Multi-time Kosambi-Cartan-Chern invariants and applications

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Abstract. In this paper we construct some multi-time geometrical extensions of the Kosambi-Cartan-Chern (KCC)-invariants, which characterize a given second-order system of PDEs on the 1-jet space $J^1(T, M)$. A theorem of characterization of these multi-time geometrical KCC-invariants is given.

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1 Introduction

Informations about the both Lyapunov and Jacobi stability of a second-order system of differential equations may be obtained by studying the five KCC invariants of the given SODEs (see [1], [2] and [6]). We interpret the Jacobi stability as the relative insensitivity to alteration of the internal parameters and the ability to adapt to changes in environment. From the point of view of the differential geometric theory of the variational equations for deviation of whole trajectories to nearby ones, the KCC invariants allow us to estimate the admissible perturbation around the steady-states of the given SODEs (see [6]).

The tangent KCC-theory was initiated in the works of D.D. Kosambi [15], E. Cartan [9] and S.S. Chern [10], and developed further in the autonomous (i.e., time-independent) Finslerian geometric framework by P.L. Antonelli and I. Bucătaru ([1], [2]). The KCC theory can be applied in biology, population genetics, engineering, ecology, plasma physics and in Belousov-Zhabotinskii reaction model in chemistry (see Balan-Nicola [4], [7] and references therein). At the same time, Antonelli, Bucătaru and Lackey underline the importance of the KCC-theory in the Volterra-Hamilton theory (see [3]) and Analytical trophodynamics (see [16]), where intrinsic properties like curvature determine the stability of production processes.

Finally, note that, in the paper [5], Balan and Neagu present the basic elements of the *single-time* KCC-theory on the particular 1-jet space

$$J^1(\mathbb{R}, M) \equiv \mathbb{R} \times TM,$$

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where $\mathbb{R}$ is the set of real numbers. From our point of view, this represents a non-autonomous (i.e., time-dependent) Finslerian framework of the KCC-theory, in a jet geometrical approach.

2 Geometrical objects on multi-time 1-jet spaces

We remind first few differential geometrical properties of the multi-time 1-jet spaces. The multi-time 1-jet bundle

$$\xi_1 = (J^1(T, M), \pi_1, T \times M)$$

is a vector bundle over the product manifold $T \times M$, having the fibre type $\mathbb{R}^{mn}$, where m (resp. n) is the dimension of the *temporal* (resp. *spatial*) manifold T (resp. M). If the temporal manifold T has the local coordinates $(t^\alpha)_{\alpha=\overline{1,m}}$ and the spatial manifold M has the local coordinates $(x^i)_{i=\overline{1,n}}$, then we denote the local coordinates of the multi-time 1-jet space $J^1(T, M)$ by $(t^\alpha, x^i, x^i_\alpha)$. These transform by the rules [17]

$$\text{(2.1)} \qquad \begin{cases} \widetilde{t}^\alpha = \widetilde{t}^\alpha(t^\beta) \\ \widetilde{x}^i = \widetilde{x}^i(x^j) \\ \widetilde{x}^i_\alpha = \dfrac{\partial \widetilde{x}^i}{\partial x^j} \dfrac{\partial t^\beta}{\partial \widetilde{t}^\alpha} x^j_\beta, \end{cases}$$

where $\det(\partial\widetilde{t}^\alpha/\partial t^\beta) \neq 0$ and $\det(\partial\widetilde{x}^i/\partial x^j) \neq 0$.

Remark 2.1. In this work the Greek indices $\alpha, \beta, \gamma, \delta, \mu, \nu...$ run over the set $\{1, 2, ..., m\}$ and the Latin indices $i, j, k, l, p, q, ...$ run over the set $\{1, 2, ..., n\}$. The Einstein convention of summation is also adopted all over this paper.

A lot of important geometrical objects on 1-jet spaces have been intensively studied by Udrişte and Neagu in the works [20] and [17]. For example, in the geometrical study of the multi-time 1-jet vector bundle, a central rôle is played by the *distinguished tensors* ($d-$tensors).

Definition 2.2. A geometrical object $D = \left(D^{\alpha i(j)(\nu)...}_{\gamma k(\beta)(l)...}\right)$ on the *1*-jet vector bundle $J^1(T, M)$, whose local components transform by the rules

$$\text{(2.2)} \qquad D^{\alpha i(j)(\nu)...}_{\gamma k(\beta)(l)...} = \widetilde{D}^{\delta p(q)(\eta)...}_{\varepsilon r(\mu)(s)...} \frac{\partial t^\alpha}{\partial \widetilde{t}^\delta} \frac{\partial x^i}{\partial \widetilde{x}^p} \left(\frac{\partial x^j}{\partial \widetilde{x}^q} \frac{\partial \widetilde{t}^\mu}{\partial t^\beta} \right) \frac{\partial \widetilde{t}^\varepsilon}{\partial t^\gamma} \frac{\partial \widetilde{x}^r}{\partial x^k} \left(\frac{\partial \widetilde{x}^s}{\partial x^l} \frac{\partial t^\nu}{\partial \widetilde{t}^\eta} \right) \dots ,$$

is called a $d-$*tensor field*.

Remark 2.3. The use of parentheses for certain indices of the local components $D^{\alpha i(j)(\nu)...}_{\gamma k(\beta)(l)...}$ of the distinguished tensor field D on the *1*-jet space is motivated by the fact that the pair of indices " $\begin{smallmatrix}(j)\\(\beta)\end{smallmatrix}$ " or " $\begin{smallmatrix}(\nu)\\(l)\end{smallmatrix}$ " behaves like a single index.

Example 2.4. The geometrical object

$$\mathbf{C} = \mathbf{C}^{(i)}_{(\alpha)} \frac{\partial}{\partial x^i_\alpha},$$

where $\mathbf{C}^{(i)}_{(\alpha)} = x^i_\alpha$, represents a $d-$tensor field on the *1*-jet space. This is called the *canonical Liouville d−tensor field* of the *1*-jet vector bundle $J^1(T, M)$, and it is a global geometrical object.

Example 2.5. Let $h = (h_{\alpha\beta}(t))$ be a Riemannian metric on the temporal manifold T. The geometrical object

$$\mathbf{J}_h = J^{(i)}_{(\alpha)\beta j} \frac{\partial}{\partial x^i_\alpha} \otimes dt^\beta \otimes dx^j,$$

where $J^{(i)}_{(\alpha)\beta j} = h_{\alpha\beta}\delta^i_j$ is a distinguished tensor field on $J^1(T, M)$, which is called the *h-normalization d−tensor field* of the *1*-jet space $J^1(T, M)$. Obviously, it is also a global geometrical object.

