Lagrange-Hamilton conjugations

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. The general theory of almost symplectic ω-conjugation was elaborated, for the first time, by P. Stavre ([7]). Its applications to the total space $E$ of a vector bundle $\xi = (E, \pi, M)$ are already studied ([3]). In the cases of the tangent bundle and of the cotangent bundle, the two authors obtained new results, with applications in Hamiltonian models ([9], [4]). Using these results and the theory of the $G$-conjugation ([8]), in the framework of the theory of Lagrange-Hamilton models, the authors obtain the case when the theory of the almost symplectic $\omega$-conjugation and the theory of the $G$-conjugation are equivalent.


Key words: Almost symplectic conjugation; distributions; linear d-connections; normal connections.

1 General notions. Theorems

Let us consider a $p$-dimensional paracompact differentiable manifold $E_p$, endowed with an almost symplectic structure.

Definition 1.1. ([7]) Let us consider the structure $L_p = (E_p, \omega, D)$ where $\omega$ is an almost symplectic one and $D$ a linear connection on $E$ and analogously $L_p = (E_p, \omega, D)$. Let us consider two one-dimensional distributions $D_1$, $D_2$:

$$(1) \quad D_1 : u \rightarrow D_1 u \subset T_u E; \quad D_2 : u \rightarrow D_2 u \subset T_u E, \quad u \in E.$$ 

If we have

$$(2) \quad \omega(V_1, V_2) = 0, \quad \forall \; V_1 \in D_1 u, \; V_2 \in D_2 u, \quad u \in E$$
at the parallel transport of the distributions \( D_1, D_2 \), with respect to \( D_1, D_2 \), then we will say that the linear connections \( D_1, D_2 \) are \( \omega \)-conjugated connections or that the spaces \( L_\omega, L_\omega \) are \( \omega \)-conjugated spaces. We will write: \( L_\omega \sim L_\omega \) or \( D_1 \sim D_2 \).

In the general case we have:

**Proposition 1.1.** In the general case the relation \( \sim \) is not an equivalence one on the set \( D(\omega) = \{ D, \omega, D \sim \} \) of the \( \omega \)-conjugated connections.

Let us consider \( M_2(D) \) the set of the linear connections which are not \( \omega \)-conjugated with any \( \omega \)-compatible connection.

**Partition Theorem 1.2.** (\([9]\)) One has the partitions:

\[
M(D) = M_1(D) \cup M_2(D); \quad M_1(D) \cap M_2(D) = \emptyset.
\]

As well, \( \{ C_\omega \} \) are equivalence classes.

It is has already been established that:

\[
(d\omega)(XYZ) = \sum_{XYZ} \{(D_X \omega)(Y, Z) + \omega T(XY, Z)\}.
\]

The following result holds true:

**Proposition 1.3.** If \( \omega \) is a symplectic structure, i.e. \( d\omega = 0 \), then there exist linear connections \( \nabla \), which are \( \omega \)-compatible \((\nabla = 0)\) and torsion free \((T = 0)\).

Starting from the above relations and Proposition 1.3, we infer:

**Proposition 1.4.** If \( \omega \) is a symplectic structure, then there exist at least one class, \( C_{\nabla} \) in the partition \((1.4)\).

**Proposition 1.5.** If a linear connection, \( D \), is \( \omega \)-compatible, then \( D \) is not \( \omega \)-conjugated with \( \nabla \).

**Proposition 1.6.** If \( D \) is a linear and symmetrical connection \((T = 0)\) then \( D \) is not \( \omega \)-conjugated with \( \nabla \).

In [7] P. Stavre gave a general theory of \( \omega \)-conjugation, based on the operator \( \nabla X \) defined by:

\[
(\nabla X \omega)(YZ) = (\nabla X \omega)(Y, Z) - \omega(X, \tau(Y, Z))
\]
where
\[ (1.7) \quad \tau(XZ) = (2) D_X Z - (1) D_X Z \]
will be called the covariant mixed \( \omega \)-derivative. The already established general relations between \( D_X \omega \) and \( D_X \omega \) allow us to state

**Proposition 1.7.** The necessary and sufficient condition for \( D \sim D \) is:
\[ (1.8) \quad (12) D_X \omega = (21) \alpha(X) \omega, \forall X \in \mathcal{X}(E) \text{ where } \alpha \in \Lambda_1(E). \]

As a corollary we have \( D_X \omega = \alpha(X) \omega \) and the converse, as well. From (1.6), (1.8), it results

**Proposition 1.8.** The set of linear connection transformations induced by \( \omega \)-conjugation relation does not, generally, form a group of transformations.

