Symmetries and conservation laws in k-symplectic geometry

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Abstract. In this paper it will be extend the study of symmetries and conservation laws from Classical Mechanics to the first-order classical field theories, both for the Lagrangian and Hamiltonian k-symplectic formalisms. More exactly, we will obtain new kinds of conservation laws for k-symplectic Hamiltonian systems and k-symplectic Lagrangian systems, without the help of a Noether type theorem, only using symmetries and pseudosymmetries.

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1 Introduction

The k-Symplectic Geometry provides the simplest geometric framework for describing certain class of first-order classical field theories. Using this description we analyze different kinds of symmetries for the Hamiltonian and Lagrangian formalisms of these field theories, including the study of conservation laws associated to them and stating Noether's Theorem ([42]) and more that we will generalize the study of symmetries and conservation laws from classical (symplectic, $k = 1$) formalism ([1], [2]) to the k-symplectic formalism (introduced by Ch. Günther, [19]) for obtain new kinds of conservation laws for k-symplectic Hamiltonian and Lagrangian systems. A similar study for the case of higher order tangent bundles geometry was done by the author in [33], [34], [37], [36]. The higher order Hamiltonians was introduced by R. Miron ([28], [29], [30], [31]).

The k-symplectic formalism is the generalization to the field theories of the standard symplectic formalism in Mechanics, which is the geometric framework for describing autonomous dynamical systems ([1], [2]). Like in the classical case, the k-symplectic formalism allow us to study toghether the Lagrangian and the Hamiltonian formalisms for field theories (using the Legendre transformation or the k-tangent structure ([22], [23], [24], [35], [42])). So, many results obtained by M. Crampin ([9],

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[10], [11]), J. Grifone ([17], [18]), J. Klein ([21]), M. de Leon ([22], [23], [24]), R. Miron and M. Anastasiei ([27], [28], [29], [30], [31]), E. Noether ([38]), for the symplectic formalism was already extend or can be extend to the k-symplectic case.

This paper is devoted to studying symmetries, conservation laws and relationship between this in the framework of k-symplectic geometry, more exactly we extend the study of symmetries and conservation laws from Classical Mechanics to the first-order classical field theories, both for the Lagrangian and Hamiltonian formalisms, using Günther's k-symplectic description, and considering only the regular case. We will find new kinds of conservation laws, nonclassical, without the help of a Noether's type theorem, using only a result who gives a relationship between symmetries, pseudosymmetries and conservation laws (G.L. Jones ([20]), M. Crâşmăreanu ([12])).

The study of symmetries and conservation laws for k-symplectic Hamiltonian systems is, like in the classical case, a topic of great interest and was developped recently by M. Salgado, N. Roman-Roy, S. Vilarino in [42] and L. Bua, I. Bucătaru, M. Salgado in [7]. More that, in the paper [43] J.C. Marrero, N. Roman-Roy, M. Salgado, S. Vilarino begin the study of symmetries and conservation laws for k-cosymplectic Hamiltonian systems, like an extension to field theories of the standard cosymplectic formalism for nonautonomous mechanics ([24], [25]). In [42] the Noether's theorem, obtained for a k-symplectic Hamiltonian system, associates conservation laws to so-called Cartan symmetries. However, these kinds of symmetries do not exhaust the set of symmetries. As is known, in mechanics and physics there are symmetries which are not of Cartan type, and which generate conserved quantities, i.e. conservation laws (see [26], [40], [41] for some examples).

So, by generalization from symplectic geometry to k-symplectic geometry, we will obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem and without the use of a variational principle, using only symmetries and pseudosymmetries associated to the k-vector fileds $\mathbf{X} = (X_1, \ldots, X_k)$ which are solutions of the equation $\sum\limits_{A=1}^{k} i_{X_A}\omega_A = dH$. The main result is a generalization from the classical case ($k = 1$) of a results of G.L. Jones ([20]) and M. Crâşmăreanu ([12]). Applications for Lagrangian and Hamiltonian k-symplectic formalisms are also presented ([35], [37]).

In the second section are presented the notions used in the next sections and the classical results who will be generalized in the last section. In the third section it will be present, shortly, the geometric elements of k-symplectic formalism who need to explain and to obtain the results from the last section. Two very interesting examples of k-symplectic Lagrangian and Hamiltonian systems are presented ([35], [37]). In section four we enounce and prove the main generalized result and, finally, we present some applications for k-symplectic Lagrangian and Hamiltonian systems.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood. The Lagrangian and Hamiltonian functions are regular.

2 Classical results

Let M be a smooth, n-dimensional manifold, $C^\infty(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $A^p(M)$ the $C^\infty(M)$-module of p-differential forms, $1 \le p \le n$. For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ we consider the system of ordinary differential equations which give the flow $\{\Phi_t\}_t$ of X, locally,

$$\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i(x^1(t), \ldots, x^n(t)), \quad i = 1, \ldots, n. \tag{2.1}$$

A dynamical system is a couple (M, X), where M is a smooth manifold and $X \in \mathcal{X}(M)$. A dynamical system is denoted by the flow of X, $\{\Phi_t\}_t$ or by the system of differential equations (2.1).

A function $f \in C^\infty(M)$ is called *conservation law* for dynamical system (M, X) if f is constant along the every integral curves of X (solutions of 2.1), that is

$$L_X f = 0, \tag{2.2}$$

where $L_X f$ means the Lie derivative of f with respect to X.

If $Z \in \mathcal{X}(M)$ is fixed, then $Y \in \mathcal{X}(M)$ is called *Z-pseudosymmetry* for (M, X) if there exists $f \in C^\infty(M)$ such that $L_X Y = fZ$. A X-pseudosymmetry for X is called *pseudosymmetry* for (M, X). $Y \in \mathcal{X}(M)$ is called *symmetry* for (M, X) if $L_X Y = 0$.

Example 2.1 ([16], [13]) The system from the theory of static $SU(2)$-monopoles is:

$$\frac{dx^1}{dt} = x^2x^3, \ \frac{dx^2}{dt} = x^3x^1, \ \frac{dx^3}{dt} = x^1x^2. \tag{2.3}$$

The vector field $X = x^2x^3\frac{\partial}{\partial x^1} + x^3x^1\frac{\partial}{\partial x^2} + x^1x^2\frac{\partial}{\partial x^3}$ is homogeneous of order two, that is $[Y, X] = X$, where $Y = \sum_{i=1}^{3} x^i\frac{\partial}{\partial x^i}$. Equivalently, $L_X Y = X$, and this means that Y is a X-pseudosymmetry for (2.3) (or pseudosymmetry for X).

Let us recall that $\omega \in A^p(M)$ is called *invariant form* for (M, X) if $L_X\omega = 0$. If (M, ω) is a symplectic manifold then the dynamical system (M, X) is said to be *a dynamical Hamiltonian system* (or, shortly, *Hamiltonian system*) if there exists a function $H \in C^\infty(M)$ (called *the Hamiltonian*) such that

$$i_X\omega = -dH, \tag{2.4}$$

where i_X denotes the interior product with respect to X.