In the Riemann-Lagrange differential geometry of 1-jet spaces developed in [17] and [18], important rôles are also played by geometrical objects as the *temporal* or *spatial semisprays*, together with the *multi-time jet nonlinear connections.*

Definition 2.6. A set of local functions $H = \left(H^{(i)}_{(\alpha)\beta}\right)$ on $J^1(T, M)$, which transform by the rules

$$2\widetilde{H}^{(i)}_{(\alpha)\beta} = 2H^{(k)}_{(\gamma)\nu} \frac{\partial \widetilde{x}^i}{\partial x^k} \frac{\partial t^\gamma}{\partial \widetilde{t}^\alpha} \frac{\partial t^\nu}{\partial \widetilde{t}^\beta} - \frac{\partial t^\mu}{\partial \widetilde{t}^\beta} \frac{\partial \widetilde{x}^i_\alpha}{\partial t^\mu}, \tag{2.3}$$

is called a *temporal semispray* on $J^1(T, M)$.

Example 2.7. Let us consider a Riemannian metric $h = (h_{\alpha\beta}(t))$ on the temporal manifold T and let

$$\kappa^\alpha_{\beta\gamma} = \frac{h^{\alpha\mu}}{2}\left(\frac{\partial h_{\beta\mu}}{\partial t^\gamma} + \frac{\partial h_{\gamma\mu}}{\partial t^\beta} - \frac{\partial h_{\beta\gamma}}{\partial t^\mu}\right)$$

be its Christoffel symbols. Taking into account that we have the transformation rules

$$\widetilde{\kappa}^\delta_{\nu\rho} = \kappa^\alpha_{\beta\gamma} \frac{\partial \widetilde{t}^\delta}{\partial t^\alpha} \frac{\partial t^\beta}{\partial \widetilde{t}^\nu} \frac{\partial t^\gamma}{\partial \widetilde{t}^\rho} + \frac{\partial \widetilde{t}^\delta}{\partial t^\varepsilon} \frac{\partial^2 t^\varepsilon}{\partial \widetilde{t}^\nu \partial \widetilde{t}^\rho}, \tag{2.4}$$

we deduce that the local components

$$\mathring{H}^{(i)}_{(\alpha)\beta} = -\frac{1}{2}\kappa^\mu_{\alpha\beta} x^i_\mu$$

define a temporal semispray $\mathring{H} = \left(\mathring{H}^{(i)}_{(\alpha)\beta}\right)$ on $J^1(T, M)$. This is called the *canonical temporal semispray associated to the temporal metric* $h_{\alpha\beta}(t)$.

Definition 2.8. A set of local functions $G = \left(G^{(i)}_{(\alpha)\beta}\right)$, which transform by the rules

$$2\widetilde{G}^{(i)}_{(\alpha)\beta} = 2G^{(k)}_{(\gamma)\nu}\frac{\partial\widetilde{x}^i}{\partial x^k}\frac{\partial t^\gamma}{\partial\widetilde{t}^\alpha}\frac{\partial t^\nu}{\partial\widetilde{t}^\beta} - \frac{\partial x^r}{\partial\widetilde{x}^s}\frac{\partial\widetilde{x}^i_\alpha}{\partial x^r}\widetilde{x}^s_\beta, \tag{2.5}$$

is called a *spatial semispray* on $J^1(T, M)$.

Example 2.9. Let $\varphi = (\varphi_{ij}(x))$ be a Riemannian metric on the spatial manifold M, and let us consider

$$\gamma^i_{jk} = \frac{\varphi^{im}}{2}\left(\frac{\partial\varphi_{jm}}{\partial x^k} + \frac{\partial\varphi_{km}}{\partial x^j} - \frac{\partial\varphi_{jk}}{\partial x^m}\right)$$

its Christoffel symbols. Taking into account that we have the transformation rules

$$\widetilde{\gamma}^p_{qr} = \gamma^i_{jk}\frac{\partial\widetilde{x}^p}{\partial x^i}\frac{\partial x^j}{\partial\widetilde{x}^q}\frac{\partial x^k}{\partial\widetilde{x}^r} + \frac{\partial\widetilde{x}^p}{\partial x^l}\frac{\partial^2 x^l}{\partial\widetilde{x}^q\partial\widetilde{x}^r}, \tag{2.6}$$

we deduce that the local components

$$\mathring{G}^{(i)}_{(\alpha)\beta} = \frac{1}{2}\gamma^i_{pq}x^p_\alpha x^q_\beta$$

define a spatial semispray $\mathring{G} = \left(\mathring{G}^{(i)}_{(\alpha)\beta}\right)$ on $J^1(T, M)$. This is called the *canonical spatial semispray associated to the spatial metric* $\varphi_{ij}(x)$.

Definition 2.10. A set of local functions $\Gamma = \left(M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}\right)$ on $J^1(T, M)$, which transform by the rules

$$\widetilde{M}^{(i)}_{(\alpha)\beta} = M^{(k)}_{(\gamma)\nu}\frac{\partial\widetilde{x}^i}{\partial x^k}\frac{\partial t^\gamma}{\partial\widetilde{t}^\alpha}\frac{\partial t^\nu}{\partial\widetilde{t}^\beta} - \frac{\partial t^\mu}{\partial\widetilde{t}^\beta}\frac{\partial\widetilde{x}^i_\alpha}{\partial t^\mu} \tag{2.7}$$

and

$$\widetilde{N}^{(i)}_{(\alpha)j} = N^{(k)}_{(\gamma)l}\frac{\partial\widetilde{x}^i}{\partial x^k}\frac{\partial t^\gamma}{\partial\widetilde{t}^\alpha}\frac{\partial x^l}{\partial\widetilde{x}^j} - \frac{\partial x^r}{\partial\widetilde{x}^j}\frac{\partial\widetilde{x}^i_\alpha}{\partial x^r}, \tag{2.8}$$

is called a *nonlinear connection* on the *1*-jet space $J^1(T, M)$.

Example 2.11. Let us consider that $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$ are Riemannian manifolds having the Christoffel symbols $\kappa^\alpha_{\beta\gamma}(t)$ and $\gamma^i_{jk}(x)$. Then, using the transformation rules (2.1), (2.4) and (2.6), we deduce that the set of local functions

$$\mathring{\Gamma} = \left(\mathring{M}^{(i)}_{(\alpha)\beta}, \mathring{N}^{(i)}_{(\alpha)j}\right),$$

where

$$\mathring{M}^{(i)}_{(\alpha)\beta} = -\kappa^\mu_{\alpha\beta}x^i_\mu, \qquad \mathring{N}^{(i)}_{(\alpha)j} = \gamma^i_{jr}x^r_\alpha,$$

represents a nonlinear connection on the *1*-jet space $J^1(T, M)$. This multi-time jet nonlinear connection is called the *canonical nonlinear connection attached to the pair of Riemannian metrics* $(h_{\alpha\beta}(t), \varphi_{ij}(x))$.

