

# Other classes of tangent bundles with general natural almost anti-Hermitian structures

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*Dedicated to the 70-th anniversary  
of Professor Constantin Udriste*

**Abstract.** We continue the study initiated by Oproiu in [21], concerning the anti-Hermitian structures of general natural lift type on the tangent bundles. We get the conditions under which these structures are in the eight classes obtained by Ganchev and Borisov in [8]. We complete the characterization of the general natural anti-Kählerian structures on the tangent bundles with necessary and sufficient conditions, then we present some results concerning the remaining classes.

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## 1 Introduction

In the last fifty years, a lot of papers were dedicated to the geometric structures obtained by lifting the metric from the base manifold to the tangent bundle. The first Riemannian metric on the tangent bundle was constructed by Sasaki in [28], but in the most cases, the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold. In the next years, the authors were interested in finding of other lifted structures on the tangent bundles, with quite interesting properties (see [1], [4], [7], [11], [12], [22], [29] - [32]).

The results concerning the natural lifts (see [11] and [12]), allowed Oproiu to introduce on the tangent bundle  $TM$ , a natural almost complex structure  $J$  and a natural metric  $G$ , both of them being obtained as diagonal lifts of the Riemannian metric  $g$  from the base manifold (see [23]). The same author generalized these lifts in [20], introducing the notion of general natural lift on the tangent bundle.

In several recent works, like [2] - [6], [10], [13], [17] - [19], there are studied some new geometric structures on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$ , obtained by considering the natural lifts of  $g$  to  $TM$ .

In [21], Oproiu gave the characterization for the anti-Hermitian structures of general natural lift type on the tangent bundle and obtained some necessary conditions under which these structures are anti-Kählerian.

In the present paper we give the complete characterization of the anti-Kählerian structures of general natural lift type on the tangent bundle and we obtain the conditions under which the general natural almost anti-Hermitian tangent bundles from [21] are in the other classes of anti-Hermitian manifolds (almost complex manifolds with Norden metric), determined by Ganchev and Borisov in [8]. Particularizing the obtained results to the case of the natural diagonal or anti-diagonal lifted structures, we get the examples constructed by Oproiu and Papaghiuc in [25] and [24].

The same authors studied in [26] the Einstein property for the quasi-anti-Kähler manifolds from [25], and in [27] Papaghiuc treated a particular case of [24].

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class  $C^\infty$  (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices  $h, i, j, k, l, m, r$ , being always  $\{1, \dots, n\}$ .

## 2 Preliminary results

Let  $(M, g)$  be a smooth  $n$ -dimensional Riemannian manifold and denote its tangent bundle by  $\tau : TM \rightarrow M$ . The total space  $TM$  has a structure of a  $2n$ -dimensional smooth manifold, induced from the smooth manifold structure of  $M$ . This structure is obtained by using local charts on  $TM$  induced from the usual local charts on  $M$ . If  $(U, \varphi) = (U, x^1, \dots, x^n)$  is a local chart on  $M$ , then the corresponding induced local chart on  $TM$  is  $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ , where the local coordinates  $x^i, y^j$ ,  $i, j = 1, \dots, n$ , are defined as follows. The first  $n$  local coordinates of a tangent vector  $y \in \tau^{-1}(U)$  are the local coordinates in the local chart  $(U, \varphi)$  of its base point, i.e.  $x^i = x^i \circ \tau$ , by an abuse of notation. The last  $n$  local coordinates  $y^j$ ,  $j = 1, \dots, n$ , of  $y \in \tau^{-1}(U)$  are the vector space coordinates of  $y$  with respect to the natural basis in  $T_{\tau(y)}M$  defined by the local chart  $(U, \varphi)$ . Due to this special structure of differentiable manifold for  $TM$ , it is possible to introduce the concept of  $M$ -tensor field on it (see [16]), called by R. Miron and his collaborators *distinguished tensor field* or *d-tensor field*. The algebra of  $d$ -tensor fields on the tangent bundle of a manifold is studied in [14] (see also [3], [15]).

