On the stability of transverse locally conformally symplectic structures

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Abstract. In this paper we define transverse locally conformally symplectic forms on foliated manifolds and using the basic Lichnerowicz cohomology on foliated manifolds, we generalize some Moser's type stability results for transverse locally conformally symplectic forms.

M.S.C. 2010: 53C12, 53C55, 53D99, 57R17, 58H15.
Key words: foliated manifold; transverse locally conformally symplectic form; basic Lichnerowicz cohomology; Moser stability.

1 Introduction

The locally conformally symplectic (l.c.s.) structures were introduced by Lee [17] and Vaisman [28]. The fundamental properties of these structures have been studied extensively by Vaisman, Banyaga, de Leon, Bande, Kotschick and many others, see for instance [2, 3, 4, 18, 28] and the references given there for a more thorough discussion.

The Lichnerowicz cohomology, also known in literature as Morse-Novikov cohomology, is a cohomology defined for a smooth manifold M and a closed 1-form θ. It is defined by twisting the usual differential of the de Rham complex $\Omega^\bullet(M)$ of M; namely, the Lichnerowicz cohomology is the cohomology of a complex $(\Omega^\bullet(M), d_\theta)$, where d_θ is defined by $d_\theta\varphi = d\varphi - \theta \wedge \varphi$. This cohomology was originally defined by Lichnerowicz [19] and Novikov [22] in the context of Poisson geometry and Hamiltonian mechanics, respectively. Lichnerowicz cohomology is naturally defined for a l.c.s. manifold with its canonical closed 1-form called the Lee form, [2, 3].

In this paper we introduce a new class of foliations called transversely locally conformally symplectic foliations (transversely l.c.s. foliations), which is a foliated version of l.c.s. manifolds roughly in the following sense: This class of foliations has a l.c.s. structure on the direction transverse to the leaves. For instance, the simple foliation defined by a C^∞ submersion $f : \mathcal{M} \to M$ of $\mathcal{M}$ onto a l.c.s. manifold M is transversely l.c.s. The case where the dimension of the leaves is zero corresponds to the original l.c.s. manifolds. The next aim of this paper is to generalize some Moser's type results concerning to stability of such transverse l.c.s. forms on foliated

BSG Proceedings, Vol. 20, 2013, pp. 1-8.

manifolds. In this sense, in the preliminary section following [8, 9, 10], we make a short review about some basic facts on foliated manifolds, basic de Rham cohomology and transverse symplectic structures. Next we define transverse l.c.s. structures and we generalize the basic Lichnerowicz cohomology and a basic Hodge decomposition for basic forms on foliated manifolds. Finally, using an argument similar to that in [4] concerning to Moser type stability of l.c.s. forms we generalize these results to the case of transverse l.c.s. forms on foliated manifolds.

2 Generalities

2.1 Foliated manifolds

Let us consider $\mathcal{M}$ an $(n+m)$-dimensional manifold which will be assumed to be connected and orientable.

Definition 2.1. A codimension n foliation $\mathcal{F}$ on $\mathcal{M}$ is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of $\mathcal{M}$;

(ii) For every $i \in I$, $\varphi_i : U_i \to M$ are submersions, where M is an n-dimensional manifold, called transversal manifold;

(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i, \tag{2.1}$$

for every $(i,j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

Every fibre of φ_i is called a *plaque* of the foliation. Condition (2.1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by φ_i and φ_j coincides. The manifold $\mathcal{M}$ is decomposed into a family of disjoint immersed connected submanifolds of dimension m; each of these submanifolds is called a *leaf* of $\mathcal{F}$.

We say that $\mathcal{F}$ is *transversely orientable* if on M can be given an orientation which is preserved by all $f_{i,j}$. By $T\mathcal{F}$ we denote the tangent bundle to $\mathcal{F}$ and $\Gamma(\mathcal{F})$ is the space of its global sections i.e. vector fields tangent to $\mathcal{F}$. We say that a differential form φ is *basic* if it satisfies $i_X\varphi = \mathcal{L}_X\varphi = 0$ for every $X \in \Gamma(\mathcal{F})$, where i_X and $\mathcal{L}_X$ denotes the the interior product and Lie derivative with respect to X, respectively. A *basic function* is a function constant on the leaves; such functions form an algebra denoted by $\mathcal{F}_b(\mathcal{M})$. The quotient $Q\mathcal{F} = T\mathcal{M}/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. A vector field $Y \in \mathcal{X}(\mathcal{M})$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$ we have $[X,Y] \in \Gamma(\mathcal{F})$; $\mathcal{X}(\mathcal{M},\mathcal{F})$ denotes the algebra of foliated vector fields on $\mathcal{M}$. The quotient $\mathcal{X}(\mathcal{M}/\mathcal{F}) = \mathcal{X}(\mathcal{M},\mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on $\mathcal{M}$.