It is known that the symplectic form ω is an invariant 2-form for (M, X) and the Hamiltonian H is a conservation law for (M, X).

A *Cartan symmetry* for Lagrangian L is a vector field $X \in \mathcal{X}(TM)$ characterized by $L_X\omega_L = 0$ and $L_X H = 0$, where $\omega_L = d\theta_L$ is the Cartan 2-form associated to the regular Lagrangian L, $\theta_L = J^*(dL)$, J^* being the adjoint of the natural tangent structure J on TM and $H = E_L = \frac{\partial L}{\partial y^i}y^i - L$ is the en energy of L. It is known that ([11]) that any Cartan symmetry for Lagrangian L is a symmetry for the canonical semispray S of L ([27]), that is $L_S X = 0$. For each Cartan symmetry X for (M, L) we have $dL_X\theta_L = 0$, which implies that $L_X\theta_L$ is a closed 1-form. If $L_X\theta_L$ is a exact

1-form, then we say that X is *exact Cartan symmetry* for (M, L). Obviously, the canonical semispray of L is an exact Cartan symmetry for Lagrangian L ([11], [27]).

In the classical case ($k = 1$), we know that Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse ([38], [11], [39], [20], [12]).

Theorem 2.1. *(Noether Theorem) If X is an exact Cartan symmetry with $L_X\theta_L = df$, then*

$$P_X = J(X)L - f$$

is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian L.
Conversely, if F is a conservation law for the Euler-Lagrange equations assocaited to the regular Lagrangian L, then the vector field X uniquely defined by

$$i_X\omega_L = -dF$$

is an exact Cartan symmetry.

The next theorem which gives the association between pseudosymmetries and conservation laws is due to M. Crâşmăreanu ([12]) and G.L. Jones ([20]). Next, using this result, we will find new kinds of conservation laws, nonclassical, without the help of Noether's type theorem.

Theorem 2.2. *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in A^p(M)$ be a invariant p-form for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetry for X, then*

$$\Phi = \omega(X, S_1, \ldots, S_{p-1}) \tag{2.5}$$

or, locally,

$$\Phi = S_1^{i_1} \cdots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 \ldots i_{p-1} i_p}$$

is a conservation laws for (M, X).

Particularly, if Y, S_1, $\ldots$, S_{p-1} are symmetries for X then Φ given by (2.5) is conservation laws for (M, X).

Now, we can apply this result to the dynamical Hamiltonian systems.

Proposition 2.3. *Let be (M, X_H) a Hamiltonian system on the symplectic manifold (M, ω), with the local coordinates (x^i, p_i). If $Y \in \mathcal{X}(M)$ is a symmetry for X_H and $Z \in \mathcal{X}(M)$ is a Y-pseudosymmetry for X_H, then*

$$\Phi = \omega(Y, Z) \tag{2.6}$$

is a conservation law for the Hamiltonian system (M, X_H).

Particularly, if Y and Z are symmetries for X_H then Φ from (2.6) is a conservation law for (M, X_H).

If $Y = Y^k \frac{\partial}{\partial x^k} + \widetilde{Y}_k \frac{\partial}{\partial p_k}$ and $Z = Z^k \frac{\partial}{\partial x^k} + \widetilde{Z}_k \frac{\partial}{\partial p_k}$ then (2.6) becomes

$$\Phi = \begin{pmatrix} Y^k & \widetilde{Y}_k \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z^k \\ \widetilde{Z}_k \end{pmatrix} = \widetilde{Y}_k Z^k - Y^k \widetilde{Z}_k. \tag{2.7}$$

Corollary 2.4. *If $Y \in \mathcal{X}(M)$ is a X_H-pseudosymmetry for X_H , then*

$$\Phi = \omega(X_H, Y) = -L_Y H \tag{2.8}$$

or

$$\Phi = \frac{\partial H}{\partial x^k} Y^k + \frac{\partial H}{\partial p_k} \widetilde{Y}_k \tag{2.9}$$

is a conservation law for (M, X_H).

Now, if we consider the Hamiltonian system (TM, S_L) on the symplectic manifold (TM, ω_L), where S_L is the canonical semispray and ω_L the Cartan 2-form associated to a regular Lagrangian L on TM (for more details see [27], [36]), then we have:

Corollary 2.5. *If $Y = Y^k \frac{\partial}{\partial x^k} + \widetilde{Y}^k \frac{\partial}{\partial y^k} \in \mathcal{X}(TM)$ is a S_L-pseudosymmetry for S_L , then*

$$\Phi = \omega_L(S_L, Y) = -L_Y E_L$$

or

$$\Phi = \frac{\partial E_L}{\partial x^k} Y^k + \frac{\partial E_L}{\partial y^k} \widetilde{Y}^k$$

is a conservation law for (TM, S_L).

An immediately consequence of this last result is the following ([12], [36]):

Corollary 2.6. *If the canonical semispray S_L associated to the regular Lagrangian L is 2-positive homogeneous with respect to velocity (S_L is a spray) and g_{ij} is the metric tensor of L, then $\Phi = g_{ij} y^i \widetilde{Y}^j$ is a conservation law for (TM, S_L).*

Taking into account that the canonical semispray S_L associated to the regular Lagrangian L is a spray if and only if $[S_L, C] = S_L$, that is $L_{S_L} C = S_L$, we have that the Liouville (*canonical*) vector field $C = y^i \frac{\partial}{\partial y^i}$ is a pseudosymmetry for S_L , and using the last corollary we obtain that $\Phi = g_{ij} y^i y^j$ is a conservation law for (TM, S_L). So we obtained the conservation of the kinetic energy $\mathcal{E}(L) = \frac{1}{2} g_{ij} y^i y^j$ of the metric g_{ij}.

Example 2.2 ([12], [13]) Let the 2-dimensional isotropic harmonic oscillator

$$\begin{array}{rcl} \ddot{q}^1 + \omega^2 q^1 & = & 0 \\ \ddot{q}^2 + \omega^2 q^2 & = & 0 \end{array} \tag{2.10}$$

a toy model for many methods to finding conservation laws. The Lagrangian is

$$L = \frac{1}{2} \left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 \right] - \frac{\omega^2}{2} \left[\left(q^1\right)^2 + \left(q^2\right)^2 \right] \tag{2.11}$$

and then applying the conservation of energy we have two conservation laws
$\Phi_1 = \left(\dot{q}^1\right)^2 + \omega^2 \left(q^1\right)^2$, $\Phi_2 = \left(\dot{q}^2\right)^2 + \omega^2 \left(q^2\right)^2$.