Denote by  $\check{\nabla}$  the Levi Civita connection of the Riemannian metric  $g$  on  $M$ . Then we have the direct sum decomposition

$$(2.1) \quad TTM = VTM \oplus HTM$$

of the tangent bundle to  $TM$  into the vertical distribution  $VTM = \text{Ker } \tau_*$  and the horizontal distribution  $HTM$  defined by  $\check{\nabla}$  (see [32]). The set of vector fields  $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$  on  $\tau^{-1}(U)$  defines a local frame field for  $VTM$  and for  $HTM$  we have the local frame field  $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$ , where  $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma_{0i}^h \frac{\partial}{\partial y^h}$ ,  $\Gamma_{0i}^h = y^k \Gamma_{ki}^h$ , and  $\Gamma_{ki}^h(x)$  are the Christoffel symbols of  $g$ .

The set  $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=1,n}$  defines a local frame on  $TM$ , adapted to the direct sum decomposition (2.1).

Consider the energy density of the tangent vector  $y$  with respect to the Riemannian metric  $g$

$$(2.2) \quad t = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k, \quad y \in \tau^{-1}(U).$$

Obviously, we have  $t \in [0, \infty)$  for all  $y \in TM$ . We shall use the following lemma, which may be proved easily.

**Lemma 2.1.** *If  $n > 1$  and  $u, v$  are smooth functions on  $TM$  such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0,$$

*on the domain of any induced local chart on  $TM$ , then  $u = 0$ ,  $v = 0$ . We used the notation  $g_{0i} = y^h g_{hi}$ .*

Vasile Oproiu introduced in [20] a natural 1-st order almost complex structure on  $TM$ , just like the natural 1-st order lifts of  $g$  to  $TM$  are obtained in [12]:

$$(2.3) \quad \begin{aligned} JX_y^H &= a_1(t)X_y^V + b_1(t)g_{\tau(y)}(X, y)y_y^V + a_4(t)X_y^H + b_4(t)g_{\tau(y)}(X, y)y_y^H, \\ JX_y^V &= a_3(t)X_y^V + b_3(t)g_{\tau(y)}(X, y)y_y^V - a_2(t)X_y^H - b_2(t)g_{\tau(y)}(X, y)y_y^H, \end{aligned}$$

$\forall X \in \mathcal{T}_0^1(TM)$ ,  $\forall y \in TM$ ,  $a_\alpha, b_\alpha$ ,  $\alpha = \overline{1, 4}$  being smooth functions of the energy density,  $t$ .

In the mentioned paper, Oproiu proved the following results.

**Theorem 2.2.** *The natural tensor field  $J$  of type  $(1, 1)$  on  $TM$ , given by (2.3) defines an almost complex structure on  $TM$ , if and only if  $a_4 = -a_3$ ,  $b_4 = -b_3$ , and the coefficients  $a_1, a_2, a_3, b_1, b_2$  and  $b_3$  are related by*

$$(2.4) \quad a_1a_2 = 1 + a_3^2, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.$$

**Theorem 2.3.** *Let  $(M, g)$  be an  $n (> 2)$ -dimensional connected Riemannian manifold. The almost complex structure  $J$  defined by (2.3) on  $TM$  is integrable if and only if  $(M, g)$  has constant sectional curvature  $c$ , and the coefficients  $b_1, b_2, b_3$  are given by:*

$$(2.5) \quad \begin{aligned} b_1 &= \frac{2c^2ta_2^2 + 2cta_1a_2' + a_1a_1' - c + 3ca_3^2}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}, & b_2 &= \frac{2ta_3^2 - 2ta_1'a_2' + ca_2^2 + 2cta_2a_2' + a_1a_2'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}, \\ b_3 &= \frac{a_1a_3' + 2ca_2a_3 + 4cta_2'a_3 - 2cta_2a_3'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}. \end{aligned}$$