In this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(x^1,\ldots,x^n,y^1,\ldots,y^m)$ on an open subset U on which the foliation is trivial and defined by the equations $dx^a = 0$, $a = 1,\ldots,n$.

In a such distinguished local chart a foliated vector field is given by

$$X = \sum_{a=1}^{n} X^a(x^1,\ldots,x^n)\frac{\partial}{\partial x^a} + \sum_{\alpha=1}^{m} X^\alpha(x^1,\ldots,x^n,y^1,\ldots,y^m)\frac{\partial}{\partial y^\alpha}. \tag{2.2}$$

Then X projects on a basic vector field $\overline{X} = X^a(x^1, \ldots, x^n)\frac{\partial}{\partial x^a}$ which is independent on the coordinates along the leaves. We also notice that a basic r-form is locally given by

$$\varphi = \sum_{(a_1 \ldots a_r)} \varphi_{a_1 \ldots a_r}(x^1, \ldots, x^n) dx^{a_1} \wedge \ldots \wedge dx^{a_r}. \tag{2.3}$$

We also recall that a *foliated diffeomorphism* f is a diffeomorphism f of $\mathcal{M}$ preserving the foliation $\mathcal{F}$, i.e., $f(F) = F$ for every leaf F of $\mathcal{F}$ and we denote by $\mathrm{Diff}(\mathcal{M}, \mathcal{F})$ the group of such foliated diffeomorphisms.

2.2 Basic de Rham cohomology

Let $\Omega^r(\mathcal{M}/\mathcal{F})$ be the space of all basic forms of degree r. It is easy to see that the exterior derivative of a basic form is also a basic form. Indeed, if $\varphi \in \Omega^r(\mathcal{M}/\mathcal{F})$ then $i_X\varphi = \mathcal{L}_X\varphi = 0$ for any $X \in \Gamma(\mathcal{F})$ and, then by Cartan's formula $\mathcal{L}_X = i_X d + d i_X$ and $d^2 = 0$ it follows that $i_X d\varphi = \mathcal{L}_X d\varphi = 0$ for any $X \in \Gamma(\mathcal{F})$. Let us denote by $d_b = d|_{\Omega^\bullet(\mathcal{M}/\mathcal{F})}$ the restriction of exterior derivative to basic forms. Then we have $d_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}) \longrightarrow \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F})$ and the differential complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_b)$ which is called the *basic de Rham complex* of $\mathcal{F}$; its cohomology is called the *basic de Rham cohomology* and denoted by $H^\bullet(\mathcal{M}/\mathcal{F})$.

2.3 Transverse symplectic structures

Definition 2.2. A transverse structure to $\mathcal{F}$ is a geometric structure on M invariant by all the local diffeomorphisms $f_{i,j}$.

A transverse structure can be considered as a geometric structure on the leaf space $\mathcal{M}/\mathcal{F}$ (which is not a manifold in general).

Definition 2.3. ([6]). A *transverse symplectic structure* on a foliated manifold $(\mathcal{M}, \mathcal{F})$ is given by a family of symplectic forms $\{\omega_i\}$ on the transversal manifold M such that $f_{i,j}$ preserves the symplectic structure, $\omega_j = f^*_{i,j}\omega_i$ on $\varphi_j(U_i \cap U_j)$.

The pullback $\varphi_i^*\omega_i$ is closed and

$$\varphi_i^*\omega_i = (f_{i,j} \circ \varphi_j)^*\omega_i = \varphi_j^* \circ f^*_{i,j}\omega_i = \varphi_j^*\omega_j$$

so that the forms $\varphi_i^*\omega_i$ agree on overlaps and so define a global closed basic 2-form $\omega \in \Omega^2(\mathcal{M}/\mathcal{F})$. Of course, ω is not non-degenerate, but its kernel is precisely the bundle $T\mathcal{F}$ of vectors tangent to the foliation. A manifold $\mathcal{M}$ is called *presymplectic*, if it is endowed with a closed 2-form ω of constant rank. Hence, due to [6] we have

Proposition 2.1. *A transverse symplectic structure on $(\mathcal{M}, \mathcal{F})$ determines in a unique manner a presymplectic structure ω on $\mathcal{M}$ such that $T\mathcal{F}$ is the kernel of the form ω.*

Proposition 2.2. *If $(\mathcal{M}, \omega)$ is a presymplectic manifold and $T\mathcal{F}$ is the kernel of ω, then there is a naturally induced transverse symplectic structure on $(\mathcal{M}, \mathcal{F})$ which corresponds to ω.*

We notice that in a recent paper [21] it is proved a Moser type stability result for transverse symplectic structures, which implies Moser's stability theorem for presymplectic forms.

