Types of pseudomonotonicity in the study of variational inequalities

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. Pseudomonotonicity is the basic tool in proving the existence of solutions of variational inequalities. There is a variety of algebraic and topological extensions corresponding to the solvability of the forms of generalized variational-like inequalities. An ordering of a part of these types of pseudomonotone set-valued mappings is presented.


Key words: Pseudomonotone operators; set-valued maps; variational inequalities.

Let $X$ be a real Banach space, $X^*$ its dual space and $(\cdot, \cdot): X^* \times X \to \mathbb{R}$ their usual pairing. For a given $f \in X^*$, a monotone-like map $A: X \to X^*$ and $C$ a nonempty closed and convex subset of $X$, the problem of finding an element $x \in C$ such that

\[ \langle Ax - f, y - x \rangle \geq 0, \forall y \in C, \]  

is called a variational inequality (VI). In the last 40 years, variational inequalities have turned out to be appropriate in mathematical modeling. To prove the existence of solutions of VIs in the case of singlevalued operators, Brezis ([1]) and Karamardian ([8]) introduced different concepts of pseudomonotonicity. Throughout the paper, we suppose that $X$ is a reflexive Banach space as infinite-dimensional commodity space, and we replace nets by the usual sequences.

Starting from the monotonicity condition of the operator

\[ \langle Ax - A y, x - y \rangle \geq 0, \forall x, y \in D(A), \]

or $\langle Ax, x - y \rangle \geq \langle A y, x - y \rangle, \forall x, y \in D(A)$, Karamardian defined the following algebraic pseudomonotonicity: $A: D(A) \to X^*$ is said to be $a$-pseudomonotone if

\[ \langle A y, x - y \rangle \geq 0 \quad \text{implies} \quad \langle A x, x - y \rangle \geq 0, \forall x, y \in D(A). \]

Clearly, $a$-pseudomonotonicity is weaker then the monotonicity condition. On the order hand, Brezis ([1]) introduced a topological pseudomonotonicity, namely, $A :
$D(A) \rightarrow X^*$ is $b$-pseudomonotone if for every sequence $\{x_n\} \subset D(A)$ the conditions $x_n \rightharpoonup x$ in $X$ and $\lim \sup (Ax_n, x_n - x) \leq 0$ imply that

$$\langle Ax, x - y \rangle \leq \lim \inf (Ax_n, x_n - y) \text{ for all } y \in D(A).$$

Recall that $A : D(A) \rightarrow X^*$ is hemi-continuous if the function $t \mapsto \langle T(x + ty), z \rangle$ is continuous on $[0,1]$ for all $x,y,z \in D(A)$. Note that, a monotone hemi-continuous operator $A$ is $b$-pseudomonotone.

The above pseudomonotonicities are extended by the following definition: an operator $A : D(A) \rightarrow X^*$ is called $c$-pseudomonotone if for every sequence $\{x_n\} \subset D(A)$ the conditions $x_n \rightharpoonup x$ in $X$ and $\langle Ax_n, (1-t)x+ty-x_n \rangle \geq 0$ for all $t \in [0,1], y \in D(A)$ and $n \in \mathbb{N}$, imply that $\langle Ax, y-x \rangle \geq 0$.

It has been proved in [4] that both any $b$-pseudomonotone and hemi-continuous $a$-pseudomonotone operators are $c$-pseudomonotone. Now, we denote by $\text{Conv} (X^*)$ the collection of all convex closed subsets of $X^*$ and introduce, for a set-valued mapping $T : D(T) \subseteq X \rightarrow \text{ Conv} (X^*)$, the upper and lower support functions by

$$[T(x), y]_+ = \sup_{x^* \in T(x)} \langle x^*, y \rangle \quad \text{and} \quad [T(x), y]_- = \inf_{x^* \in T(x)} \langle x^*, y \rangle,$$

where the upper norm on Conv $(X^*)$ defined by $\|T(x)\|_+ = \sup_{x^* \in T(x)} \|x^*\|_{X^*}$. We distinguish $T(x)$ and $\overline{\text{co}}T(x)$, the minimal closed convex set which contains $T(x)$, and let the graph $G(\overline{\text{co}}T) = \{(x, x^*) \in D(T) \times X^* : x^* \in \overline{\text{co}}T(x)\}$.

Using the above notations, we will be able to present, in a simpler way, three extensions of the pseudomonotonicity proposed by Inoan and Kolumban ([7]) in the case of a set-valued mapping $T : D(T) \subseteq X \rightarrow 2^{X^*}$.