In what follows, let us expose the geometrical relations between *temporal* or *spatial semisprays* and *multi-time nonlinear connections* on the 1-jet vector bundle $J^1(T,M)$. In this direction, using the local transformation laws (2.3), (2.7) and (2.1), respectively the transformation laws (2.5), (2.8) and (2.1), by direct local computations, we find the following geometrical results:

Theorem 2.1. *The temporal semisprays* $H=\left(H^{(i)}_{(\alpha)\beta}\right)$ *and the sets of temporal components of nonlinear connections* $\Gamma_{\text{temporal}}=\left(M^{(i)}_{(\alpha)\beta}\right)$ *are in one-to-one correspondence on the* 1*-jet space* $J^1(T,M)$*, via:*

$$M^{(i)}_{(\alpha)\beta}=2H^{(i)}_{(\alpha)\beta},\qquad H^{(i)}_{(\alpha)\beta}=\frac{1}{2}M^{(i)}_{(\alpha)\beta};$$

Theorem 2.2. **(i)** *If* $G^{(i)}_{(\alpha)\beta}$ *are the components of a spatial semispray on the* 1*-jet space* $J^1(T,M)$*, where* $(T,h_{\alpha\beta}(t))$ *is a Riemannian manifold, then the components*

$$N^{(i)}_{(\alpha)j}=\frac{\partial G^i}{\partial x^j_\mu}h_{\alpha\mu},$$

where $G^i=h^{\delta\varepsilon}G^{(i)}_{(\delta)\varepsilon}$*, represent a spatial nonlinear connection on* $J^1(T,M)$*.*

(ii) *Conversely, the spatial nonlinear connection* $\Gamma_{\text{spatial}}=\left(N^{(i)}_{(\alpha)j}\right)$ *produces the spatial semispray components*

$$G^{(i)}_{(\alpha)\beta}=\frac{1}{2}N^{(i)}_{(\alpha)r}x^r_\beta.$$

3 Jet multi-time geometrical KCC-theory

In this Section we construct some multi-time generalizations on the 1-jet space $J^1(T,M)$ of the basic objects of the tangent KCC-theory (see [1], [2], [7], [19]). More exactly, in the present paper we give a natural *multi-time extension* of the results exposed in [5]. This means that we extend the jet ordinary KCC geometrical framework from [5] for second-order systems of partial (multi-time or multi-parameter) differential equations.

In this respect, let us consider on $J^1(T,M)$ a second-order system of partial differential equations of local form

$$\frac{\partial^2 x^i}{\partial t^\alpha\partial t^\beta}+F^{(i)}_{(\alpha)\beta}(t^\gamma,x^k,x^k_\gamma)=0,\quad \alpha,\beta=\overline{1,m},\quad i=\overline{1,n},\tag{3.1}$$

where $x^k_\gamma=\partial x^k/\partial t^\gamma$, $F^{(i)}_{(\alpha)\beta}=F^{(i)}_{(\beta)\alpha}$ and the local components $F^{(i)}_{(\alpha)\beta}(t^\gamma,x^k,x^k_\gamma)$ transform, under a change of coordinates (2.1), by the rules

$$\widetilde{F}^{(i)}_{(\alpha)\beta}=2F^{(k)}_{(\gamma)\nu}\frac{\partial\widetilde{x}^i}{\partial x^k}\frac{\partial t^\gamma}{\partial\widetilde{t}^\alpha}\frac{\partial t^\nu}{\partial\widetilde{t}^\beta}-\frac{\partial t^\mu}{\partial\widetilde{t}^\beta}\frac{\partial\widetilde{x}^i_\alpha}{\partial t^\mu}-\frac{\partial x^r}{\partial\widetilde{x}^s}\frac{\partial\widetilde{x}^i_\alpha}{\partial x^r}\widetilde{x}^s_\beta.\tag{3.2}$$

Remark 3.1. The second-order system of partial differential equations (3.1) is invariant under a change of coordinates (2.1).

Example 3.2. Let us consider that $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$ are Riemannian manifolds having the Christoffel symbols $\kappa^\alpha_{\beta\gamma}(t)$ and $\gamma^i_{jk}(x)$. Then, the local components

$$\mathring{F}^{(i)}_{(\alpha)\beta} = -\kappa^\mu_{\alpha\beta} x^i_\mu + \gamma^i_{pq} x^p_\alpha x^q_\beta$$

transform under a change of coordinates (2.1) by the rules (3.2). In this particular case, the PDE system (3.1) becomes

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - \kappa^\mu_{\alpha\beta} x^i_\mu + \gamma^i_{pq} x^p_\alpha x^q_\beta = 0, \quad \alpha, \beta = \overline{1, m}, \quad i = \overline{1, n}, \tag{3.3}$$

that is the PDE system of the affine maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$. We recall that these affine maps carry the geodesics of the temporal Riemannian manifold $(T, h_{\alpha\beta}(t))$ into the geodesics of the spatial Riemannian manifold $(M, \varphi_{ij}(x))$. Moreover, the $h-$trace of the equations (3.3) produces the equations of the harmonic maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$. For more details, see [14].

Using a temporal Riemannian metric $h_{\alpha\beta}(t)$ on T and taking into account the transformation rules (2.3), (2.5) and (3.2), we can rewrite the PDE system (3.1) in the following form:

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - \kappa^\mu_{\alpha\beta} x^i_\mu + 2G^{(i)}_{(\alpha)\beta}(t^\gamma, x^k, x^k_\gamma) = 0, \quad \alpha, \beta = \overline{1, m}, \quad i = \overline{1, n},$$

where

$$G^{(i)}_{(\alpha)\beta} = \frac{1}{2} F^{(i)}_{(\alpha)\beta} + \frac{1}{2} \kappa^\mu_{\alpha\beta} x^i_\mu$$

are the components of a spatial semispray on $J^1(T, M)$. The coefficients of the spatial semispray $G^{(i)}_{(\alpha)\beta}$ produce the spatial components $N^{(i)}_{(\alpha)j}$ of a nonlinear connection Γ on the 1-jet space $J^1(T, M)$, by putting

$$N^{(i)}_{(\alpha)j} = \frac{h^{\mu\nu} \partial G^{(i)}_{(\mu)\nu}}{\partial x^j_\gamma} h_{\gamma\alpha} = \frac{h^{\mu\nu}}{2} \frac{\partial F^{(i)}_{(\mu)\nu}}{\partial x^j_\gamma} h_{\gamma\alpha} + \frac{h^{\mu\nu}}{2} \kappa^\gamma_{\mu\nu} h_{\gamma\alpha} \delta^i_j.$$