The cases of subsets of transformations induced by " \( \sim \) " , which form a group of transformations, are already studied. We further present several consequences of the above results.

### 2 Natural almost symplectic structures

The almost symplectic structures and the integrable almost symplectic structures play an important role in the theory of the geometrical models of the Hamiltonian mechanical systems ([2]).

Let us consider a real, \( n \)-dimensional \( C^\infty \) differentiable manifold \( M_n = (M, [A], R^n) \), which is paracompact, connected. Let \( \xi = (E = TM, \omega, M) \) be the tangent bundle and let \( \xi^* = (E^* = T^* M, \pi^*, M) \) be the cotangent bundle. In Hamiltonian mechanics, \( M \) is called the configuration space, \( TM \) is the speed space and \( T^* M \) is the phase space.

Generally speaking, a nondegenerate differential 2 form on a differentiable manifold is an almost symplectic structure. The manifold dimension must be an even number, so that the nondegenerate differential 2-form could exist. For \( \omega \) to be integrable, from topological point of view, the manifold must be orientable. We have \( \dim TM = 2n \) and \( \dim T^* M = 2n \). Moreover, we can state the following result:

**Proposition 2.1.** On the phase space \( T^* M \), there exists an integrable, almost symplectic structure \( \omega \).

Because \( TM \) is paracompact, connected and \( C^\infty \)-differentiable manifold and \( \dim TM = 2n \), there exists a metric structure, \( g \) and an almost complex structure \( F \) \( (F^2 = -I) \).

We define:
\[ (2.1) \quad G(X, Y) = g(X, Y) + g(FX, FY), \quad \forall X, Y \in \mathcal{X}(TM). \]

It results:
\[ (2.2) \quad G(FX, FY) = G(X, Y) \]
hence this represents an almost Hermitian $G$ structure. We define $\omega$ by

\begin{equation}
(2.3) \quad \omega(X, Y) = G(X, FY);
\end{equation}

and we infer $\omega(X, Y) = -\omega(Y, X)$ hence $\omega$ is a nondegenerate, differential 2-form. It will be called natural almost symplectic structure, associated to $(G, F)$. Let us take a natural, almost symplectic structure $\omega$, defined by (2.3). We obtain:

**Proposition 2.2.** We have, for any linear connection $D$ on $E = TM$:

\begin{equation}
(2.4) \quad (D\omega)(Y, Z) = (D_XG)(Y, FZ) + G(Y, (D_XF)(Z)).
\end{equation}

**Definition 2.1.** A linear connection, $D$, on $TM$ with the property $DF = 0$ will be called $F$-connection.

We will denote by $D(F)$ the set of the $F$-connections on $E = TM$. We have ([4]):

\begin{equation}
(2.5) \quad (D\omega)(Y, Z) = (D_XG)(Y, FZ), \quad \forall \ D \in D(F).
\end{equation}

Let us associate the Nijenhuis tensor to the almost complex structure $F$:

\begin{equation}
(2.6) \quad N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y], \quad (F^2 = -I).
\end{equation}

**Proposition 2.3.** We have:

\begin{equation}
(2.7) \quad N_F(X, Y) = (D_{FX}F)(Y) - (D_{FY}F)(X) - F(D_XF)(Y) + F(D_YF)(X)
\end{equation}

\begin{equation}
\quad - \{T(FX, FY) - FT(FX, Y) - FT(X, FY) - T(X, Y)\}.
\end{equation}

**Proposition 2.4.** If there exists a linear torsion-free (symmetrical) connection, $D \in D(F)$, then the almost complex structure $F$ is integrable. So $F$ is a complex structure.