A straightforward computation give that the complet lift of $X = q^2 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2}$ is an exact Cartan symmetry with $f = 0$ and then the associated classical Noetherian conservation law is

$$\Phi_3 = P_X = J(X) L = X^i \frac{\partial L}{\partial \dot{q}^i} = q^2 \dot{q}^1 - q^1 \dot{q}^2 \, .$$

But we can obtain a nonclassical conservation law with symmetries taking into account that the canonical spray of L is

$$S = \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^2 \frac{\partial}{\partial q^2} - \omega^2 q^1 \frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2 \frac{\partial}{\partial \dot{q}^2}$$

and another computation gives that

$$Y = \dot{q}^2 \frac{\partial}{\partial q^1} + \dot{q}^1 \frac{\partial}{\partial q^2} - \omega^2 q^2 \frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1 \frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S. Also, because S is total 1-homogeneous, that means that S is 1-homogeneous with respect to all variables $(q, \dot{q})$, it result that

$$Z = q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} + \dot{q}^1 \frac{\partial}{\partial \dot{q}^1} + \dot{q}^2 \frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for S. Next, we have $L_Y H = 0$, $L_Z H = 2H$ and then $\Phi = \omega_L(S, Y) = 0$, $\Phi = \omega_L(S, Z) = 2H$, that means that we not have new conservation law applying Theorem 2.2. But $\Phi_4 = \omega_L(Y, Z) = \dot{q}^1 \dot{q}^2 + \omega^2 q^1 q^2$ is a new conservation law given by Theorem 2.2 or by their corollaries.

We remark that Φ_4 is a nonclassical conservation law, obtained by symmetries, and Φ_4 represent the energy of a new Lagrangian of (2.10), $\tilde{L} = \dot{q}^1 \dot{q}^2 - \omega^2 q^1 q^2$ ([45]).

3 Geometric framework of Günther k-symplectic formalism

In this section we present, shortly, the basic geometric elements of Günther k-symplectic formalism ([19]) necessarly for obtain and explain the results from the next section.

3.1 The tangent bundle of k^1-velocities of a manifold M

An almost tangent structure J on a $2n$-dimensional manifold M is tensor field of type $(1, 1)$ of constant rank n such that $J^2 = 0$. The manifold M is then called an almost tangent manifold. Almost tangent structures were introduced by Clark and Bruckheimer [8] and Eliopoulos [15] around 1960 and have been studied by many authors (see [5, 9, 10, 17, 18, 21]).

The canonical model of these structures is the tangent bundle $\tau_M : TM \to M$ of an arbitrary manifold M. The canonical tangent structure J on TM is locally given by

$$J = \frac{\partial}{\partial v^i} \otimes dx^i \tag{3.1}$$

with respect the bundle coordinates on TM. This tensor J can be regarded as the vertical lift of the identity tensor on M to TM ([32]).

The almost k-tangent structures were introduced as generalization of the almost tangent structures ([22, 23]).

Definition 3.1. An almost k-tangent structure J on a manifold M of dimension $n+kn$ is a family $(J^1,\dots,J^k)$ of tensor fields of type $(1,1)$ such that

$$(3.2)\quad J^A\circ J^B=J^B\circ J^A=0,\qquad rank\,J^A=n,\qquad Im\,J^A\cap(\oplus_{B\neq A}Im\,J^B)=0,$$

for $1\le A,B\le k$. In this case the manifold M is then called an almost k-tangent manifold.

The canonical model of these structures is the k-tangent vector bundle $T^1_kM=J^1_0(\mathbb{R}^k,M)$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with source at $0\in\mathbb{R}^k$ and with projection map $\tau:T^1_kM\to M$, $\tau(j^1_0\sigma)=\sigma(0)$. This bundle is also known as the *tangent bundle of k^1-velocities* of M [32].

The manifold T^1_kM can be canonically identified with the Whitney sum of k copies of TM, that is

$$\begin{array}{rcl} T^1_kM & \equiv & TM\oplus\cdots\oplus TM,\\ j^1_0\sigma & \equiv & (j^1_0\sigma_1=v_1,\dots,j^1_0\sigma_k=v_k)\end{array}$$

where $\sigma_A=\sigma(0,\dots,t,\dots,0)$ with $t\in\mathbb{R}$ at position A and $v_A=(\sigma_A)_*(0)(\frac{d}{dt}_0)$.

If (x^i) are local coordinates on $U\subseteq M$ then the induced local coordinates (x^i,v^i_A), $1\le i\le n$, $1\le A\le k$, on $\tau^{-1}(U)\equiv T^1_kU$ are given by

$$x^i(j^1_0\sigma)=x^i(\sigma(0)),\qquad v^i_A(j^1_0\sigma)=\frac{d}{dt}(x^i\circ\sigma_A)|_{t=0}=v_A(x^i)\,.$$

Definition 3.2. For a vector X_x at M we define its vertical A-lift $(X_x)^A$ as the vector on T^1_kM given by

$$(X_x)^A(j^1_0\sigma)=\frac{d}{dt}((v_1)_x,\dots,(v_{A-1})_x,(v_A)_x+tX_x,(v_{A+1})_x\dots,(v_k)_x)|_{t=0}\in T_{j^1_0\sigma}(T^1_kM)$$

for all points $j^1_0\sigma\equiv((v_1)_x,\dots,(v_k)_x))\in T^1_kM$.

In local coordinates we have $(X_x)^A=\displaystyle\sum_{i=1}^n a^i\frac{\partial}{\partial v^i_A}$, for a vector $X_x=a^i\,\partial/\partial x^i$. The *canonical vertical vector fields* C^A_B on T^1_kM are defined by $C^A_B(x,X_1,X_2,\dots,X_k)=(X_B)^A$ and are locally given by $C^A_B=v^i_B\frac{\partial}{\partial v^i_A}$.

The canonical k-tangent structure $(J^1,\dots,J^k)$ on T^1_kM is defined by $J^A(Z_{j^1_0\sigma})=(\tau_*(Z_{j^1_0\sigma}))^A$, for all vectors $Z_{j^1_0\sigma}\in T_{j^1_0\sigma}(T^1_kM)$. In local coordinates we have

$$(3.3)\qquad J^A=\frac{\partial}{\partial v^i_A}\otimes dx^i$$

The tensors J^A can be regarded as the $(0,\dots,1_A,\dots,0)$-lift of the identity tensor on M to T^1_kM defined in [32].

3.2 The cotangent bundle of k^1-covelocities of M and $(T^1_k)^*M$

Almost cotangent structures were introduced by Bruckheimer [6]. *An almost cotangent structure* on a $2m$-dimensional manifold M consists of a pair (ω,V) where ω is a symplectic form and V is a distribution such that

$$(i)\quad \omega\rfloor_{V\times V}=0,\qquad (ii)\quad \ker\omega=\{0\}$$

The canonical model of this structure is the cotangent bundle $\tau^*_M : T^*M \to M$ of an arbitrary manifold M, where ω is the canonical symplectic form $\omega_0 = -d\theta_0$ on T^*M and V is the vertical distribution. The Liouville form θ_0 in T^*M is defined by $\theta_0(\alpha)(\tilde{X}_\alpha) = \alpha((\tau^*_M)_*(\alpha)(\tilde{X}_\alpha))$, for all vectors $\tilde{X}_\alpha \in T_\alpha(T^*M)$. In local coordinates (x^i, p_i) on T^*M

$$\theta_0 = p_i dx^i, \qquad \omega_0 = dx^i \wedge dp_i, \qquad V = \langle \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \rangle. \tag{3.4}$$

Definition 3.3. ([3, 4]) A k-symplectic structure on a manifold M of dimension $N = n + kn$ is a family $(\omega_A, V; 1 \leq A \leq k)$, where each ω_A is a closed 2-form and V is an nk-dimensional distribution on M such that

$$(i) \quad \omega_{A_{\rfloor V \times V}} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker \omega_A = \{0\}.$$

In this case (M, ω_A, V) is called a k-symplectic manifold.

