A characterization of $H$-strictly convex hypersurfaces in hyperbolic space by the Ricci curvatures

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In previous papers the first author obtained some upper bound estimations for the Ricci curvatures of the hypersurfaces in a sphere ([6]) and in a hyperbolic manifold ([4], [5]) by the extremum principle. In the present paper, we introduce an $H$-strictly convex hypersurface in the hyperbolic space $N^{(n+1)}$ and using a result given by B.Y.Chen in [2] we give a lower bound approximation for the Ricci curvature of a $H$-strictly convex hypersurface in $N^{(n+1)}$.

Key words: Ricci curvature; $H$-convex hypersurface; hyperbolic space.

1 Introduction

Let $M$ be an $n$-dimensional oriented hypersurface in an $(n+1)$-dimensional Riemannian manifold $N$, that is, $M$ is an $n$-dimensional Riemannian submanifold embedded in $N$ with the induced Riemannian structure. We will denote by $g$ resp. ($\tilde{g}$) the Riemannian metric tensor of $M$ (resp. $N$). Let $T_xM^\perp$ be the normal space to $M$ at $x$. We denote by $\nabla$ (resp.$\tilde{\nabla}$) the covariant differentiation on $M$ (resp. $N$). Then, for tangent vector fields $X, Y$ and the unit normal field $\zeta$ on $M$, as is well known, the formulas of Gauss and Weingarten are

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)
\end{equation}

\begin{equation}
\tilde{\nabla}_X \zeta = -A_\zeta(X)
\end{equation}

where $\sigma$ is the second fundamental form of $M$ and satisfies $\sigma(X,Y) = \sigma(Y,X)$ and $A_\zeta$ is the symmetric linear transformation on each tangent space to $M$, which is called the shape operator. Since $M$ is a hypersurface we may write

\begin{equation}
\sigma(X,Y) = h(X,Y)\zeta.
\end{equation}
Then we easily see that

\[(1.4) \quad h(X, Y) = \tilde{g}(\sigma(X, Y), \zeta) = g(A_\zeta(X), Y)\]

The eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\) of the shape operator \(A_\zeta\) are called \textit{principal curvatures} of \(M\), and the vectors of an orthonormal basis \(\{e_1, e_2, \ldots, e_n\}\) such that

\[A_\zeta e_i = \lambda_i e_i\]

are called \textit{principal vectors} on \(M\). In this case,

\[\lambda_i = h(e_i, e_i), \quad i = 1, 2, \ldots, n,\]

Furthermore, the mean curvature vector of the hypersurface \(M\) is defined by \(H = \overline{\text{trace}}\sigma\) and \(K_n = \lambda_1 \lambda_2 \ldots \lambda_n\) is called the \textit{Gaussian curvature} of \(M\).

The second fundamental form \(\sigma\) is said to be semidefinite at \(x \in M\) if \(\sigma(X, X) \geq 0\) or \(\sigma(X, X) \leq 0\) for all nonzero vectors \(X \in T_x M\), that is, \(h\) is either positive semidefinite or negative semidefinite. It is well known that if \(M\) is convex at \(x \in M\), then the \(h\) is semidefinite at the point \(x\). The second fundamental form \(\sigma\) is said to be definite at \(x \in M\) if \(\sigma(X, X) \neq 0\) for all nonzero vectors \(X \in T_x M\), that is, \(h\) is either positive definite or negative definite. In this case the hypersurface \(M\) is said to be strictly convex at the point \(x\). \(\sigma\) is said to be non-degenerate at \(x\) if \(h\) is non-degenerate at \(x\). B.Y. Chen defined a convex submanifold in a Riemannian space form in [2], also see [7], [8] and [6].

**Definition 1.1.** A Riemannian submanifold is said to be \(H\)-Strictly convex submanifold if the shape operator \(A_H\) is positive definite at each point of the submanifold.

Denote by \(R\) the Riemannian curvature tensor of \(M\). Then the equation of Gauss is given by

\[(1.5) \quad R(X, Y; Z, W) = c(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W))\]

for vector fields \(X, Y, Z, W\) tangent to \(M\), where \(c\) is the constant sectional curvature of the Riemannian space form \(N\). For the hypersurface \(M\), denote by \(K(\pi)\) the sectional curvature of a 2-plane section \(\pi \subset T_x M\), \(x \in M\) and choose an orthonormal basis \(e_1, e_2, \ldots, e_n\) of \(T_x M\) such that \(e_1 = X\), then we may define the \textit{Ricci curvature} of \(T_x M\) at \(x\) by

\[(1.6) \quad \text{Ric}(X) = \sum_{j=2}^{n} K_{ij}\]

where \(K_{ij}\) denotes the sectional curvature of the 2-plane section spanned by \(e_i, e_j\). The \textit{scalar curvature} \(\tau\) of the hypersurface \(M\) is defined by

\[(1.7) \quad \tau(M) = \sum_{1 \leq i \leq j \leq n} K_{ij},\]
2 Ricci curvature of the $H$-convex hypersurface in hyperbolic space

Now, following [2] by B.Y.Chen, we may introduce a Riemannian invariant on the hypersurface $M$ of a $(n+1)$-dimensional hyperbolic space $N$ of constant sectional curvature $-1$ defined by

$$\theta_n(x) = \frac{1}{n-1} \inf \text{Ric}(X), (X \in T_x M, \|X\| = 1, x \in M).$$

