 0$)

$$\begin{aligned} F &= \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2} + \varepsilon u^1 = \sqrt{(s^1)^2 + (s^2)^2 + \frac{(s^3)^2}{x^2}} + \varepsilon s^1 \\ &= \sqrt{g_{ij} s^i s^j} + \sum_{i=1}^{3} b^i s^i, \end{aligned}$$

(Einstein's summation) where $b^1 = \varepsilon$, $b^2 = 0$, $b^3 = 0$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/x^2 \end{pmatrix}.$$

The Lagrangian has the form $L = \frac{1}{2}F^2$ and using [4] we obtain the Hamiltonian

$$H = \frac{1}{2}\left(\sqrt{\widetilde{g}_{ij}p_ip_j} - \widetilde{b}^i p_i\right), \tag{2.4}$$

where

$$\widetilde{g}_{ij} = \frac{1}{1-b^2}g^{ij} + \frac{1}{(1-b^2)^2}b^ib^j, \quad \widetilde{b^i} = \frac{1}{1-b^2}b^i, \quad b = \sqrt{g_{ij}b^ib^j},$$

and g^{ij} is the inverse of the matrix g_{ij}. In these conditions we obtain that

$$b^2 = \varepsilon^2, \quad \widetilde{b}^1 = \frac{\varepsilon}{1-\varepsilon^2}, \quad \widetilde{b}^2 = 0, \quad \widetilde{b}^2 = 0,$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2 \end{pmatrix},$$

and it results

$$\widetilde{g}^{ij} = \begin{pmatrix} \frac{1}{(1-\varepsilon^2)^2} & 0 & 0 \\ 0 & \frac{1}{1-\varepsilon^2} & 0 \\ 0 & 0 & \frac{x^2}{1-\varepsilon^2} \end{pmatrix}.$$

From (2.4) we obtain

$$H = \frac{1}{2}\left(\sqrt{\frac{p_1^2}{(1-\varepsilon^2)^2} + \frac{p_2^2}{1-\varepsilon^2} + \frac{p_3^2x^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2}\right)^2, \tag{2.5}$$

or, in the equivalent form

$$H = \frac{\left(1+\varepsilon^2\right)p_1^2}{2\left(1-\varepsilon^2\right)^2} + \frac{p_2^2 + p_3^2x^2}{2\left(1-\varepsilon^2\right)} - \frac{\varepsilon p_1}{1-\varepsilon^2}\sqrt{\frac{p_1^2}{(1-\varepsilon^2)^2} + \frac{p_2^2+p_3^2x^2}{1-\varepsilon^2}}.$$

I have to remark that in the case $x = 0$ we obtain

$$L = \frac{1}{2}F^2 = \sqrt{\left(u^1\right)^2 + \left(u^2\right)^2} + \varepsilon u^1,$$

with the constraint $\dot{z} = 0$. Using the Lagrange multipliers we obtain

$$L_1 = L + \lambda\,\dot{z},$$

and from Legendre transformation by direct computation it results

$$H_1 = \frac{1}{2}\left(\sqrt{\frac{p_1^2}{(1-\varepsilon)^2} + \frac{p_2^2}{1-\varepsilon}} - \frac{\varepsilon p_1}{1-\varepsilon^2}\right)^2,$$

which leads to the equality

$$H|_{x=0} = H_1.$$

Next, if we denote

$$\Theta = \frac{p_1^2}{(1-\varepsilon^2)^2} + \frac{p_2^2 + p_3^2 x^2}{1-\varepsilon^2},$$

then the Hamilton's equations (2.1) lead to the following differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_1} = \frac{(1+\varepsilon^2)p_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon}{1-\varepsilon^2}\sqrt{\Theta} - \frac{\varepsilon p_1^2}{(1-\varepsilon^2)^3}\frac{1}{\sqrt{\Theta}}, \tag{2.6}$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial p_2} = \frac{p_2}{1-\varepsilon^2} - \frac{\varepsilon p_1 p_2}{(1-\varepsilon^2)^2}\frac{1}{\sqrt{\Theta}}, \tag{2.7}$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_3} = \frac{p_3 x^2}{1-\varepsilon^2} - \frac{\varepsilon p_1 p_3 x^2}{(1-\varepsilon^2)^2}\frac{1}{\sqrt{\Theta}}, \tag{2.8}$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = -\frac{p_3^2 x}{1-\varepsilon^2} + \frac{\varepsilon p_1 p_3^2 x}{(1-\varepsilon^2)^2}\frac{1}{\sqrt{\Theta}}, \tag{2.9}$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = a = const.$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial z} = 0 \Rightarrow p_3 = b = const.$$

Without lose the generality we can consider $a^2 = 1-\varepsilon^2$. In these conditions we consider the following change of variables:

$$x(t) = \frac{\sqrt{1-\varepsilon^2}\sqrt{r^2-1}\sin A\theta(t)}{b}, \quad p_1(t) = \left(1-\varepsilon^2\right)\sqrt{r^2-1}\cos A\theta(t). \tag{2.10}$$

It results $\Theta = r^2(t)$ and from (2.6) we get

$$\frac{dx}{dt} = \frac{(1+\varepsilon^2)\sqrt{r^2-1}\cos A\theta}{1-\varepsilon^2} - \frac{\varepsilon r}{1-\varepsilon^2} - \frac{\varepsilon\left(r^2-1\right)\cos^2 A\theta}{r\left(1-\varepsilon^2\right)}.$$