**Proof.** From (2.8) with $T = 0$ and $DF = 0$, it results $N_F = 0$. So $F$ is a complex structure and, therefore, $(G, F)$ is a Hermitian structure. \hfill $\square$

**Proposition 2.5.** If $D \in D(F)$ and $T = 0$ then:

a) The structure $(G, F)$ is a Hermitian structure.

b) We have:

\begin{equation}
(2.8) \quad (d\omega)(X, Y, Z) = \sum_{(XYZ)} (D_XG)(Y, FZ).
\end{equation}

**Definition 2.2.** A linear connection $D$ will be called $\omega$-compatible if we have:

\begin{equation}
(2.9) \quad D_X \omega = 0, \quad \forall \ X.
\end{equation}

It results:

**Proposition 2.6.** If $D$ is $\omega$-compatible and torsion free, then $\omega$ is a symplectic structure.

**Proof.** From (2.8) it follows $d\omega = 0$, so $\omega$ is a symplectic structure. \hfill $\square$

$(G, F, \omega)$ is an almost Kählerian structure.

**Proposition 2.7.** The Levi-Civita connection, $\nabla$, is an $F$-connection if and only if the structure $(G, F, \omega)$ is a Kählerian structure.
3 Natural almost symplectic conjugations and \(\omega\)-compatible models

Any geometric model of an Hamiltonian mechanical system contains a differentiable manifold \(E\), an almost symplectic structure, \(\omega\), and a linear connection, \(D\), on \(E\). So it is a space with linear connection, \((1)\), denoted by \((E, D)\) endowed with an almost symplectic structure \(\omega\). This model will be denoted by \(L = (E, \omega, (1)D)\).

If we change \((1)D\) with \((2)D\) we will obtain another model, \(L = (E, \omega, (2)D)\).

The linear and symmetrical connections \(D\) on \(E\) play an important role in mathematical modelling. There exist such connections. But a linear connection \(D\) of the model can be chosen such that \(D\) is \(\omega\)-compatible, i.e., \(D\omega = 0\). In Lagrange models, based on a Riemannian metric \(G\) and on a relativistic (pseudo-Riemannian) metric \(G\), respectively, there always exists a linear connection \(\nabla\), which is symmetrical and \(G\)-compatible \((\nabla G = 0)\). This is the beforementioned Levi-Civita connection.

**Proposition 3.1.** ([5]). Let us consider \((E, \omega)\), where \(\omega\) is an almost symplectic structure.

a) There exist symmetrical linear connections \(D\).

b) There exist \(\omega\)-compatible, linear connections \(D\).

c) There exist no linear connections \(D\), on \(E\), which have both properties a) and b).

But generally speaking, we have \(d\omega \neq 0\) for almost symplectic structures.

**Example 3.1.** If \(\omega\) is natural, i.e., it is defined by (2.3), then the Levi-Civita connection \(\nabla\), is torsion free: \((T = 0)\) but is not \(\omega\)-compatible, provided that \(\omega\) is not integrable.

Having two models \(L = (E, \omega, (1)D), (2)L = (E, \omega, (2)D)\) of a Hamiltonian mechanical systems, there should exist a natural criterion of comparison (compatibility) for these systems.

**Definition 3.1.** We say that the two models \(L = (E, \omega, (1)D), (2)L = (E, \omega, (2)D)\) are \(\omega\)-compatible if \(D, (2)D\) are \(\omega\)-conjugated. Then we shall write \(L \sim (2)L\) or \((1)D \sim (2)D\).

In [7] the first author gave, for the first time, a general theory of geometries of spaces with \(\omega\)-conjugated linear connections. If we apply this theory to the above case, it will result:

**Proposition 3.2.** Two models \(L, (2)L\) are not \(\omega\)-compatible if \((1)D\) is \(\omega\)-compatible \((D\omega = 0; (2)D\omega = 0)\).

**Proposition 3.3.** Let us consider \(L = (E, \omega, (1)D), (2)L = (E, \omega, (2)D)\) such that \(T = (2)T\). Then \(L, (1)L\) are not \(\omega\)-compatible if \((1)D\) is \(\omega\)-compatible or \((2)D\) is \(\omega\)-compatible.

As a consequence we obtain:
Proposition 3.4. Let us consider two Hamiltonian mechanical models, with the symplectic structure \( \omega \) (\( d\omega = 0 \)). These two models are not \( \omega \)-compatible if \( T^1 = T^2 = 0 \) and \( D^1 \) is \( \omega \)-compatible.