The canonical model of this structure is the k-cotangent bundle $(T^1_k)^*M = J^1(M, \mathbb{R}^k)_0$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with target at $0 \in \mathbb{R}^k$, and projection $\tau^*(j^1_{x,o}\sigma) = x$.

The manifold $(T^1_k)^*M$ can be canonically identified with the Whitney sum of k copies of T^*M, say

$$\begin{array}{rcl} (T^1_k)^*M & \equiv & T^*M \oplus \cdots \oplus T^*M, \\ j_{x,0}\sigma & \equiv & (j^1_{x,0}\sigma^1, \dots, j^k_{x,0}\sigma^k) \end{array}$$

where $\sigma^A = \pi_A \circ \sigma : M \longrightarrow \mathbb{R}$ is the A-th component of σ.

The canonical k-symplectic structure $(\omega_0)_A, V; 1 \leq A \leq k)$, on $(T^1_k)^*M$ is defined by

$$\begin{array}{c} (\omega_0)_A = (\tau^*_A)^*(\omega_0) \\ V(j^1_{x,0}\sigma) = \ker(\tau^*)_*(j^1_{x,0}\sigma) \end{array}$$

where $\tau^*_A = (T^1_k)^*M \to T^*M$ is the projection on the A^{th}-copy T^*M of $(T^1_k)^*M$, and ω_0 is the canonical symplectic structure of T^*M.

One can also define the 2-forms ω_A by $\omega_A = -d\theta_A$ where $(\theta_0)^A = (\tau^*_A)^*\theta_0$.

If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, p^A_i), $1 \leq i \leq n$, $1 \leq A \leq k$ on $(T^1_k)^*U = (\tau^*)^{-1}(U)$ are given by

$$x^i(j^1_{x,0}\sigma) = x^i(x), \qquad p^A_i(j^1_{x,0}\sigma) = d_x\sigma^A \left(\left. \frac{\partial}{\partial x^i} \right|_x \right).$$

Then the canonical k-symplectic structure is locally given by

$$(\omega_0)^A = \sum_{i=1}^n dx^i \wedge dp^A_i, \qquad V = \langle \frac{\partial}{\partial p^1_i}, \dots, \frac{\partial}{\partial p^k_i} \rangle \quad , 1 \leq A \leq k, \tag{3.5}$$

and $(\theta_0)^A = p^A_i \, dx^i$.

3.3 Second Order Partial Differential Equations on $T^1_k M$

Let M be an arbitrary manifold and $\tau : T^1_k M \longrightarrow M$ its tangent bundle of k^1-velocities.

Definition 3.4. A section $\mathbf{X} : M \longrightarrow T^1_k M$ of the projection τ will be called a k-vector field on M.

Since $T^1_k M$ can be canonically identified with the Whitney sum $T^1_k M \equiv TM \oplus \cdots \oplus TM$ of k copies of TM, we deduce that a k-vector field $\mathbf{X}$ defines a family of vector fields $X_1, \ldots, X_k$ on M. Günther in [19] introduce the following definition.

Definition 3.5. An integral section of the k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ passing through a point $x \in M$ on M is a map $\phi : U_0 \subset \mathbb{R}^k \to M$, defined on some neighborhood U_0 of $0 \in \mathbb{R}^k$, such that

$$\phi(0) = x, \quad \phi_*(t)(\frac{\partial}{\partial t^A}) = X_A(\phi(t)) \quad \forall t \in U, \quad 1 \leq A \leq k,$$

or equivalently, ϕ satisfies

$$X \circ \phi = \phi^{(1)}, \tag{3.6}$$

where $\phi^{(1)}$ is the first prolongation of ϕ defined by

$$\begin{array}{rccl} \phi^{(1)} : & U_0 \subset \mathbb{R}^k & \longrightarrow & T^1_k M \\ & t & \longrightarrow & \phi^{(1)}(t) = j^1_0 \phi_t \end{array}$$

where $\phi_t(s) = \phi(s+t)$ for all $t, s \in \mathbb{R}^k$ such that $s + t \in U_0$.

In local coordinates,

$$\phi^{(1)}(t^1, \ldots, t^k) = (\phi^i(t^1, \ldots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \ldots, t^k)), \qquad 1 \leq A \leq k \,,\, 1 \leq i \leq n \,. \tag{3.7}$$

Definition 3.6. We say that a k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ on M is integrable if there is an integral section passing through each point of M.

Remark 3.7. If ϕ is an integral section of a k-vector field $(X_1, \ldots, X_k)$ then each curve on M defined by $\phi_A = \phi \circ h_A$, where $h_A : \mathbb{R} \to \mathbb{R}^k$ is the natural inclusion $h_A(t) = (0, \ldots, t, \ldots, 0)$, is an integral curve of the vector field X_A on M, with $1 \leq A \leq k$.

Definition 3.8. A k-vector field on $T^1_k M$, that is, a section $\xi \equiv (\xi_1, \ldots, \xi_k) : T^1_k M \to T^1_k(T^1_k M)$ of the projection $\tau_{T^1_k M} : T^1_k(T^1_k M) \to T^1_k M$, is a Second Order Partial Differential Equation (SOPDE) if and only if it is also a section of the vector bundle $T^1_k(\tau) : T^1_k(T^1_k M) \to T^1_k M$, where $T^1_k(\tau)$ is defined by $T^1_k(\tau)(j^1_0 \sigma) = j^1_0(\tau \circ \sigma)$.

Let (x^i) be a coordinate system on M and (x^i, v^i_A) the induced coordinate system on $T^1_k M$. From the definition we deduce that the local expression of a SOPDE ξ is

$$\xi_A(x^i, v^i_A) = v^i_A \frac{\partial}{\partial x^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k. \tag{3.8}$$

Proposition 3.1. *Let ξ an integrable k-vector field on T^1_kM. The necessary and sufficient condition for ξ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is*

$$\xi_A(\phi^{(1)}(t)) = (\phi^{(1)})_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \dots, k$. These maps ϕ will be called solutions of the SOPDE ξ.