But

$$\frac{dx}{dt} = \frac{\sqrt{1-\varepsilon^2}}{b}\left(\frac{r\,\dot{r}}{\sqrt{r^2-1}}\sin A\theta + \sqrt{r^2-1}A\,\dot{\theta}\cos A\theta\right),$$

and it results

$$c_1\left(\frac{r\,\dot{r}}{\sqrt{r^2-1}}\sin A\theta + \sqrt{r^2-1}A\,\dot{\theta}\cos A\theta\right) \tag{2.11}$$

$$= \sqrt{r^2-1}(1+\varepsilon^2)\cos A\theta - \varepsilon r - \frac{\varepsilon(r^2-1)\cos^2 A\theta}{r},$$

where we have denoted $c_1 = \dfrac{\left(1-\varepsilon^2\right)\sqrt{1-\varepsilon^2}}{b}$. The equation (2.9) yields

$$\frac{dp_1}{dt} = -\frac{b\sqrt{r^2-1}\sin A\theta}{\sqrt{1-\varepsilon^2}} + \frac{\varepsilon b\left(r^2-1\right)\cos A\theta\sin A\theta}{r\sqrt{1-\varepsilon^2}}.$$

But

$$\frac{dp_1}{dt} = \left(1-\varepsilon^2\right)\left(\frac{r\,\dot{r}}{\sqrt{r^2-1}}\cos A\theta - \sqrt{r^2-1}A\,\dot{\theta}\,\sin A\theta\right),$$

which leads to

$$c_1\left(\frac{r\,\dot{r}}{\sqrt{r^2-1}}\cos A\theta - \sqrt{r^2-1}A\,\dot{\theta}\,\sin A\theta\right) \tag{2.12}$$

$$= -\sqrt{r^2-1}\sin A\theta + \frac{\varepsilon\left(r^2-1\right)\cos A\theta\sin A\theta}{r}.$$

The equation (2.11) multiplied by $\cos A\theta$, minus the equation (2.12) multiplied by $\sin A\theta$ lead to the equation

$$c_1 A\frac{d\theta}{dt} = \left(\varepsilon\cos A\theta - \frac{r}{\sqrt{r^2-1}}\right)\left(\varepsilon\cos A\theta - \frac{\sqrt{r^2-1}}{r}\right). \tag{2.13}$$

Moreover, the equation (2.11) multiplied by $\sin A\theta$, plus the equation (2.12) multiplied by $\cos A\theta$ lead to the equation

$$c_1 r\frac{dr}{dt} = \varepsilon\left(r^2-1\right)\sin A\theta\left(\varepsilon\cos A\theta - \frac{r}{\sqrt{r^2-1}}\right). \tag{2.14}$$

Using (2.10) by direct computation, the Hamiltonian become

$$H = \frac{1}{2}\left(r - \varepsilon\sqrt{r^2-1}\cos A\theta\right)^2.$$

Considering the integral curves parameterized by arclength, that corresponds to fix the level $\frac{1}{2}$ of the Hamiltonian, we obtain

$$r - \varepsilon\sqrt{r^2-1}\cos A\theta = 1,$$

and it result

$$r = \frac{1+\varepsilon^2\cos^2 A\theta}{1-\varepsilon^2\cos^2 A\theta}. \tag{2.15}$$

In these conditions, from (2.15) we obtain

$$x(t) = \frac{\varepsilon\sqrt{1-\varepsilon^2}}{b}\frac{\sin 2A\theta}{1-\varepsilon\cos A\theta}. \tag{2.16}$$

The equation (2.13) leads to

$$c_1 A\frac{d\theta}{dt} = \frac{\left(1-\varepsilon^2\cos^2 A\theta\right)^2}{2\left(1+\varepsilon^2\cos^2 A\theta\right)} \tag{2.17}$$

The differential equation (2.7) yields

$$\frac{dy}{dt} = \frac{a}{1-\varepsilon^2} - \frac{a\varepsilon\left(1-\varepsilon^2\right)\sqrt{r^2-1}\cos A\theta}{(1-\varepsilon^2)^2}\frac{1}{r},$$

and it results

$$\frac{dy}{dt} = \frac{r - \varepsilon\sqrt{r^2 - 1}\cos A\theta}{r\sqrt{1-\varepsilon^2}} = \frac{1}{r\sqrt{1-\varepsilon^2}},$$

or

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-\varepsilon^2}} \frac{1 - \varepsilon^2\cos^2 A\theta}{1 + \varepsilon^2\cos^2 A\theta}.$$

Using (2.17) we obtain

$$y(t) = \frac{2(1-\varepsilon^2)A}{b} \int \frac{d\theta}{1 - \varepsilon^2\cos^2 A\theta}.$$

The equations (2.8) yields

$$\frac{dz}{dt} = \frac{\left(r^2 - 1\right)\sin^2 A\theta}{br},$$

and using (2.17) we get

$$z(t) = \frac{2A\varepsilon^2(1-\varepsilon^2)^{3/2}}{b^2} \int \frac{\sin^2 2A\theta}{(1 - \varepsilon^2\cos^2 A\theta)^3} d\theta.$$

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