In order to find the basic jet multi-time geometrical invariants of the PDE system (3.1) (see Kosambi [15], Cartan [9] and Chern [10]) under the coordinate transformations (2.1), we define the *$h-$KCC-covariant derivative of a $d-$tensor of type* $T^{(i)}_{(\alpha)}(t^\gamma, x^k, x^k_\gamma)$ on the 1-jet space $J^1(T, M)$, via

$$\begin{aligned} \frac{\overset{h}{\nabla} T^{(i)}_{(\alpha)}}{\partial t^\beta} &= \frac{\partial T^{(i)}_{(\alpha)}}{\partial t^\beta} + N^{(i)}_{(\alpha)r} T^{(r)}_{(\beta)} - \kappa^\mu_{\alpha\beta} T^{(i)}_{(\mu)} = \\ &= \frac{\partial T^{(i)}_{(\alpha)}}{\partial t^\beta} + \frac{h^{\mu\nu}}{2} \frac{\partial F^{(i)}_{(\mu)\nu}}{\partial x^r_\gamma} h_{\gamma\alpha} T^{(r)}_{(\beta)} + \frac{h^{\mu\nu}}{2} \kappa^\gamma_{\mu\nu} h_{\gamma\alpha} T^{(i)}_{(\beta)} - \kappa^\mu_{\alpha\beta} T^{(i)}_{(\mu)}. \end{aligned}$$

Remark 3.3. The $h-$KCC-covariant derivative components $\dfrac{\overset{h}{\nabla} T^{(i)}_{(\alpha)}}{\partial t^\beta}$ transform under a change of coordinates (2.1) as a $d-$tensor of type $\mathcal{T}^{(i)}_{(\alpha)\beta}$.

In such a geometrical context, if we use the notation $x_\alpha^i = \partial x^i/\partial t^\alpha$, then the PDE system (3.1) can be rewritten in the following distinguished tensorial form:

$$\begin{aligned}\frac{\overset{h}{\nabla} x_\alpha^i}{\partial t^\beta} &= -F_{(\alpha)\beta}^{(i)}(t^\gamma, x^k, x_\gamma^k) + N_{(\alpha)r}^{(i)} x_\beta^r - \kappa_{\alpha\beta}^\mu x_\mu^i = \\ &= -F_{(\alpha)\beta}^{(i)} + \frac{h^{\mu\nu}}{2}\frac{\partial F_{(\mu)\nu}^{(i)}}{\partial x_\gamma^r} h_{\gamma\alpha} x_\beta^r + \frac{h^{\mu\nu}}{2}\kappa_{\mu\nu}^\gamma h_{\gamma\alpha} x_\beta^i - \kappa_{\alpha\beta}^\mu x_\mu^i.\end{aligned}$$

Definition 3.4. The distinguished tensor

$$\overset{h}{\varepsilon}{}_{(\alpha)\beta}^{(i)} = -F_{(\alpha)\beta}^{(i)} + \frac{h^{\mu\nu}}{2}\frac{\partial F_{(\mu)\nu}^{(i)}}{\partial x_\gamma^r} h_{\gamma\alpha} x_\beta^r + \frac{h^{\mu\nu}}{2}\kappa_{\mu\nu}^\gamma h_{\gamma\alpha} x_\beta^i - \kappa_{\alpha\beta}^\mu x_\mu^i$$

is called the *first multi-time* $h-$*KCC-invariant* on the *1*-jet space $J^1(T, M)$ of the PDEs (3.1). Sometimes, this can be interpreted as a multi-time *external force* (see [7]).

Example 3.5. For the second-order PDE system (3.3), which gives the affine maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$, the first multi-time $h-$KCC-invariant is zero.

Example 3.6. For the particular first order PDE system

$$\frac{\partial x^i}{\partial t^\alpha} = X_{(\alpha)}^{(i)}(t^\gamma, x^k) \Rightarrow \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} = \frac{\partial X_{(\alpha)}^{(i)}}{\partial t^\beta} + \frac{\partial X_{(\alpha)}^{(i)}}{\partial x^r} x_\beta^r, \tag{3.4}$$

where $X_{(\alpha)}^{(i)}(t, x)$ is a given $d-$tensor on $J^1(T, M)$, the first multi-time $h-$KCC-invariant has the form

$$\overset{h}{\varepsilon}{}_{(\alpha)\beta}^{(i)} = \frac{\partial X_{(\alpha)}^{(i)}}{\partial t^\beta} + \frac{1}{2}\frac{\partial X_{(\alpha)}^{(i)}}{\partial x^r} x_\beta^r + \frac{h^{\mu\nu}}{2}\kappa_{\mu\nu}^\gamma h_{\gamma\alpha} x_\beta^i - \kappa_{\alpha\beta}^\mu x_\mu^i.$$

In what follows, let us vary the solutions $x^i(t^\gamma)$ of the PDE system (3.1) by the nearby smooth maps $(\overline{x}^i(t^\gamma, s))_{s\in(-\varepsilon,\varepsilon)}$, where $\overline{x}^i(t^\gamma, 0) = x^i(t^\gamma)$. Then, if we consider the *variation* $d-$*tensor field*

$$\xi^i(t^\gamma) = \left.\frac{\partial \overline{x}^i}{\partial s}\right|_{s=0},$$

we get the *variational equations*

$$\frac{\partial^2 \xi^i}{\partial t^\alpha \partial t^\beta} + \frac{\partial F_{(\alpha)\beta}^{(i)}}{\partial x^k}\xi^k + \frac{\partial F_{(\alpha)\beta}^{(i)}}{\partial x_\mu^r}\frac{\partial \xi^r}{\partial t^\mu} = 0, \quad \alpha, \beta = \overline{1, m}, \quad i = \overline{1, n}. \tag{3.5}$$

It is obvious that the equations (3.5) imply the $h-$*trace variational equations*

$$h^{\alpha\beta}\frac{\partial^2 \xi^i}{\partial t^\alpha \partial t^\beta} + \frac{\partial F^i}{\partial x^k}\xi^k + \frac{\partial F^i}{\partial x_\mu^r}\frac{\partial \xi^r}{\partial t^\mu} = 0, \quad i = \overline{1, n}, \tag{3.6}$$

where $F^i = h^{\alpha\beta}F^{(i)}_{(\alpha)\beta}$. To find other multi-time geometrical invariants for the PDE system (3.1), we also introduce the *$h-$KCC-covariant derivative of a $d-$tensor of type* $\xi^i(t^\gamma)$ on the 1-jet space $J^1(T, M)$, via

$$\frac{\overset{h}{\nabla}\xi^i}{\partial t^\alpha} = \frac{\partial \xi^i}{\partial t^\alpha} + N^{(i)}_{(\alpha)r}\xi^r = \frac{\partial \xi^i}{\partial t^\alpha} + \frac{1}{2}\frac{\partial F^i}{\partial x^r_\gamma}h_{\gamma\alpha}\xi^r + \frac{1}{2}\kappa^\gamma h_{\gamma\alpha}\xi^i, \tag{3.7}$$

where $\kappa^\gamma = h^{\mu\nu}\kappa^\gamma_{\mu\nu}$.