From (3.7) and (3.8) we have

Proposition 3.2. *$\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE $\xi = (\xi_1, \dots, \xi_k)$, locally given by (3.8), if and only if*

$$\frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{(1)}(t)), \qquad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\xi_A)^i_B(\phi^{(1)}(t)).$$

If $\xi : T^1_kM \to T^1_kT^1_kM$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to T^1_kM$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$, where $\tau : T^1_kM \to M$ is the canonical projection.

Now we show how to characterize the SOPDE using the canonical k-tangent structure of T^1_kM.

Definition 3.9. The Liouville (or canonical) vector field C on T^1_kM is the infinitesimal generator of the one parameter group

$$\begin{array}{ccc} \mathbb{R} \times (T^1_kM) & \longrightarrow & T^1_kM \\ (s, (x^i, v^i_B)) & \longrightarrow & (x^i, e^s\, v^i_B)\,. \end{array}$$

Thus C is locally expressed as follows:

$$C = \sum_B C_B = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B}, \tag{3.9}$$

where each C_B corresponds with the canonical vector field on the B-th copy of TM on T^1_kM.

From (3.3), (3.8) and (3.9) we deduce the next result:

Proposition 3.3. *A k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on T^1_kM is a SOPDE if and only if*

$$J^A(\xi_A) = C_A, \qquad \forall\, 1 \leq A \leq k,$$

where $(J^1, \dots, J^k)$ is the canonical k-tangent structure on T^1_kM.

3.4 Hamiltonian and Lagrangian formalisms

The role played by symplectic manifolds in classical mechanics is here played by the k-symplectic manifolds (see Günther, [19]). Let $(M, \omega_A, V; 1 \leq A \leq k)$ be a k-symplectic manifold. Let us consider the vector bundle morphism defined by Günther ([19]):

$$\begin{array}{cccc} \Omega^\sharp : & T^1_kM & \longrightarrow & T^*M \\ & (X_1, \dots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \dots, X_k) = \displaystyle\sum_{A=1}^k i_{X_A}\,\omega_A\,. \end{array} \tag{3.10}$$

Definition 3.10. Let $H : M \longrightarrow \mathbb{R}$ be a function on M. Any k-vector field $(X_1, \dots, X_k)$ on M such that

$$\Omega^\sharp(X_1, \dots, X_k) = dH$$

will be called an evolution k-vector field on M associated with the Hamiltonian function H.

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [24] that there always exists an *evolution k-vector field* associated with a Hamiltonian function H.

We denote by $\mathcal{X}^k_H(M)$ the set of k-vector fields $\mathbf{X} = (X_1, \dots, X_k)$ on M which are the solutions of the equation

$$\sum_{A=1}^{k} i_{X_A}\,\omega_A = dH. \tag{3.11}$$

Let (x^i, p^A_i) be a local coordinate system on M. Then we have:

Proposition 3.4. *If $\mathbf{X} = (X_1, \dots, X_k)$ is an integrable evolution k-vector field associated to H, i.e. $\mathbf{X} \in \mathcal{X}^k_H(M)$, then its integral sections*

$$\begin{array}{rccc} \sigma : & \mathbb{R}^k & \longrightarrow & M \\ & (t^B) & \longrightarrow & (\sigma^i(t^B), \sigma^A_i(t^B)), \end{array}$$

are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [44]:

$$\frac{\partial H}{\partial x^i} = -\sum_{A=1}^{k} \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \le i \le n,\ 1 \le\ A \le k\,. \tag{3.12}$$

If we consider the canonical k-symplectic structure $((\omega_0)_A, V; 1 \le A \le k)$ on $(T^1_k)^*M$ and $H : (T^1_k)^*M \to \mathbb{R}$ be a Hamiltonian function on $(T^1_k)^*M$, then the family $\big((T^1_k)^*M, (\omega_0)_A, H\big)$ is called *a k-symplectic Hamiltonian system* and the equations (3.12) are called *the Hamilton-de Donder-Weyl equations* associated to this system.

Example 3.1 ([35]) We shall use the above formalism to obtain an intrinsic version for the electrostatic equations. Let us consider $\mathbb{R}^3$ with a metric g with components g_{ij}. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}$ be the electric potential and $P = (P_1, P_2, P_3) : \mathbb{R}^3 \to \mathbb{R}^3$ the electric field. We denote by (t^1, t^2, t^3) the standard coordinates on $\mathbb{R}^3$ and we set $\sqrt{g} = \sqrt{\det g_{ij}}$. By $r(t)$ we denote the scalar function which gives the density of the electric charge on $\mathbb{R}^3$. In this example we suppose that $r(t)$ is constant, $r(t) = r$, that is the distribution of the electric charge is constant on $\mathbb{R}^3$ and, also, we suppose that the metric g on $\mathbb{R}^3$ is the Euclidian metric.

Let us consider on $M = (T^1_3)^*\mathbb{R}$ the canonical polysymplectic structure $((\omega_0)_1, (\omega_0)_2, (\omega_0)_3)$. We denote by (q, p^1, p^2, p^3) the local coordinates on $M = (T^1_3)^*\mathbb{R}$ induced by the standard coordinates (q) on $\mathbb{R}$, and we define a Hamiltonian function $H : (T^1_3)^*\mathbb{R} \to \mathbb{R}$ by $H(q, p^1, p^2, p^3) = 4\pi r q + \frac{1}{2} \sum\limits_{A=1}^{3} (p^A)^2$.

Consider the equation

$$\Omega^\sharp(X_1, X_2, X_3) = \sum_{A=1}^{3} i_{X_A}(\omega_0)_A = dH, \tag{3.13}$$

where (X_1, X_2, X_3) is a 3-vector field on $(T^1_3)^*\mathbb{R}$.

Let $\phi : \mathbb{R}^3 \to (T^1_3)^*\mathbb{R}$, $\phi(t) = (\psi(t), \psi^1(t), \psi^2(t), \psi^3(t))$ be an integral section of an evolution 3-vector field which is a solution of (3.13). Then we obtain the Hamilton-de Donder-Weyl equations associated to this 3-symplectic Hamiltonian system:

$$4\pi r = -\left(\frac{\partial \psi^1}{\partial t^1} + \frac{\partial \psi^2}{\partial t^2} + \frac{\partial \psi^3}{\partial t^3}\right), \quad \psi^A = \frac{\partial \psi}{\partial t^A}, \; A = 1, 2, 3,$$

which are the electrostatic equations, and then the components $\psi(t)$ and $(\psi^1(t), \psi^2(t), \psi^3(t))$ of ϕ are the electric potential σ and the electric field $P = (P_1, P_2, P_3)$ on $\mathbb{R}^3$, respectively. So, the equation (3.13) is a geometric version of the electrostatic equations.