Let  $G$  be the 1-st order natural semi-Riemannian metric  $G$ , of signature  $(n, n)$  on  $TM$ , considered by Oproiu in [21]:

$$(2.6) \quad \begin{aligned} G(X_y^H, Y_y^H) &= c_1(t)g_{\tau(y)}(X, Y) + d_1(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^V) &= c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^H) &= c_3(t)g_{\tau(y)}(X, Y) + d_3(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y). \end{aligned}$$

$\forall X, Y \in \mathcal{T}_0^1(TM)$ ,  $\forall y \in TM$ , where  $c_\alpha, d_\alpha$ ,  $\alpha = \overline{1, 3}$  are six smooth functions of the energy density on  $TM$ . The conditions for  $G$  to be nondegenerate are assured if

$$c_1c_2 - c_3^2 \neq 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 \neq 0.$$

The almost anti-Hermitian or Norden metric with respect to the almost complex structure  $J$ , was defined in [8] and [24] as the semi-Riemannian metric  $G$  satisfying

$$(2.7) \quad G(JX, JY) = -G(X, Y), \quad \forall X, Y \in \mathcal{T}_0^1(TM).$$

The study of the conditions under which the metric  $G$  is almost anti-Hermitian with respect to the almost complex structure  $J$ , led in [21] to the result:

**Theorem 2.4.** *The general natural semi-Riemannian metric  $G$  is almost anti-Hermitian with respect to the general natural almost complex structure  $J$  on the tangent bundle  $TM$ , if and only if the coefficients of  $J$  and  $G$  satisfy*

$$(2.8) \quad a_2c_1 + a_1c_2 = 2a_3c_3,$$

$$(2.9) \quad (a_2 + 2tb_2)(c_1 + 2td_1) + (a_1 + 2tb_1)(c_2 + 2td_2) = 2(a_3 + 2tb_3)(c_3 + 2td_3).$$

In the same paper, Oproiu initiated the study of the anti-Kähler structures of general natural lift type on  $TM$ , obtaining some necessary conditions under which the almost anti-Hermitian structures characterized by theorem 2.4 are anti-Kähler.

Since in the case of the almost anti-Hermitian structures of natural diagonal lift type on  $TM$  the coefficients  $a_3, b_3, c_3, d_3$  vanish, the relations (2.8) and (2.9) become simpler, allowing the complete characterization of the natural diagonal anti-Kähler structures on  $TM$  (see [21]), and the characterizations of all the other classes of almost anti-Hermitian structures of natural diagonal lift type on  $TM$  (see [25]).

In the following sections, we shall get some necessary and sufficient conditions under which the general natural almost anti-Hermitian structures on  $TM$ , obtained in [21], are in the eight classes determined by Ganchev and Borisov in [8].

### 3 Anti-Kähler structures of general natural lift type on $TM$

The first class of almost anti-Hermitian structures obtained by Ganchev and Borisov in [8] are the anti-Kähler structures, studied by the same authors in [9].

In this section we shall give the complete characterization of the general natural anti-Kählerian structures on the tangent bundle  $(TM, G)$  of a Riemannian manifold  $(M, g)$ . To this aim, we shall introduce a  $(0, 3)$ -tensor field  $F$  on  $TM$ , depending on the Levi Civita connection  $\nabla$  of  $G$ :

$$(3.1) \quad F(X, Y, Z) = G((\nabla_X J)Y, Z), \quad \forall X, Y, Z \in \mathcal{T}_0^1(TM),$$

with the properties  $F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ)$ .

$(TM, G, J)$  is anti-Kählerian if and only if  $F = 0$ . Since  $\nabla$  is torsion free, the condition  $F = 0$  implies that  $J$  is integrable, hence  $(M, g)$  is a space form.