**Definition 1.** The map $T$ is said to be $\alpha$-pseudomonotone if, $[T(y), x - y]_+ \geq 0$ implies $[T(x), x - y]_+ \geq 0$, for every $x,y \in D(T)$.

This definition represents an extension of the algebraic pseudomonotonicity introduced by Karamardian [8].

**Definition 2.** The map $T$ is $\beta$-pseudomonotone if, for every sequence $\{x_n\} \subset D(T)$ the conditions $x_n \rightharpoonup x$ in $X$ and

$$\lim_{n \rightarrow \infty} [T(x_n), x_n - x]_+ \leq 0$$

imply that to each $y \in D(T)$ there corresponds $y^* \in T(x)$ such that

$$\langle y^*, x - y \rangle \leq \lim_{n \rightarrow \infty} [T(x_n), x_n - y]_-.$$

This definition extends the topological pseudomonotonicity considered by Brezis [1].

**Definition 3.** The map $T$ is $\gamma$-pseudomonotone if, for every $x,y \in D(T)$ and every sequence $\{x_n\} \subset D(A)$, the conditions $x_n \rightharpoonup x$ in $X$ and

$$[T(x_n), x_n - (1-t)x + ty]_- \leq 0, \forall t \in [0,1], \forall n \in \mathbb{N},$$

imply that $[T(x), y - x]_+ \geq 0$. 
Likewise, it has proved in [7] that both \(\alpha\)-pseudomonotone and \(\beta\)-pseudomonotone mappings are \(\gamma\)-pseudomonotone, for certain classes of multivalued mappings.

We are in position to introduce a general definition of pseudomonotonicity useful to the solution of variational inequalities. Let \(C\) be a closed convex set of a real reflexive Banach space \(X\).

**Definition 4.** The mapping \(T : C \to 2^{X^*}\) is said to be pseudomonotone if the following conditions hold:

(i) For each \(x \in C\), the image \(T(x)\) is nonempty bounded closed and convex subset in \(X^*\);

(ii) For each finite-dimensional subspace \(F\) of \(X\), the map \(T\) is upper semicontinuous from \(C \cap F\) into \(2^{X^*}\), with \(X^*\) given its weak topology;

(iii) If \(\{x_n\}\) is a sequence in \(C\) converging weakly to \(x \in C\) and \(\lim_{n \to \infty} \|T(x_n) - T(x)\| \leq 0\), to each \(y \in C\) there corresponds an element \(y^* \in T(x)\), with the property that

\[
\lim_{n \to \infty} \langle T(x_n), y - x_n \rangle \leq \langle y^*, y - x \rangle.
\]

Of course, the condition (iii) can be extended by the \(\gamma\)-pseudomonotonicity of the mapping \(T\).

Moreover, Hess and Browder ([3]) introduced a more general concept:

**Definition 5.** The mapping \(T : C \to 2^{X^*}\) is said to be generalized pseudomonotone if it verifies (i) - (ii) while (iii) will be replaced by

(iii') for any sequence \(\{(x_n, x_n^*)\} \subset \text{co}(T)\) such that \((x_n, x_n^*) \to (x, x^*)\) in \(X \times X^*\) and

\[
\lim_{n \to \infty} \langle x_n^*, x - x_n \rangle \geq 0,
\]

it follows that \(x^* \in \text{co}(T(x))\) and \(\langle x_n^*, x \rangle \to \langle x^*, x \rangle\).

It is easy to check that every pseudomonotone mapping is generalized pseudomonotone and a bounded generalized pseudomonotone mapping is pseudomonotone [11]. Along with the initial problem (1.1), we will investigate the multivalued variational inequality (MVI), i.e., finding an element \(u \in C\) such that

\[
\langle T(u), y - u \rangle \geq 0 \quad \forall y \in C,
\]

where \(C \subset D(T)\) be a nonempty closed convex subset of a reflexive Banach space \(X\), (see, e.g. [14]).

In proving the existence of solutions (VMI)'s we are based on the following:

**Theorem 1.** (The Fixed Point Theorem [2]) Let \(K\) be a nonempty compact and convex set in a locally convex space \(X\) and \(S : K \to 2^K\) a multivalued mapping with the properties that \(S(x)\) is a nonempty convex subset in \(X^*\) for all \(x \in K\) and the preimages \(S^{-1}(y)\) are relatively open with respect to \(K\) for all \(y \in K\). Then \(S\) has a fixed point.