3 Transverse locally conformally symplectic forms and their stability

3.1 Transverse locally conformally symplectic structures

Following a definition for transverse locally conformal Kähler (l.c.K.) foliation given in [5], we consider here an analogue definition for transverse locally conformally symplectic (l.c.s.) structure.

Definition 3.1. A *transverse locally conformally symplectic structure* (or transverse l.c.s. structure) on a foliated manifold $(\mathcal{M},\mathcal{F})$ is a non-degenerate basic 2-form ω which is locally conformal to a transverse symplectic form. The triplet $(\mathcal{M},\mathcal{F},\omega)$ is called *transverse l.c.s. foliation.*

In other words, $(\mathcal{M},\mathcal{F},\omega)$ is a transverse l.c.s. structure if there exists an open covering with foliated charts $\{U_i,\varphi_i\}$ of $(\mathcal{M},\mathcal{F})$ and a smooth positive basic function $f_i\in\mathcal{F}_b(U_i)$ such that $f_i\omega|_{U_i}$ is a transverse symplectic structure on U_i.

Equivalently, there exists on $(\mathcal{M},\mathcal{F})$ a closed basic 1-form θ, called *the basic Lee form*, such that ω satisfies the integrability condition

$$d_b\omega=\theta\wedge\omega. \tag{3.1}$$

Indeed, by $d_b\theta=0$ and Poincaré Lemma, there is an open cover $\{U_i\}_{i\in I}$ of $\mathcal{M}$ and a family $\{\sigma_i\}_{i\in I}$ of C^∞ functions $\sigma_i:U_i\to\mathbb{R}$ so that $\theta=d\sigma_i$ on U_i. In particular, as θ is basic, the functions σ_i must be basic, as well, i.e. $\sigma_i\in\mathcal{F}_b(U_i)$. Then $\omega_i=e^{-\sigma_i}\omega|_{U_i}$ is a transverse symplectic structure on U_i.

When θ vanishes identically, the form ω is transverse symplectic.

Example 3.2. A simple foliation defined by a C^∞ submersion $f:\mathcal{M}\to M$ of $\mathcal{M}$ onto a l.c.s. manifold M is a transverse l.c.s. foliation. The case where the dimension of the leaves is zero corresponds to the original l.c.s. manifolds.

Two transverse l.c.s. forms ω and $\omega^{'}$ are said to be (conformally) equivalent if there exists some positive basic function f such that $\omega=f\omega^{'}$. A transverse locally conformally symplectic structure is an equivalence class of transverse l.c.s. forms for this relation. Note that the basic de Rham cohomology class of the basic Lee form is an invariant of the transverse l.c.s. structure because a conformal rescaling of ω changes θ by the addition of an exact basic form.

If a transverse l.c.s. structure contains a transverse symplectic representative, then it is *transverse globally conformally symplectic structure.* This is the case if and only if the basic Lee form is exact.

3.2 Basic Lichnerowicz cohomology

Let $(\mathcal{M},\mathcal{F})$ be a transversely foliation and $\theta\in\Omega^1(\mathcal{M}/\mathcal{F})$ be a closed basic 1-form. Denote by $d_{b,\theta}:\Omega^r(\mathcal{M}/\mathcal{F})\to\Omega^{r+1}(\mathcal{M}/\mathcal{F})$ the map $d_{b,\theta}=d_b-\theta\wedge$.

Since $d_b\theta=0$, we easily obtain that $d^2_{b,\theta}=0$. The differential complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}),d_{b,\theta})$ is called the *basic Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$; its cohomology groups $H^\bullet_\theta(\mathcal{M}/\mathcal{F})$ are called the *basic Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

This is a basic version of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [19] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic manifolds geometry, see [3, 18]. We also notice that Vaisman in [27] studied it under the name of "adapted cohomology" on locally conformal Kähler l.c.K. manifolds. A generalization of basic Lichnerowicz cohomology on transversally l.c.K. foliations is given in [15]. Some other notions concerning to a such basic Lichnerowicz cohomology may be found in [14].