Remark 3.7. The $h-$KCC-covariant derivative components $\dfrac{\overset{h}{\nabla}\xi^i}{\partial t^\alpha}$ transform, via a change of coordinates (2.1), as a $d-$tensor of type $\mathrm{T}^{(i)}_{(\alpha)}$.

In this geometrical context, the $h-$trace variational equations (3.6) can be rewritten in the following distinguished tensorial form:

$$h^{\alpha\beta}\frac{\overset{h}{\nabla}}{\partial t^\beta}\left[\frac{\overset{h}{\nabla}\xi^i}{\partial t^\alpha}\right] = \overset{h}{P}{}^i_r\xi^r,$$

where

$$\begin{aligned}\overset{h}{P}{}^i_j \;=\; & -\frac{\partial F^i}{\partial x^j} + \frac{1}{2}\frac{\partial^2 F^i}{\partial t^\gamma \partial x^j_\gamma} + \frac{1}{2}\frac{\partial^2 F^i}{\partial x^r \partial x^j_\gamma}x^r_\gamma - \frac{1}{2}\frac{\partial^2 F^i}{\partial x^j_\mu \partial x^r_\gamma}F^{(r)}_{(\gamma)\mu} + \\ & + \frac{h_{\gamma\mu}}{4}\frac{\partial F^i}{\partial x^r_\gamma}\frac{\partial F^r}{\partial x^j_\mu} + \frac{h^{\gamma\eta}}{2}\frac{\partial h_{\mu\gamma}}{\partial t^\eta}\frac{\partial F^i}{\partial x^j_\mu} + \\ & + \left[\frac{1}{2}\frac{\partial \kappa^\gamma}{\partial t^\gamma} + \frac{h^{\gamma\eta}}{2}\frac{\partial h_{\mu\gamma}}{\partial t^\eta}\kappa^\mu - \frac{h_{\gamma\mu}}{4}\kappa^\gamma\kappa^\mu\right]\delta^i_j.\end{aligned}$$

Definition 3.8. The $d-$tensor $\overset{h}{P}{}^i_j$ is called the *second multi-time $h-$KCC-invariant* on the *1*-jet space $J^1(T, M)$ of the PDE system (3.1), or the *multi-time $h-$deviation curvature $d-$tensor* (see [19]).

Example 3.9. If we consider the second-order PDE system of the affine maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$, system which is given by (3.3), then the second multi-time $h-$KCC-invariant has the form

$$\overset{h}{P}{}^i_j = -h^{\alpha\beta}\mathfrak{R}^i_{pqj}x^p_\alpha x^q_\beta,$$

where

$$\mathfrak{R}^i_{pqj} = \frac{\partial \gamma^i_{pq}}{\partial x^j} - \frac{\partial \gamma^i_{pj}}{\partial x^q} + \gamma^r_{pq}\gamma^i_{rj} - \gamma^r_{pj}\gamma^i_{rq}$$

are the components of the curvature of the spatial Riemannian metric $\varphi_{ij}(x)$. Consequently, the $h-$trace variational equations (3.6) become the following *multi-time $h-$Jacobi field equations:*

$$h^{\alpha\beta}\left\{\frac{\overset{h}{\nabla}}{\partial t^\beta}\left[\frac{\overset{h}{\nabla}\xi^i}{\partial t^\alpha}\right] + \mathfrak{R}^i_{pqr}x^p_\alpha x^q_\beta\xi^r\right\} = 0,$$

where

$$\frac{\overset{h}{\nabla}\xi^i}{\partial t^\alpha}=\frac{\partial \xi^i}{\partial t^\alpha}+\gamma^i_{pr}x^p_\alpha\xi^r.$$

Example 3.10. For the particular first order PDE system (3.4), the multi-time $h-$deviation curvature $d-$tensor is given by

$$\begin{aligned}\overset{h}{P}{}^i_j &= \frac{h^{\alpha\beta}}{2}\left[\frac{\partial^2 X^{(i)}_{(\alpha)}}{\partial t^\beta\partial x^j}+\frac{\partial^2 X^{(i)}_{(\alpha)}}{\partial x^j\partial x^r}x^r_\beta+\frac{1}{2}\frac{\partial X^{(i)}_{(\alpha)}}{\partial x^r}\frac{\partial X^{(r)}_{(\beta)}}{\partial x^j}\right]+\\ &+\left[\frac{1}{2}\frac{\partial \kappa^\gamma}{\partial t^\gamma}+\frac{h^{\gamma\eta}}{2}\frac{\partial h_{\mu\gamma}}{\partial t^\eta}\kappa^\mu-\frac{h_{\gamma\mu}}{4}\kappa^\gamma\kappa^\mu\right]\delta^i_j.\end{aligned}$$

Definition 3.11. The distinguished tensors

$$\overset{h}{R}{}^{i\alpha}_{jk}=\frac{1}{3}\left[\frac{\partial \overset{h}{P}{}^i_j}{\partial x^k_\alpha}-\frac{\partial \overset{h}{P}{}^i_k}{\partial x^j_\alpha}\right],\qquad \overset{h}{B}{}^{i\alpha(\beta)}_{jk(l)}=\frac{\partial \overset{h}{R}{}^{i\alpha}_{jk}}{\partial x^l_\beta}$$

and

$$D^{(i)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(j)(k)(l)}=\frac{\partial^3 F^{(i)}_{(\alpha)\beta}}{\partial x^j_\gamma\partial x^k_\varepsilon\partial x^l_\mu}$$

are called the *third*, *fourth* and *fifth multi-time $h-$KCC-invariant* on the *1*-jet vector bundle $J^1(T,M)$ of the PDE system (3.1).

Remark 3.12. Taking into account the transformation rules (3.2) of the components $F^{(i)}_{(\alpha)\beta}$, we immediately deduce that the components $D^{(i)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(j)(k)(l)}$ behave like a $d-$tensor on the *1*-jet space $J^1(T,M)$.

Remark 3.13. The distinguished tensor

$$D^{(i)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(j)(k)(l)}$$

may be called the *multi-time Douglas $d-$tensor* of the second-order system of partial differential equations (3.1). This is because, in the particular single-time case $T=\mathbb{R}$, the $d-$tensor $D^{(i)\ \ (1)(1)(1)}_{(1)1(j)(k)(l)}$ can be identified with the *classical Douglas tensor* D^i_{jkl} of the tangent semispray $F^{(i)}_{(1)1}:=G^i(t,x^k,y^k_1)$ (see [2] and [11]).