Now we consider an auxiliary multivalued variational inequality for a mapping \(T : K \to 2^{X^*}\). Find a pair such that

\[
\langle g, v - u \rangle \geq 0 \quad \text{for all} \quad v \in K.
\]

As a consequence of Browder's theorem, we extend a result from [6] to multivalued mappings:
Theorem 2. Assume that the mapping $T : K \to 2^{X^*}$ is locally bounded, upper semi-continuous and generalized pseudomonotone. Then the variational inequality (1.2) has a solution $(u, g) \in K \times T(u)$.

Proof. In the contrary case, to each $h \in T(u)$ would correspond an element $w \in K$ such that

$$\langle h, w - u \rangle < 0.$$  

Define the multivalued map $S : K \to 2^K$ by

$$S(u) := \{ w \in K : \langle h, w - u \rangle < 0 \}.$$  

According to assumption (1.3), the set $S(u)$ is nonempty for all $u \in K$. Moreover, $S(u)$ is convex. We show that its preimage

$$S^{-1}(w) := \{ u \in K : \langle h, w - u \rangle < 0 \}$$

is relatively open in $K$. Let $\{u_n\}$ be a sequence in $K - S^{-1}(w)$ with $u_n \to z$ and $h_n \in T(u_n)$, so that

$$\langle h_n, w - u_n \rangle \geq 0 \quad \text{or} \quad \langle h_n, u_n - w \rangle \leq 0, \; \forall n \in \mathbb{N}.$$  

From the local boundedness of $T$, we can assume that $h_n \to h$ in $X^*$. The generalized pseudomonotonicity of $T$ implies that $h \in T(z)$. Thus, $K - S^{-1}(w)$ is relatively closed and $S^{-1}(w)$ is relatively open in $K$.

In virtue of Browder’s Theorem, there exists a fixed point $u \in S(u)$. This leads to the contradiction $\langle h, u - u \rangle < 0$. Hence, there exists a $g \in T(u)$ with $u \in K$, satisfying of the variational inequality (1.2).

Similar to [3, Proposition 16], using a finite-dimensional procedure, we can replace the compactness condition on $C$ by the coerciveness of $T$. Considering a reflexive Banach space $X$ and a closed convex subset $C$ of $X$, a multivalued map $T : C \to 2^{X^*}$ is said to be coercive if there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$, such that $\langle f, x \rangle \geq c(||x||) ||x||$ for all $x \in C$ and $f \in T(x)$. In a manner similar to [3, Proof of Theorem 15], we establish:

Theorem 3. Let $C$ be a nonempty closed convex subset of a real reflexive Banach space $X$. Assume that $0 \in C$ and $T : C \to 2^{X^*}$ is a coercive generalized pseudomonotone mapping. Then for each $f \in X^*$, there exists a solution $u \in C$ of the multivariational inequality

$$[T(u), y - u]_+ \geq \langle f, y - u \rangle, \; y \in C.$$  

Related to the condition (iii) in Definition 4, a subclass of multivalued monotone maps has been introduced by:

Definition 6. [12]. The mapping $T : C \to 2^{X^*}$ is of type (S) if it satisfies (i) - (ii) while (iii) will be replaced by

(iii") If $\{x_n\}$ is a sequence in $C$ converging weakly to $x \in X$ such that

$$\lim \sup \langle f_n, x_n - x \rangle \leq 0 \quad \text{for some} \; f_n \in Tx_n,$$

it follows that $x_n \longrightarrow x \in C$ and $\{f_n\}$ has a subsequence which converges weakly to $f \in Tx$ in $X^*$.
Using the above notations, we have also:

**Definition 7.** [7]. The mapping \( T : C \to \text{Conv}(X^*) \) is of type \((\alpha)\) if for any sequence \( \{x_n\} \subset C \) converging weakly to \( x \in X \) such that

\[
\lim_{n \to \infty} [T(x_n), x_n - x]_+ \leq 0,
\]

there exists a subsequence \( \{x_m\} \subset \{x_n\} \) such that \( x_m \to x \) in \( C \).

The alikeness between the mappings of type \((S)\) and of type \((\alpha)\) are discussed in Skrypnik’s book [13]. The mappings of type \((S)\) are a good substitute of compactness in proving the existence of solutions of the nonlinear problems, in particular, of the general variational inequalities.

The similarity with compact operators directed the first author ([5]) to introduce a topological degree for the multivalued mappings of type \((S)\). On the other hand, the variational inequalities can be converted to inclusions defined by sums between mappings of monotone-like and a subdifferentials. The second author ([10]) surveyed the actual progress in defining an appropriate topological degree for operators \((S)\) with maximal monotone perturbations.

**References**


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