The components of  $F$  with respect to the local adapted frame  $\{\delta_i, \partial_j\}_{i,j=1,\dots,n}$ , are denoted by  $F$  followed by sequences of  $X$  and  $Y$ , to indicate horizontal or vertical argument on a certain position. For example, we have

$$\begin{aligned} FYYY_{ijk} &= F(\partial_i, \partial_j, \partial_k) = G(\nabla_{\partial_i} J\partial_j, \partial_k) - G(\nabla_{\partial_i} \partial_j, J\partial_k), \\ FXYX_{ijk} &= F(\delta_i, \partial_j, \delta_k) = G(\nabla_{\delta_i} J\partial_j, \delta_k) - G(\nabla_{\delta_i} \partial_j, J\delta_k). \end{aligned}$$

Using the expression of the Levi-Civita connection of the metric  $G$  (see [5]), and the relations (2.3), (2.6), which give the components of  $J$  and  $G$ , we obtain

$$\begin{aligned} FYYY_{ijk} &= (a'_3c_2 - a'_2c_3 + a_2d_3)g_{jk}g_{0i} + (2b_3c_2 - 2b_2c_3 + a_2c'_3 \\ &+ a_2d_3 + 2b_3c'_2t)(g_{ik}g_{0j} + g_{ij}g_{0k}) + (b'_3c_2 - b_3c'_2 - b'_2c_3 + b_2c'_3 + a'_3d_2 \\ &+ 2b_3d_2 - a'_2d_3 + a_2d'_3 + 2b'_3d_2t - 2b'_2d_3t + 2b_2d'_3t)g_{0i}g_{0j}g_{0k}. \end{aligned}$$

The other components,  $FYYX_{ijk} = FYXY_{ikj}$ ,  $FYXX_{ijk}$ ,  $FXXY_{ikj} = FXYX_{ijk}$ ,  $FXY Y_{ijk}$ ,  $FXX X_{ijk}$ , depend on the components  $R_{kij}^h$  of the curvature tensor field of the connection  $\dot{\nabla}$  on  $M$ . If  $(TM, G, J)$  is an anti-Kähler manifold, then  $(M, g)$  must have constant sectional curvature  $c$ , i.e. the components of the curvature tensor field of  $\dot{\nabla}$  must be given by

$$(3.2) \quad R_{kij}^h = c(\delta_i^h g_{kj} - \delta_j^h g_{ki}).$$

Taking (3.2) into account, and using the relations (2.3) and (2.6), we obtain

$$\begin{aligned} FYXX_{ijk} &= \frac{1}{2}(2b_1c_3 - 2b_3c_1 + a_1c'_3 + a_1d_3 + 2b_3cc_2t + 2b_1c'_3t - 2b_3d_1t \\ &+ 2b_1d_3t)(g_{ik}g_{0j} + g_{ij}g_{0k}) + (a'_1c_3 - a'_3c_1 + a_1d_3)g_{jk}g_{0i} + (b'_1c_3 - b'_3c_1 - b_3cc_2 \\ &- a'_3d_1 - b_3d_1 + a'_1d_3 + 3b_1d_3 + a_1d'_3 - 2b'_3d_1t + 2b'_1d_3t + 2b_1d'_3t)g_{0j}g_{0k}g_{0i}. \end{aligned}$$

Multiplying the above relation by  $y^i$ ,  $y^j$ , and  $y^k$ , successively, and taking into account that  $g_{0i}y^i = 2t$ , it follows, by using lemma 2.1, that  $FYXX_{ijk} = 0$  is equivalent to the vanishing condition for all its coefficients. From the coefficient of  $g_{jk}g_{0i}$ , we obtain that  $d_3$  has the expression:

$$(3.3) \quad d_3 = \frac{a'_3c_1 - a'_1c_3}{a_1}.$$

The final expression of  $FXY Y_{ijk}$  is

$$\begin{aligned} FXY Y_{ijk} &= -a_2(cc_2 - d_1)g_{jk}g_{0i} + \frac{1}{2}(a_2c'_1 + a_2cc_2 + a_2d_1 \\ &+ 2b_2cc_2t + 2b_3c'_3t + 2b_2d_1t - 2b_3d_3t)(g_{ik}g_{0j} + g_{ji}g_{0k}) \\ &+ (b_2c'_1 - b_2cc_2 - b_3c'_3 + b_2d_1 + a_2d'_1 + b_3d_3 + 2b_2d'_1t)g_{0i}g_{0j}g_{0k}. \end{aligned}$$

Multiplying the relation  $FXY Y_{ijk} = 0$ , by  $y^i$ ,  $y^j$ ,  $y^k$ , successively, we obtain by using lemma 2.1, a simple expression for  $d_1$ :

$$(3.4) \quad d_1 = cc_2.$$

The final expressions of the components of  $F$ , become quite long after replacing the values of the coefficients obtained from the conditions for  $(TM, G, J)$  to be anti-Hermitian.