Example 3.2. Let us consider the natural, almost symplectic structure \( \omega \) defined by (2.3). Generally speaking, we have \( d\omega \neq 0 \). So it does not exist a symmetrical, \( \omega \)-compatible connection \( D \), but there exist \( \omega \)-compatible connections \( D \). If we apply Proposition 3.2 and Proposition 3.3 we will obtain:

Proposition 3.5. Let us consider \( L^1 = (E, \omega, D^1) \), \( L^2 = (E, \omega, D^2) \) two models with \( T^1 = 0 \), \( T^2 = 0 \). They can be \( \omega \)-compatible and \( D^1 \), \( D^2 \) are mutually determined.

In the following we will emphasize the cases of the spaces \( TM \), \( T^*M \), endowed with such structures. Let us consider an almost Hermitian structure \( G \), on \( E = TM \) and a natural, almost symplectic structure \( \omega \), which is defined by (2.3). Let us consider the nonlinear connection \( N \), given by the condition ([5]):

\[
G(hX, vY) = 0, \quad \forall \; X, Y \in \mathcal{X}(TM).
\]

Then we have:

\[
T_u E = H_u E \oplus V_u E, \quad \forall \; u \in E \quad (E = TM),
\]

and it results \( G = hG + vG \).

A general theory of the linear connections \( D \overset{\sim}{\approx} D^2 \) (or \( G \)-conjugated connections) was given by P. Stavre in [8]. Afterwards the theory was applied to the total space \( E \) of a vector bundle and, as a particular case, to the case of the total space of the tangent bundle, with numerous applications, when \( G \) is a pseudo-Riemannian one. For \( n = 4 \), the Einstein model (with torsion, which was elaborated by Einstein) is included in the general theory of the \( G \)-conjugated models. In the above conditions we investigated the equivalence between the \( G \)-conjugated models and the natural \( \omega \)-conjugated Hamiltonian models.

If we define, in the same way, the \( G \)-conjugation of the linear connections \( D \), \( D \) ([8]), we obtain:

Proposition 3.6. We have:

a) \( (D^1 \chi \omega)(Y, Z) = (D^1 \chi G)(Y, FZ) + G(Y, (D^1 X F)(Z)) \);

b) \( (D^2 \chi \omega)(Y, Z) = (D^2 \chi G)(Y, FZ) + G(Y, (D^2 X F)(Z)) \).

Using these properties we can solve the important problem of giving the conditions for an equivalence between the \( G \)-conjugation and the \( \omega \)-conjugation. The normal linear \( d \)-connections play an important role in the theory of the tangent bundle.

It is already known that, if \( D \) is a normal linear \( d \)-connection then we have:

\[
D \chi F = 0.
\]

From Proposition 3.6 and (3.2), it results:
Proposition 3.7. In the above conditions we have:

\[ (12) D_X \omega)(Y, Z) = (12) D_X G)(Y, FZ) \text{ if and only if } \hat{D}_X F = 0 \]

\[ (21) D_X \omega)(Y, Z) = (21) D_X G)(Y, FZ) \text{ if and only if } \hat{D}_X F = 0. \]

From the relation (2.3) and from Proposition 3.7 it results:

The equivalence theorem. Let us consider two Hamiltonian models \( L(\omega) = (TM, G, \omega, D) \), \( L(\omega) = (TM, G, \omega, D) \), where \( D, \hat{D} \) are normal, linear \( d \)-connections on \( E = TM \). Let us consider the corresponding \( G \)-models \( L(G) = (TM, G, D) \), \( \hat{L}(G) = (TM, G, \hat{D}) \). We have \( L \sim \hat{L} \) if and only if \( \hat{L} \sim \hat{L} \).

Therefore the relation of \( \omega \)-conjugation is equivalent with the \( G \)-conjugation for normal linear \( d \)-connections. This result is very important for the Lagrange-Hamilton duality and also in applications, especially for the case when \( G \) is defined by a Lagrangian. Some extensions and the curvature invariants of the \( G \)-conjugated, respectively \( \omega \)-conjugated connections have recently been obtained.

References


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