Example 3.14. For the second-order PDE system (3.3) of the affine maps between the Riemannian manifolds $(T,h_{\alpha\beta}(t))$ and $(M,\varphi_{ij}(x))$, the third, fourth and fifth multi-time $h-$KCC-invariants are given by

$$\overset{h}{R}{}^{i\alpha}_{jk}=h^{\alpha\mu}\mathfrak{R}^i_{pjk}x^p_\mu,\qquad \overset{h}{B}{}^{i\alpha(\beta)}_{jk(l)}=h^{\alpha\beta}\mathfrak{R}^i_{ljk},\qquad D^{(i)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(j)(k)(l)}=0.$$

Example 3.15. For the first order PDE system (3.4) the third, fourth and fifth multi-time $h-$KCC-invariants are zero.

Example 3.16 (Burgers equation). The Burgers equation (see [13]) is a fundamental PDE from fluid mechanics, which is used for describing wave processes in acoustics and hydrodynamics. It may be thought as a nonlinear version of the heat equation. The Burgers equation is given by $u_t + uu_x = \mu u_{xx}$, where $u = u(t, x)$ is the velocity of the fluid, and μ is the viscosity coefficient. Making the changes of variables $x = -\mu x'$ and $t = \mu t'$, the Burgers equation becomes (primes are omitted)

$$u_t - uu_x = u_{xx}. \tag{3.8}$$

In our geometrical context, the Burgers equation (3.8) can be regarded as the second-order PDE system

$$\frac{\partial^2 x^1}{\partial t^\alpha \partial t^\beta} + F^{(1)}_{(\alpha)\beta}(t^\gamma, x^1, x^1_\gamma) = 0, \quad \alpha, \beta = \overline{1, 2}, \tag{3.9}$$

where

$$\left(t^1 := t,\ t^2 := x,\ x^1 := u,\ x^1_1 := u_t,\ x^1_2 := u_x\right)$$

are the coordinates on the *1*-jet space $J^1(\mathbb{R}^2, \mathbb{R})$,

$$F^{(1)}_{(1)1} = F^{(1)}_{(1)1}(t^1, t^2) := -u_{tt}, \quad F^{(1)}_{(2)1} = F^{(1)}_{(1)2} = F^{(1)}_{(1)2}(t^1, t^2) := -u_{tx}$$

are arbitrary local functions, and we have

$$u_{xx} := F^{(1)}_{(2)2} \stackrel{def}{=} x^1 \cdot x^1_2 - x^1_1.$$

Note that the second order PDE system (3.9) is a particular case of the PDE system (3.1), setting $T = \mathbb{R}^2$ and $M = \mathbb{R}$. Taking now the Euclidian metric on $\mathbb{R}^2$ (that is, we take $(T, h_{\alpha\beta}) = (\mathbb{R}^2, \delta_{\alpha\beta})$), then the formulas for the five KCC-invariants of the PDE system (3.9) simplify as follows:

$$\overset{\delta}{\varepsilon}{}^{(1)}_{(\alpha)\beta} = -F^{(1)}_{(\alpha)\beta} + \frac{1}{2}\frac{\partial F^1}{\partial x^1_\alpha} x^1_\beta,$$

$$\overset{\delta}{P}{}^1_1 = -\frac{\partial F^1}{\partial x^1} + \frac{1}{2}\frac{\partial^2 F^1}{\partial t^\gamma \partial x^1_\gamma} + \frac{1}{2}\frac{\partial^2 F^1}{\partial x^1 \partial x^1_\gamma} x^1_\gamma - \frac{1}{2}\frac{\partial^2 F^1}{\partial x^1_\mu \partial x^1_\gamma} F^{(1)}_{(\gamma)\mu} + \frac{1}{4}\left(\frac{\partial F^1}{\partial x^1_\mu}\right)^2,$$

$$\overset{\delta}{R}{}^{1\alpha}_{11} = \frac{1}{3}\left[\frac{\partial \overset{\delta}{P}{}^1_1}{\partial x^1_\alpha} - \frac{\partial \overset{\delta}{P}{}^1_1}{\partial x^1_\alpha}\right] = 0, \qquad \overset{\delta}{B}{}^{1\alpha(\beta)}_{11(1)} = \frac{\partial \overset{\delta}{R}{}^{1\alpha}_{11}}{\partial x^1_\beta} = 0,$$

$$D^{(1)\ \ (\gamma)(\varepsilon)(\mu)}_{(\alpha)\beta(1)(1)(1)} = \frac{\partial^3 F^{(1)}_{(\alpha)\beta}}{\partial x^1_\gamma \partial x^1_\varepsilon \partial x^1_\mu} = 0,$$

where

$$F^1 = F^{(1)}_{(1)1} + F^{(1)}_{(2)2} = F^{(1)}_{(1)1} + x^1 \cdot x^1_2 - x^1_1.$$

Consequently, by direct computations, the first and the second KCC-invariants attached to the Burgers equation (3.8) have the expressions:

1. the *Burgers first multi-time* $\delta-$*KCC-invariant*

$$\overset{\delta}{\varepsilon} := \left(\overset{\delta}{\varepsilon}{}^{(1)}_{(\alpha)\beta}\right)_{\alpha,\beta=\overline{1,2}} = \begin{pmatrix} u_{tt} - \dfrac{u_t}{2} & u_{tx} - \dfrac{u_x}{2} \\ u_{xt} + \dfrac{uu_t}{2} & u_t - \dfrac{uu_x}{2} \end{pmatrix};$$

2. the *Burgers multi-time* $\delta-$*deviation curvature* $d-$*tensor*

$$\overset{\delta}{P}{}^1_1 = \frac{1}{4}\left(u^2 - 2u_x + 1\right).$$

Theorem 3.1 (of characterization of the multi-time KCC-invariants). *Let* (T, h) *be a Riemannian manifold, where* $m = \dim T \geq 3$*. If the first and the fifth KCC-invariants of the PDE system (3.1) are zero on* $J^1(T, M)$*, then there exist on* $J^1(T, M)$ *some local functions* $\Gamma^i_{pq}(t, x)$*, where* $i, p, q = \overline{1, n}$*,* $n = \dim M$*, and* $\mathtt{S}^{i\nu}_{\alpha pq}(t, x)$*,* $\alpha \neq \nu \in \{1, 2, ..., m\}$*,* $i = \overline{1, n}$*,* $p \neq q \in \{1, 2, ..., n\}$*, which have the properties*

$$\Gamma^i_{pq} = \Gamma^i_{qp}, \qquad \mathtt{S}^{i\nu}_{\alpha pq} + \mathtt{S}^{i\nu}_{\alpha qp} = 0$$