Next, if we consider a Lagrangian function $L : T^1_k M \to \mathbb{R}$, $L = L(x^i, v^i_A)$, then we obtain, by using a variational principle, the *generalized Euler-Lagrange equations* for L:

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left(\frac{\partial L}{\partial v^i_A}\right) - \frac{\partial L}{\partial x^i} = 0, \qquad v^i_A = \frac{\partial x^i}{\partial t^A}. \tag{3.14}$$

Following the ideas of Günther [19], we will describe the above equations (3.14) in terms of the geometry of k-tangent structures. In classical mechanics the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play complementary roles [9, 10, 17, 18, 21]. Also, that the k-symplectic structures and the k-tangent structures play similarly complementary roles. So, we construct a k–symplectic structure on the manifold $T^1_k M$, using its canonical k–tangent structure for each $1 \leq A \leq k$.

Let us consider the 1–forms $(\beta_L)_A = dL \circ J^A$, $1 \leq A \leq k$. In a local coordinate system (x^i, v^i_A) we have

$$(\theta_L)_A = \frac{\partial L}{\partial v^i_A} dx^i, \; 1 \leq A \leq k. \tag{3.15}$$

Definition 3.11. A Lagrangian L is called regular if

$$det\left(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right) \neq 0, \qquad 1 \leq i, j, \leq n, \quad 1 \leq A, B \leq k\,. \tag{3.16}$$

By introducing the following 2–forms

$$(\omega_L)_A = -d(\theta_L)_A\,, \; 1 \leq A \leq k, \tag{3.17}$$

one can easily prove the following.

Proposition 3.5. *$L : T^1_k M \to \mathbb{R}$ is a regular Lagrangian if and only if $((\omega_L)_1, \ldots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$, where V denotes the vertical distribution of $\tau : T^1_k M \to M$.*

Let $L : T^1_k M \to \mathbb{R}$ be a regular Lagrangian and let us consider the k-symplectic structure $((\omega_L)_1, \dots, (\omega_L)_k, V)$ on $T^1_k M$ defined by L. Let $\Omega^\sharp_L$ be the morphism defined by this k-symplectic structure

$$\Omega^\sharp_L : T^1_k(T^1_k M) \longrightarrow T^*(T^1_k M).$$

Thus, we can set the following equation:

$$\Omega^\sharp_L(X_1, \dots, X_k) = dE_L, \tag{3.18}$$

where $E_L = C(L) - L = \sum_{i,A} v^i_A \dfrac{\partial L}{\partial v^i_A} - L.$

The family $\left(T^1_k M, (\omega_L)_A, E_L\right)$ is called *a k-symplectic Lagrangian system.* As in the Hamiltonian case, we will denote by $\mathcal{X}^k_L(T^1_k M)$ the set of k -vector fields $\xi = (\xi_1, \cdots, \xi_k)$ on $T^1_k M$ which are the solutions of the equation (3.18).

Proposition 3.6. *Let L be a regular Lagrangian. If $\xi = (\xi_1, \cdots, \xi_k)$ is a solution of (3.18) then it is a SOPDE. In addition, if ξ is integrable then the solutions of ξ are solutions of the generalized Euler-Lagrange equations (3.14).*

This k-symplectic structure, associated to a regular Lagrangian L, was also introduced by Günther([19]) using the Legendre transformation, as follow.

The Legendre map $FL : T^1_k M \to (T^1_k)^* M$ was introduced by Günther ([19]) and was rewritten in [35] as follow: if $(v_{1x}, \dots, v_{kx}) \in (T^1_k)_x M$, then

$$\left[FL(v_{1x}, \dots, v_{kx})\right]^A (u_x) = \frac{d}{ds}|_{s=0}\, L(v_{1x}, \dots, v_{Ax} + s u_x, \dots, v_{kx}), \tag{3.19}$$

for each $A = 1, \dots, k$ and $u_x \in T_x M$. Locally, FL is given by

$$FL(x^i, v^i_A) = \left(x^i, \frac{\partial L}{\partial v^i_A}\right). \tag{3.20}$$

In fact, from (3.15), (3.17) and (3.20) we obtain the following propositions:

Proposition 3.7. *For all $A = 1, \dots, k$, $(\omega_L)_A = (FL)^*(\omega_0)_A$, where $((\omega_0)_1, \dots, (\omega_0)_k)$ are the 2-forms of the canonical k -symplectic structure on $(T^1_k)^* M$.*

Proposition 3.8. *Let L be a Lagrangian. The following conditions are equivalent:*

(1) L is regular.

(2) FL is a local diffeomorphism.

(3) $((\omega_L)_1, \dots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$.

Example 3.2 ([35]) In this example we consider the theory of a vibrating string. Coordinates (t^1, t^2) are interpreted as the time and the distance along the string, respectively. If $\phi(t^1, t^2)$ denotes the displacement of each point of the string as function of the time t^1 and the position t^2, the motion equations are

$$\sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2} = 0, \tag{3.21}$$

where σ and τ are certain constants of the mechanical system.

We shall show that the equations (3.21) can be described as the generalized Euler-Lagrange equations associated to a Lagrangian L defined on the jet bundle $T^1_k M$ with $M = \mathbb{R}$ and $k = 2$. Let us denote by (x, v_1, v_2) the coordinates on $T^1_2\mathbb{R}$ and consider the Lagrangian $L : T^1_2\mathbb{R} \to \mathbb{R}$, $(x, v_1, v_2) \to \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$.

Since L is regular there exists a k-symplectic structure $((\omega_L)_1, (\omega_L)_2)$, associated to L, given in local coordinates by $(\omega_L)_1 = \sigma dv_1 \wedge dx$, $(\omega_L)_1 = -\tau dv_2 \wedge dx$. The energy $E_L = C(L) - L$ is locally given by $E_L = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$ and $dE_L = \sigma v_1 dv_1 - \tau v_2 dv_2$.

Now, we consider the map $\Omega^\sharp_L : T^1_2(T^1_2\mathbb{R}) \longrightarrow T^*(T^1_2\mathbb{R})$ and let us suppose that there exists (ξ_1, ξ_2) a solution of the equation

$$\Omega^\sharp_L(\xi_1, \xi_2) = i_{X_1}(\omega_L)_1 + i_{X_2}(\omega_L)_2 = dE_L. \tag{3.22}$$

Then, from the Proposition (3.6), we know that (ξ_1, ξ_2) is a SOPDE. Let us suppose that $(\xi_1, \xi_2) \in T^1_2(T^1_2\mathbb{R})$ are locally given by $\xi_A = v_A \frac{\partial}{\partial x} + (\xi_A)_1 \frac{\partial}{\partial v_1} + (\xi_A)_2 \frac{\partial}{\partial v_2}$, $\quad A = 1, 2$.

If (ξ_1, ξ_2) is a solution of $\Omega^\sharp_L(\xi_1, \xi_2) = dE_L$, then we have $\sigma(\xi_1)_1 - \tau(\xi_2)_2 = 0$. So, if we consider $\phi : \mathbb{R}^2 \to \mathbb{R}$, $\phi = \phi(t^1, t^2)$, a solution of $\xi = (\xi_1, \xi_2)$, then we obtain

$$0 = \sigma(\xi_1)_1 - \tau(\xi_2)_2 = \sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2}.$$

Thus, the equation (3.22) is a geometric version for the equations (3.21).