We notice that, locally, the basic Lichnerowicz cohomology complex becomes the basic de Rham complex after a change $\varphi \mapsto e^f\varphi$ with f a basic function which satisfies $d_b f = \theta$, namely $d_{b,\theta}$ is the unique differential in $\Omega^\bullet(\mathcal{M}/\mathcal{F})$ which makes the multiplication by the smooth basic function e^f an isomorphism of cochain basic complexes $e^f : (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta}) \to (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_b)$.

Proposition 3.1. *The basic Lichnerowicz cohomology depends only on the basic class of θ. In fact, we have the following isomorphism $H^r_{\theta-d_bf}(\mathcal{M}/\mathcal{F}) \approx H^r_\theta(\mathcal{M}/\mathcal{F})$.*

In the case when θ is the basic Lee form of a transverse l.c.s. form ω, equation (3.1) shows that ω is $d_{b,\theta}$-closed and so defines a class in $H^2_\theta(\mathcal{M}/\mathcal{F})$, which is an anologue of Morse-Novikov class of l.c.K. manifolds, see [23]. If we consider the transverse l.c.s. structure defined by ω, and $\omega^{'} = f\omega$ with $f \in \mathcal{F}_b(\mathcal{M})$, then the basic Lee form of $\omega^{'}$ is just $\theta^{'} = \theta + d_b \ln f$ and the class $[\omega] \in H^2_\theta(\mathcal{M}/\mathcal{F})$ is mapped to $[\omega^{'}] \in H^2_{\theta'}(\mathcal{M}/\mathcal{F})$ by the isomorphism from Proposition 3.1.

3.3 Hodge decomposition

For our purposes it is useful that Hodge theory applies to the basic $d_{b,\theta}$-cohomology. For this reason in the sequel of this paper we suppose that the transversal manifold M is Riemannian and oriented and all the $f_{i,j}$ are isometries. In this case $\mathcal{F}$ is said to be *transversally Riemannian.* This means that the normal bundle $Q\mathcal{F}$ is equipped with a Riemannian metric g_Q which is "invariant along the leaves". We also notice that $\mathcal{L}_X g_Q = 0$ for all leafwise vector fields $X \in \Gamma(\mathcal{F})$. This condition is characterized by the existence of a unique metric and torsion-free connection ∇ on $Q\mathcal{F}$, see [20, 26]. We often assume that the manifold $\mathcal{M}$ is endowed with the additional structure of a *bundle-like metric* [25], i.e. the metric g on $\mathcal{M}$ induces the metric on $Q\mathcal{F} \approx T\mathcal{F}^\perp$. Every Riemannian foliation admits bundle-like metrics that are compatible with a given $(\mathcal{M}, \mathcal{F}, g_Q)$ structure. Many researchers have studied the basic Laplacian and the Hodge theory for basic forms on Riemannian foliations with bundle-like metrics (see [1, 8, 10, 16, 24, 25, 26]). The basic Laplacian Δ_b for a given bundle-like metric is a version of the Laplace operator that preserves the basic forms and that is essentially self-adjoint on the L^2-closure of the space of basic forms. Let us consider $\bar{*} : \Omega^r(\mathcal{M}) \to \Omega^{n-r}(\mathcal{M})$ the pointwise Hodge star operator on all forms given by

$$\bar{*}\varphi = (-1)^{n(n-r)} * (\varphi \wedge \chi_\mathcal{F}),$$

with $\chi_\mathcal{F}$ being the leafwise volume form, the characteristic form of the foliation, and $*$ being the ordinary Hodge star operator. The operator $\bar{*}$ maps basic forms to basic forms and it has the property that $*\varphi = \bar{*}\varphi \wedge \chi_\mathcal{F}$ for a basic form φ, see [26]. Note that $\bar{*}^2 = (-1)^{n(n-r)} Id$ on r-forms. All that is required for the formula above to be well-defined is that the Riemannian foliation is transversally oriented. Then the basic Laplacian is defined by $\Delta_b = d_b\delta_b + \delta_b d_b$, where

$$\delta_b = d_b^* + (k_b\wedge)^*.$$

Here d_b^* is the formal adjoint (with respect to g_Q) of the basic exterior derivative and $(k_b\wedge)^*$ is the pointwise adjoint of the operator $k_b\wedge$, where k_b is the basic component of the mean curvature one-form, see [1]. Clearly, $(k_b\wedge)^*$ depends on the choice of bundle-like metric g, not simply on the transverse metric g_Q. We also notice that a Hodge decomposition of basic forms with respect to the basic Laplacian holds, see [24].