and (no sum by α *or* ν*)*

$$2\mathtt{S}^{i\nu}_{\alpha pq} = \sum_{\varepsilon \in \overline{1,m}\setminus\{\nu\}} \left[h^{\varepsilon\varepsilon}\mathtt{S}^{i\nu}_{\varepsilon pq} - h^{\nu\nu}\mathtt{S}^{i\varepsilon}_{\nu pq}\right] h_{\varepsilon\alpha}, \tag{3.10}$$

such that (no sum by α *or* β*)*

$$F^{(i)}_{(\alpha)\beta} = \Gamma^i_{pq}x^p_\alpha x^q_\beta - \kappa^\mu_{\alpha\beta}x^i_\mu + 2\delta_{\alpha\beta}\left(\sum_{\nu\in\overline{1,m}\setminus\{\alpha\}}\ \sum_{p\neq q\in\overline{1,n}} \mathtt{S}^{i\nu}_{\alpha pq}x^p_\alpha x^q_\nu\right), \tag{3.11}$$

where $\delta_{\alpha\beta}$ *is the Kronecker symbol and* $\kappa^\gamma_{\alpha\beta}$ *are the Christoffel symbols of the Riemannian metric* $h_{\alpha\beta}(t)$*.*

Proof. By integration, the relations

$$D^{(i)\ (\gamma)(\varepsilon)(\sigma)}_{(\alpha)\beta(j)(k)(l)} = \frac{\partial^3 F^{(i)}_{(\alpha)\beta}}{\partial x^j_\gamma \partial x^k_\varepsilon \partial x^l_\sigma} = 0,$$

where $F^{(i)}_{(\alpha)\beta} = F^{(i)}_{(\beta)\alpha}$, subsequently lead to

$$\begin{aligned} \frac{\partial^2 F^{(i)}_{(\alpha)\beta}}{\partial x^j_\gamma \partial x^k_\varepsilon} &= 2\Gamma^{(i)\ (\gamma)(\varepsilon)}_{(\alpha)\beta(j)(k)}(t, x) \quad \Rightarrow \\ &\Rightarrow \quad \frac{\partial F^{(i)}_{(\alpha)\beta}}{\partial x^j_\gamma} = 2\Gamma^{(i)\ (\gamma)(\nu)}_{(\alpha)\beta(j)(q)}x^q_\nu + \mathcal{U}^{(i)\ (\gamma)}_{(\alpha)\beta(j)}(t, x) \Rightarrow \\ &\Rightarrow \quad F^{(i)}_{(\alpha)\beta} = \Gamma^{(i)\ (\mu)(\nu)}_{(\alpha)\beta(p)(q)}x^p_\mu x^q_\nu + \mathcal{U}^{(i)\ (\nu)}_{(\alpha)\beta(q)}x^q_\nu + \mathcal{V}^{(i)}_{(\alpha)\beta}(t, x), \end{aligned}$$

where

$$\text{(3.12)}\qquad \begin{array}{ll} \Gamma^{(i)\ \ (\gamma)(\varepsilon)}_{(\alpha)\beta(j)(k)} = \Gamma^{(i)\ \ (\gamma)(\varepsilon)}_{(\beta)\alpha(j)(k)}, & \Gamma^{(i)\ \ (\gamma)(\varepsilon)}_{(\alpha)\beta(j)(k)} = \Gamma^{(i)\ \ (\varepsilon)(\gamma)}_{(\alpha)\beta(k)(j)}, \\ \mathcal{U}^{(i)\ \ (\gamma)}_{(\alpha)\beta(j)} = \mathcal{U}^{(i)\ \ (\gamma)}_{(\beta)\alpha(j)}, & \mathcal{V}^{(i)}_{(\alpha)\beta} = \mathcal{V}^{(i)}_{(\beta)\alpha}. \end{array}$$

The equalities $\overset{h}{\varepsilon}{}^{(i)}_{(\alpha)\beta} = 0$ on $J^1(T,M)$ lead us to

$$\text{(3.13)}\qquad \begin{array}{lll} \Gamma^{(i)\ \ (\mu)(\nu)}_{(\alpha)\beta(p)(q)} & = & \dfrac{1}{2}\left[\Gamma^{i(\eta)(\nu)}_{\ \ (p)(q)}\delta^{\mu}_{\beta} + \Gamma^{i(\eta)(\mu)}_{\ \ (q)(p)}\delta^{\nu}_{\beta}\right] h_{\eta\alpha}, \\ \mathcal{U}^{(i)\ \ (\nu)}_{(\alpha)\beta(q)} & = & \dfrac{1}{2}\mathcal{U}^{i(\eta)}_{\ \ (q)} h_{\eta\alpha}\delta^{\nu}_{\beta} + \dfrac{1}{2}\kappa^{\eta} h_{\eta\alpha}\delta^{\nu}_{\beta}\delta^{i}_{q} - \kappa^{\nu}_{\alpha\beta}\delta^{i}_{q}, \\ \mathcal{V}^{(i)}_{(\alpha)\beta} & = & 0, \end{array}$$

where

$$\Gamma^{i(\mu)(\nu)}_{\ \ (p)(q)} = h^{\varepsilon\rho}\Gamma^{(i)\ \ (\mu)(\nu)}_{(\varepsilon)\rho(p)(q)} \text{ and } \mathcal{U}^{i(\nu)}_{\ \ (q)} = h^{\varepsilon\rho}\mathcal{U}^{(i)\ \ (\nu)}_{(\varepsilon)\rho(q)}.$$

Applying an $h-$trace to the second relation of (3.13), we deduce that we have

$$\mathcal{U}^{(i)\ \ (\nu)}_{(\alpha)\beta(q)} = -\kappa^{\nu}_{\alpha\beta}\delta^{i}_{q}.$$

The first relation of (3.13) and the first symmetry properties of (3.12) imply the following equalities:

1. for every $\alpha \neq \beta$ we have (no sum by α or β):

   (a) $\mu,\nu \notin \{\alpha,\beta\} \Rightarrow \Gamma^{(i)\ \ (\mu)(\nu)}_{(\alpha)\beta(p)(q)} = 0$;

   (b) $\mu = \alpha,\ \nu = \alpha \Rightarrow \Gamma^{(i)\ \ (\alpha)(\alpha)}_{(\alpha)\beta(p)(q)} = 0$;

   (c) $\mu = \alpha,\ \nu = \beta \Rightarrow \Gamma^{(i)\ \ (\alpha)(\beta)}_{(\alpha)\beta(p)(q)} = \dfrac{1}{2}\Gamma^{i(\eta)(\alpha)}_{\ \ (q)(p)} h_{\eta\alpha} := \mathbb{S}^{i}_{\alpha pq} = \mathbb{T}^{i}_{\beta pq}$;