An example of an integrable SOPDE solution $\xi = (\xi_1, \xi_2)$ of (3.21) is given by (see [7])

$$\begin{aligned} \xi_1 &= v_1 \frac{\partial}{\partial x} + \tau(\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_1} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_2}\,, \\ \xi_2 &= v_2 \frac{\partial}{\partial x} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_1} + \sigma(\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_2}\,. \end{aligned}$$

Thus any solution ϕ of the SOPDE $\xi = (\xi_1, \xi_2)$ in the formulae above is a solution of the vibrating string equation (3.21).

4 Generalized results and applications to k-symplectic Hamiltonian and Lagrangian systems

In this section we will present a result which allow us to obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem and without the use of a variational principle, using only symmetries and pseudosymmetries associated to the k-vector fileds $\mathbf{X} = (X_1, \ldots, X_k)$ which are solutions of the equation $\sum\limits_{A=1}^{k} i_{X_A}\omega_A = dH$ ([37]). This result is a generalization from the classical case ($k = 1$) of a results of G.L. Jones ([20]) and M. Crâşmăreanu ([12]). Applications for Lagrangian and Hamiltonian k-symplectic formalisms are also presented ([35], [37]).

In the classical case ($k = 1$), let us recall that Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse ([11], [12], [20], [38], [39]).

For the higher order case the problem was be solved by L. Bua, I.Bucătaru and M. Salgado in [7]. So, for $k > 1$ the Noether Theorem is also true, that is each Cartan symmetry induces a conservation law (defined for a regular Lagrangian on $T^1_k M$, like in [7]). However, the converse of Noether Theorem may not be true and in [7] is provided some examples of conservation laws that are not induced by Cartan symmetries.

Using the notions from the previous section, we have:

Definition 4.1. ([7]) A vector field $X \in \mathcal{X}(T^1_k M)$ is called a Cartan symmetry for the regular Lagrangian L, if $L_X(\omega_L)_A = 0$, for all $1 \leq A \leq k$ and $L_X E_L = 0$, where $E_L = v^i_A \frac{\partial L}{\partial v^i_A} - L$.

Let us remark that $L_X(\omega_L)_A = 0$ implies that, locally, we have $i_X(\omega_L)_A = df_A$, for all $1 \leq A \leq k$ ([7]).

Definition 4.2. Let $(M, \omega_A, V; 1 \leq A \leq k)$ be a k-symplectic manifold.
The map $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k) : M \longrightarrow \mathbb{R}^k$ is called conservation law
for a k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ on M if

$$\sum_{A=1}^{k} L_{X_A} \Phi_A = 0. \tag{4.1}$$

In [7], [42] it is presented an equivalent definition for conservation law on $T^1_k M$.

Theorem 4.1. *(Noether Theorem, [7]) Let be L a regular Lagrangian on $T^1_k M$. If $X \in \mathcal{X}(T^1_k M)$ is a Cartan symmetry for L such that there exists (locally defined) functions f_A on $T^1_k M$ with*

$$L_X(\theta_L)_A = df_A \ , 1 \leq A \leq k \, , \tag{4.2}$$

then the following functions

$$\Phi_A = (\theta_L)_A(X) - f_A \ , 1 \leq A \leq k \, , \tag{4.3}$$

give a conservation law for the Euler-Lagrange equations associated to L, i.e. for an integrable evolution k-vector field associated to $H = E_L$, the energy of L.

Next result show when a conservation law for a Lagrangian induces and are induced by a Cartan symmetry.

Theorem 4.2. *([7]) Let be L a regular Lagrangian on $T^1_k M$, the functions $f_A \in C^\infty(T^1_k M)$, $1 \leq A \leq k$, and a vector field $X \in \mathcal{X}(T^1_k M)$ such that*

$$i_X(\omega_L)_A = df_A \ , 1 \leq A \leq k \, . \tag{4.4}$$

Then $F = (f_1, \ldots, f_k)$ is a conservation law for L if and only if X is a Cartan symmetry.

Example 4.1 ([35], [7]) The following two functions $\Phi_1, \Phi_2 : T^1_2\mathbb{R} \to \mathbb{R}$,

$$\Phi_1(v_1, v_2) = -2\sigma v_1 v_2 \, , \qquad \Phi_2(v_1, v_2) = \sigma(v_1)^2 + \tau(v_2)^2 \tag{4.5}$$

give a conservation law $\Phi = (\Phi_1, \Phi_2)$ for the evolution 2-vector field associated with the Hamiltonian E_L from example 3.2. We can say that $\Phi = (\Phi_1, \Phi_2)$ is a conservation law for the Euler-Lagrange equations (3.21) of the vibrating string, or $\Phi = (\Phi_1, \Phi_2)$ give a conservation law for an integrable evolution k-vector field associated to $H = E_L$, where E_L is the energy of L. More that, this conservation law is not induced by a Cartan symmetry, and hence it will show that the converse of the Noether Theorem 4.1 is not true, unless the assumptions (4.4) are satisfied ([7]).

Now, let us introduce the notions:

Definition 4.3. A k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M is called symmetry for $\mathbf{X} = (X_1, \ldots, X_k)$ if

$$L_{X_A} Y_A = 0, \ for\ all \quad A = 1, \ldots, k. \tag{4.6}$$

Definition 4.4. If we fixed a k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M, then a k-vector field $\mathbf{Z} = (Z_1, \ldots, Z_k)$ is called $\mathbf{Y}$-pseudosymmetry for $\mathbf{X} = (X_1, \ldots, X_k)$ if, for all $A = 1, \ldots, k$, there is a function $f_A \in C^\infty(M)$ such that

$$L_{X_A} Z_A = f_A \; Y_A. \tag{4.7}$$

A $\mathbf{X}$-pseudosymmetry for $\mathbf{X}$ is called *pseudosymmetry* for $\mathbf{X}$. It is clear that a $\mathbf{O}$-pseudosymmetry for $\mathbf{X} = (X_1, \ldots, X_k)$ is a symmetry for $\mathbf{X}$.

Next, we will present the main result, which allow us to obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem and without the use of a variational principle. The proof of this theorem can be found in [37].

Theorem 4.3. *([37]) Let be $\mathbf{X} = (X_1, \ldots, X_k)$ a k-vector field on M and $(\omega_1, \ldots, \omega_k)$ be a family of p-forms on M, invariant for $\mathbf{X}$, i.e. $L_{X_A} \omega_A = 0$, for all $A = 1, \ldots, k$. If the k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M is a symmetry for $\mathbf{X}$ and the $p-1$ k-vector fields $\mathbf{S}^1 = (S_A^1)_{A=\overline{1,k}}$, ..., $\mathbf{S}^{p-1} = (S_A^{p-1})_{A=\overline{1,k}}$ are $\mathbf{Y}$-pseudosymmetries for $\mathbf{X}$, then*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)\,, \tag{4.8}$$

is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$, where $\Phi_A = \omega_A \left(S_A^1, \ldots, S_A^{p-1}, Y_A\right)$, for all $A = 1, \ldots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}^1$, ..., $\mathbf{S}^{p-1}$ are symmetries for $\mathbf{X}$ then $\mathbf{\Phi}$ given by (4.8) is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$.