We will prove the following theorem for the hypersurfaces in a hyperbolic space

**Theorem 2.1.** Let $M$ be a hypersurface of a $(n+1)$-dimensional hyperbolic space $N$, then for any point $x \in N$, we have

1. If $\theta_n(x) \neq -1$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}(\theta_n(x) + 1)I, x \in M. \tag{2.2}$$

2. If $\theta_n(x) = -1$, then $A_H \geq 0$ at $x$.

**Proof.** Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. Considering (1.6), (1.7) and (2.1) we have

$$\tau(x) \geq \frac{n(n-1)}{2} \theta_n(x). \tag{2.3}$$

Then by using the Lemma 1 in [1] we get

$$H^2(x) \geq \frac{2}{n(n-1)} \tau(x) + 1. \tag{2.4}$$

Now, from (2.3) and (2.4) we obtain $H^2(x) \geq \theta_n(x) + 1$. This shows that only when $\theta_n(x) \leq -1, H(x) = 0$ and in this case 1) and 2) is clearly satisfied, so we may assume that $H(x) \neq 0$. Choose an orthonormal basis $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ at $x$ such that $e_{n+1}$ is in the direction of the mean curvature vector $H(x)$ and $\{e_1, e_2, \ldots, e_n\}$ diagonalize the shape operator $A_H$. Then we have

$$A_H = \begin{bmatrix}
    a_1 & 0 & \cdots & 0 \\
    0 & a_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_n
\end{bmatrix}. \tag{2.5}$$

From the equation of Gauss we get

$$a_i a_j = K_{ij} + 1. \tag{2.6}$$

and from (2.6) we obtain

$$a_1(a_2 + \ldots + a_n) = \text{Ric}(e_1) + (n - 1). \tag{2.7}$$
if we take $e_1 = X$ in (2.6), taking into account (2.1) we may write

\begin{equation}
(2.8) \quad a_1(a_2 + \ldots + a_n) \geq (n - 1)(\theta_n(x) + 1) + a_1^2.
\end{equation}

In a similar way we get the following equalities for any $j = 1, 2, \ldots, n$,

\begin{equation}
(2.9) \quad a_j(a_1 + \ldots + a_{j-1} + a_{j+1} + \ldots + a_n) \geq (n - 1)(\theta_n(x) + 1) + a_j^2,
\end{equation}

which yields

\begin{equation}
(2.10) \quad A_H \geq \frac{n - 1}{n}(\theta_n(x) + 1)I.
\end{equation}

Here the equality case only occurs when one of the vectors $e_1, e_2, \ldots, e_n$ is in the null space, but for the hypersurfaces this is impossible so the inequality (2.10) must be sharp, that is

\begin{equation}
(2.11) \quad A_H > \frac{n - 1}{n}(\theta_n(x) + 1)I.
\end{equation}

From Theorem 2.1. we get the following

**Corollary 2.1.** Let $M$ be a hypersurface of an $(n + 1)$-dimensional hyperbolic space $N$ of constant sectional curvature $-1$, if the Ricci curvature of $M$ is positive then $M$ is a $H$-strictly convex hypersurface immersed in $N$.

In the previous paper, [3], the first author proved the following theorem.

**Theorem 2.2.** Let $M$ be a complete hypersurface in the hyperbolic space $N^{n+1}$, such that all sectional curvatures of $M$ are bounded away from $-\infty$. If the largest geodesic ball which is contained in $M$ has the radius $\rho$, then we have

\begin{equation}
(2.12) \quad \lim \inf_{X \in T_xM, \|X\|=1} x \in M \text{Ric}(X, X) \leq (1 - n)(1 - \frac{1}{2\sinh(\frac{\rho}{2})} - \sinh(\frac{\rho}{2})),
\end{equation}

where $\rho = 2\arcsin(\frac{1}{2})$.

Now, we would like to calculate a sharp result for the best possible approximation of the minimum Ricci curvature of a $H$-strictly convex hypersurface immersed in $N^{n+1}$. According to the Corollary 2.1 for such an hypersurface the left side of the inequality (2.12) must be positive and so by setting $\sinh(\frac{\rho}{2}) = \frac{1}{2} = y$, we get $(n - 1)(y + \frac{1}{2y} - 1) \geq 0$ and for $n \geq 2$ we must solve the inequality $y + \frac{1}{2y} - 1 \geq 0$. Let us write $y + \frac{1}{2y} = a$, for being satisfied of the inequality, we have $a \geq 1$ and to get the least value of $a$, if we solve the equation $y + \frac{1}{2y} = a$ or $y^2 + ay + \frac{1}{2} = 0$, $y \neq 0$, we find $a \geq \sqrt{2}$ and therefore $a = y + \frac{1}{2y} = \sqrt{2}$ is the minimum value. This proves the following

**Theorem 2.3.** Let $M$ be a $H$-strictly convex hypersurface with constant sectional curvatures of the hyperbolic space $N^{n+1}$. Then for any point $x \in M$ the best possible approximation for the minimum Ricci curvature of $M$ is $(\sqrt{2} - 1)(n - 1)$.
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References


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