   (d) $\mu = \beta,\ \nu = \alpha \Rightarrow \Gamma^{(i)\ \ (\beta)(\alpha)}_{(\alpha)\beta(p)(q)} = \dfrac{1}{2}\Gamma^{i(\eta)(\alpha)}_{\ \ (p)(q)} h_{\eta\alpha} := \mathbb{T}^{i}_{\alpha pq} = \mathbb{S}^{i}_{\beta pq}$;

   (e) $\mu = \beta,\ \nu = \beta \Rightarrow \Gamma^{(i)\ \ (\beta)(\beta)}_{(\alpha)\beta(p)(q)} = \Gamma^{(i)\ \ (\beta)(\beta)}_{(\beta)\alpha(p)(q)} =$

   $$= \frac{1}{2}\left[\Gamma^{i(\eta)(\beta)}_{\ \ (p)(q)} h_{\eta\alpha} + \Gamma^{i(\eta)(\beta)}_{\ \ (q)(p)} h_{\eta\alpha}\right] = \mathtt{S}^{i\beta}_{\alpha pq} + \mathtt{T}^{i\beta}_{\alpha pq} = 0;$$

2. for every $\alpha = \beta \in \{1, 2, ..., m\}$ we obtain (no sum by α):

   (a) $\mu \neq \alpha, \nu \neq \alpha \Rightarrow \Gamma^{(i)\ \ (\mu)(\nu)}_{(\alpha)\alpha(p)(q)} = 0$;

   (b) $\mu = \alpha, \nu \neq \alpha \Rightarrow \Gamma^{(i)\ \ (\alpha)(\nu)}_{(\alpha)\alpha(p)(q)} = \dfrac{1}{2}\Gamma^{i(\eta)(\nu)}_{\ \ (p)(q)} h_{\eta\alpha} := \mathtt{S}^{i\nu}_{\alpha pq} = \mathtt{T}^{i\nu}_{\alpha qp}$;

   (c) $\mu \neq \alpha, \nu = \alpha \Rightarrow \Gamma^{(i)\ \ (\mu)(\alpha)}_{(\alpha)\alpha(p)(q)} = \dfrac{1}{2}\Gamma^{i(\eta)(\mu)}_{\ \ (q)(p)} h_{\eta\alpha} := \mathtt{T}^{i\mu}_{\alpha pq} = \mathtt{S}^{i\mu}_{\alpha qp}$;

(d) $\mu = \alpha,\ \nu = \alpha \Rightarrow \Gamma^{(i)\ \ (\alpha)(\alpha)}_{(\alpha)\alpha(p)(q)} = \Gamma^{(i)\ \ (\alpha)(\alpha)}_{(\alpha)\alpha(q)(p)} =$

$$= \frac{1}{2}\left[\Gamma^{i(\eta)(\alpha)}_{\ (p)(q)} h_{\eta\alpha} + \Gamma^{i(\eta)(\alpha)}_{\ (q)(p)} h_{\eta\alpha}\right] = \mathbb{T}^i_{\alpha pq} + \mathbb{S}^i_{\alpha pq};$$

The first symmetry condition from (3.12), together with 1. (c) and 1. (d), give us ($m = \dim T \geq 3$)

$$\begin{array}{ccccccccccccc}
\mathbb{S}^i_{1pq} & = & \mathbb{T}^i_{2pq} & = & \mathbb{T}^i_{3pq} & = & \mathbb{T}^i_{4pq} & = & \cdot & \cdot & = & \mathbb{T}^i_{mpq} & := \dfrac{1}{2}\Gamma^i_{pq} \\
\mathbb{S}^i_{2pq} & = & \mathbb{T}^i_{1pq} & = & \mathbb{T}^i_{3pq} & = & \mathbb{T}^i_{4pq} & = & \cdot & \cdot & = & \mathbb{T}^i_{mpq} & := \dfrac{1}{2}\Gamma^i_{pq} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\
\mathbb{S}^i_{mpq} & = & \mathbb{T}^i_{1pq} & = & \mathbb{T}^i_{2pq} & = & \mathbb{T}^i_{3pq} & = & \cdot & \cdot & = & \mathbb{T}^i_{(m-1)pq} & := \dfrac{1}{2}\Gamma^i_{pq}.
\end{array}$$

Consequently, for every $\alpha \neq \beta \in \{1, 2, ..., m\}$ we have

$$\Gamma^{(i)\ \ (\mu)(\nu)}_{(\alpha)\beta(p)(q)} = \frac{1}{2}\Gamma^i_{pq}\left[\delta^\mu_\alpha\delta^\nu_\beta + \delta^\nu_\alpha\delta^\mu_\beta\right]$$

and for every $\alpha \in \{1, 2, ..., m\}$ we have $\Gamma^{(i)\ \ (\alpha)(\alpha)}_{(\alpha)\alpha(p)(q)} = \Gamma^i_{pq} = \Gamma^{(i)\ \ (\alpha)(\alpha)}_{(\alpha)\alpha(q)(p)} = \Gamma^i_{qp}$.

Using now all the preceding properties, together with the equality 2. (b), we find the equations (3.10). Moreover, for every $\alpha \neq \nu \in \{1, 2, ..., m\}$, it is obvious that we have (no sum by p) $\mathtt{S}^{i\nu}_{\alpha pq} + \mathtt{S}^{i\nu}_{\alpha qp} = 0 \Rightarrow \mathtt{S}^{i\nu}_{\alpha pp} = 0$.

All the preceding situations can be briefly written in the general formula

$$\Gamma^{(i)\ \ (\mu)(\nu)}_{(\alpha)\beta(p)(q)} = \frac{1}{2}\Gamma^i_{pq}\left[\delta^\mu_\alpha\delta^\nu_\beta + \delta^\nu_\alpha\delta^\mu_\beta\right] + \mathtt{S}^{i\nu}_{\alpha pq}\delta_{\alpha\beta}\delta^\mu_\alpha\left[1 - \delta^\nu_\alpha\right] + \mathtt{S}^{i\mu}_{\alpha qp}\delta_{\alpha\beta}\delta^\nu_\alpha\left[1 - \delta^\mu_\alpha\right].$$

In conclusion, we obtain the equalities (3.11) on the 1-jet space $J^1(T, M)$. □

Open problem. If we fix the indices i and $p \neq q$ in the set $\{1, 2, ..., n\}$, then we deduce that the system of equations (3.10) is a homogenous linear system of order $m(m-1)$. Consequently, it has at least the null solution. Because the coefficients of the system depend only by the metric $h_{\alpha\beta}(t)$, the following question naturally arises:
– There exists a temporal Riemannian metric $h_{\alpha\beta}(t)$ such that the system of equations (3.10) to admit only the zero solution?

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