If $(\mathcal{M}, \mathcal{F}, g)$ is a Riemannian foliation with bundle-like metric g compatible with the transversally Riemannian structure $(\mathcal{M}, \mathcal{F}, g_Q)$, then in a recent paper [13], is states a Hodge decomposition theorem for basic forms with respect to the modified basic Laplacian $\widetilde{\Delta}_b = \widetilde{d}_b\widetilde{\delta}_b + \widetilde{\delta}_b\widetilde{d}_b$, where $\widetilde{d}_b = d_b - \frac{1}{2}k_b\wedge$ and $\widetilde{\delta}_b = \delta_b - \frac{1}{2}(k_b\wedge)^*$.

In fact the Hodge decomposition theorem for basic forms holds with respect to a more modified basic Laplacian of Lichnerowicz type: $\Delta_{b,\theta} = d_{b,\theta}\delta_{b,\theta} + \delta_{b,\theta}d_{b,\theta}$, where

$$\delta_{b,\theta}\varphi = \delta_b\varphi - (-1)^{n(r-1)}(\overline{*}(\theta \wedge \overline{*}\varphi)),\ \forall\, \varphi \in \Omega^r(\mathcal{M}/\mathcal{F}), \tag{3.2}$$

is the formal L^2-adjoint of $d_{b,\theta}$ on the space of basic forms. Indeed, following an argument similar that in [7, 12], we can prove that $\Delta_{b,\theta}$ is an elliptic operator, so the general theory of elliptic operators leads to the following Hodge decomposition of basic forms on $(\mathcal{M}, \mathcal{F}, g)$:

$$\Omega^r(\mathcal{M}/\mathcal{F}) = \ker \Delta^r_{b,\theta} \oplus \operatorname{im} d^{r-1}_{b,\theta} \oplus \operatorname{im} \delta^{r+1}_{b,\theta}. \tag{3.3}$$

3.4 Moser type stability for transverse l.c.s. forms

In this subsection we consider families ω_t of transverse locally conformally symplectic forms depending smoothly on a parameter $t \in [0,1]$ and we study some problems concerning their stability giving a generalization of some result from [4] in the case of locally conformally symplectic forms. The uniqueness of the basic Lee form θ_t implies that this depends smoothly on t as well. The proofs follow using the same methods as in [4], taking into account that a time-dependent vector field X_t obtained by differentiating a foliation preserving isotopy ϕ_t is foliated, see [11], and thus it projects to a basic vector field $\overline{X}_t$.

Theorem 3.2. *Let ω_t be a family of transverse l.c.s. forms, depending smoothly on $t \in [0,1]$, on a smooth, closed $(n+m)$-dimensional manifold $\mathcal{M}$ endowed with a regular foliation $\mathcal{F}$ of codimension n. Denote by θ_t the basic Lee form of ω_t. There exists a foliation preserving isotopy ϕ_t with $\phi_t^*\omega_t$ conformally equivalent to ω_0 for all t if and only if there are positive smooth basic functions f_t on $(\mathcal{M}, \mathcal{F})$, varying smoothly with t, such that the time derivative $\frac{d}{dt}(f_t\omega_t)$ of the conformally rescaled family $f_t\omega_t$ is $d_{b,\theta_t'}$-exact for every t, where $\theta_t' = \theta_t + d_b \ln f_t$ is the basic Lee form of $f_t\omega_t$.*

If we assume that the basic Lee forms are independent of t, then Theorem 3.2 implies the following results for transverse l.c.s. forms corresponding to the classical results for l.c.s. forms given by Banyaga in [2].

Corollary 3.3. *Let ω_t be a smooth family of transverse l.c.s. forms on a compact foliated manifold $(\mathcal{M}, \mathcal{F})$ having the same basic Lee form θ. If $\omega_t - \omega_0$ is $d_{b,\theta}$-exact for all t, then there exist a family of basic functions f_t and a foliation preserving isotopy ϕ_t such that $\phi_t^*(\omega_t) = f_t\omega_t$.*

Corollary 3.4. *Let ω_t be a smooth family of transverse l.c.s. forms on a compact foliated manifold $(\mathcal{M}, \mathcal{F})$ such that the corresponding basic Lee forms θ_t have the same basic de Rham cohomology class. Suppose there exists a smooth family of basic 1-forms ψ_t such that $\omega_t = d_b\psi_t - \theta_t \wedge \psi_t$. Then there exists a foliation preserving isotopy ϕ_t such that $\phi_t^*\omega_t$ is conformally equivalent to ω_0 for all t.*

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