As an immediate consequence of the previous theorem, we have the result:

Theorem 4.4. *Let $(M, \omega_A, V; 1 \leq A \leq k)$ be a k–symplectic manifold and $H : M \longrightarrow \mathbb{R}$ be a function on M. Let $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable evolution k-vector field associated to H, i.e. $\mathbf{X} \in \mathcal{X}_H^k(M)$. If we suppose that $L_{X_A} \omega_A = 0$, for all $A = 1, \ldots, k$, then for any k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M, which is a symmetry for $\mathbf{X}$ and for any k-vector field $\mathbf{S} = (S_1, \ldots, S_k)$, which is a $\mathbf{Y}$-pseudosymmetry for $\mathbf{X}$, we have that*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)\,,$$

is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$, where $\Phi_A = \omega_A\,(S_A, Y_A)$, for all $A = 1, \ldots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}$ are symmetries for $\mathbf{X}$ then $\mathbf{\Phi}$ given above is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$.

Remark 4.5. a) Obviously, for any k-vector field $\mathbf{X} \in \mathcal{X}_H^k(M)$, using (3.11), we have $\sum_{A=1}^{k} L_{X_A}\omega_A = 0$. But, for our purpose we have need more that, that is we need that $L_{X_A}\omega_A = 0$, for all $A = 1, \ldots, k$.

b) Obviously, the Hamiltonian function H is not a conservation law for an integrable evolution k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathcal{X}_H^k(M)$. Neither the map $\mathbf{H} = (H, \ldots, H) : M \longrightarrow \mathbb{R}^k$ is not a conservation law for any integrable evolution k-vector field $\mathbf{X} \in \mathcal{X}_H^k(M)$, because $\sum_{A=1}^{k} L_{X_A} H \neq 0$.

Now, using this last result we can obtain new kinds of conservation laws for k-symplectic Lagrangian systems and k-symplectic Lagrangian systems.

Corollary 4.5. *Let $\left((T_k^1)^*M, (\omega_0)_A, H\right)$ be a k-symplectic Hamiltonian system and $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable evolution k-vector field associated to H, i.e. $\mathbf{X} \in \mathcal{X}_H^k((T_k^1)^*M)$. If we suppose that $L_{X_A}(\omega_0)_A = 0$, for all $A = 1, \ldots, k$, then for any k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on $(T_k^1)^*M$, which is a symmetry for $\mathbf{X}$ and for any k-vector field $\mathbf{S} = (S_1, \ldots, S_k)$, which is a $\mathbf{Y}$-pseudosymmetry for $\mathbf{X}$, we have that*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$, where $\Phi_A = (\omega_0)_A (S_A, Y_A)$, for all $A = 1, \ldots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}$ are symmetries for $\mathbf{X}$ then $\mathbf{\Phi}$ given above is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$.

Corollary 4.6. *Let $\left(T_k^1 M, (\omega_L)_A, E_L\right)$ be a k-symplectic Lagrangian system and $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable evolution k-vector field associated to $H = E_L$, i.e. $\mathbf{X} \in \mathcal{X}_L^k(T_k^1 M)$. If we suppose that $L_{X_A}(\omega_L)_A = 0$, for all $A = 1, \ldots, k$, then for any k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on $T_k^1 M$, which is a symmetry for $\mathbf{X}$ and for any k-vector field $\mathbf{S} = (S_1, \ldots, S_k)$, which is a $\mathbf{Y}$-pseudosymmetry for $\mathbf{X}$, we have that*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$, where $\Phi_A = (\omega_L)_A (S_A, Y_A)$, for all $A = 1, \ldots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}$ are symmetries for $\mathbf{X}$ then $\mathbf{\Phi}$ given above is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$.

Remark 4.6. If each vector fields $X_1, \ldots, X_k$ of $\mathbf{X} \in \mathcal{X}_L^k(T_k^1 M)$ are Cartan symmetries for L, then we have $L_{X_A}(\omega_L)_A = 0$, for all $A = 1, \ldots, k$, and then we can apply the last corollary for this k-vector field $\mathbf{X}$. Moreover, we have that $(H, \ldots, H)$ is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$, where $H = E_L$.

Example 4.2 ([7], [39]) a) If we consider the Lagrangians $L_1, L_2 : T_2^1\mathbb{R} \to \mathbb{R}$, defined by $L_1(x, v_1, v_2) = \frac{1}{2}(\sigma(v_1)^2 - \tau(v_2)^2)$, $L_2(x, v_1, v_2) = \sqrt{1 + (v_1)^2 + (v_2)^2}$, then the vector field $X = \frac{\partial}{\partial x}$ is a Cartan symmetry, and the induced conservation laws are $\mathbf{\Phi} = (\Phi_1 = \sigma v_1, \Phi_2 = -\tau v_2)$ for L_1 and $\mathbf{\Phi} = (\Phi_1 = \frac{v_1}{\sqrt{1+(v_1)^2+(v_2)^2}}, \Phi_2 = \frac{v_2}{\sqrt{1+(v_1)^2+(v_2)^2}})$ for L_2.

Let us observe that the above Lagrangians corresponde to the vibrating string equations and, respectively to the equations of minimal surfaces.

b) For the Lagrangian $L : T^1_3\mathbb{R} \to \mathbb{R}$, $L(x, v_1, v_2, v_3) = \frac{1}{2}\left((v_1)^2 + (v_2)^2 + (v_3)^2\right)$, the vector field $X = \frac{\partial}{\partial x}$ is a Cartan symmetry, and the induced conservation law is $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$, where $\Phi_i = v_i$, $i = 1, 2, 3$. The Euler-Lagrange equations corresponding to L are the Laplace's equations.

c) For the Lagrangian $L : T^1_2\mathbb{R}^2 \to \mathbb{R}$, defined by

$$L(x^1, x^2, v^1_1, v^1_2, v^2_1, v^2_2) = \left(\frac{1}{2}\lambda + \nu\right)\left[(v^1_1)^2 + (v^2_2)^2\right] + \frac{1}{2}\nu\left[(v^1_2)^2 + (v^2_1)^2\right] + (\lambda+\nu)v^1_1 v^2_2\,,$$

the vector field $X = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ is a Cartan symmetry, and the induced conservation law is $\mathbf{\Phi} = (\Phi_1, \Phi_2)$, where $\Phi_1 = (\lambda + 2\nu)v^1_1 + \nu v^2_1 + (\lambda + \nu)v^2_2$, $\Phi_2 = (\lambda + \nu)v^1_1 + \nu v^1_2 + (\lambda + 2\nu)v^2_2$. The Euler-Lagrange equations corresponding to L are the Navier's equations.

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