From the vanishing conditions of the coefficients of  $g_{ik}g_{0j}$  and  $g_{ij}g_{0k}$  in the final expression of  $FXX Y_{ijk}$ , we obtain the values of  $c'_1$  and  $c'_3$ , which make vanish all the components of  $F$ . Thus we proved the following theorem:

**Theorem 3.1.** *The tangent bundle  $TM$  endowed with the general natural anti-Hermitian structure  $(G, J)$ , is an anti-Kähler manifold, if and only if the almost complex structure  $J$  is integrable, the coefficients  $d_1$ ,  $d_3$  from the definition of  $G$  are given by (3.4) and (3.3), respectively, and  $c'_1$ ,  $c'_3$ , have the values*

$$\begin{aligned} c'_1 &= -2ca_1^2 \frac{2c_3[a_1(a_3+a'_3t)-2a'_1a_3t]-c_1(1+a_3^2)}{a_1^4+4a_1^2(a_3^2-1)ct+4(1+a_3^2)^2c^2t^2} \\ &- 2c(1+a_3^2) \frac{cc_1t[2a_1(1+a_3^2+4a_3a'_3t)-4a'_1(1+a_3^2)t]+2a_1^2t(a'_1c_1-2a_3cc_3t)}{a_1^4+4a_1^2(a_3^2-1)ct+4a_1(1+a_3^2)^2c^2t^2}, \\ c'_3 &= 2ca_1 \frac{c_3\{a_1+a_3[4t(a'_1a_3-a_1a'_3)]-3a_1a_3\}+2a_3c_1(1+a_3^2+2a_3a'_3t)}{a_1^4+4a_1^2(a_3^2-1)ct+4(1+a_3^2)^2c^2t^2} \\ &+ \frac{(a'_3c_1-a'_1c_3)[a_1^4-4(1+a_3^2)^2c^2t^2]-4a_1(1+a_3^2)c[2a'_1a_3c_1+(1+a_3^2)cc_3]t}{a_1^4+4a_1^2(a_3^2-1)ct+4a_1(1+a_3^2)^2c^2t^2}. \end{aligned}$$

If the sectional curvature of the base manifold  $M$  is a positive constant,  $c$ , the anti-Kählerian structure is defined on the whole tangent bundle  $TM$ , and if  $c$  is strictly negative, the anti-Kählerian structure is defined only in the condition

$$(a_1^2 - 2ct)^2 \neq -4a_3^2 ct(a_1^2 + 2ct + a_3^2 ct).$$

In a similar way, we may prove the following theorem, which characterizes the anti-Kählerian tangent bundles of natural diagonal lift type.

**Theorem 3.2.** *The almost anti-Hermitian manifold  $(TM, G, J)$  of diagonal lift type is an anti-Kähler manifold if and only if*

1) *The natural diagonal almost complex structure  $J$  is integrable (i. e. the base manifold  $M$  is of constant sectional curvature,  $c$ , and  $b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}$ ), and*

$$(3.5) \quad c_1' = \frac{2cc_1(a_1 - 2a_1' t)}{a_1(a_1^2 - 2ct)}, \quad d_1 = cc_2 = -c \frac{c_1}{a_1^2}, \quad a_1^2 \neq 2ct.$$

2) *The base manifold has constant sectional curvature  $c$ , and the essential coefficients have the expressions:*

$$a_1 = \sqrt{B + 2ct}, \quad b_1 = 0, \quad c_1 = A(B + 2ct), \quad d_1 = -cA,$$

where  $A$  is a nonzero real constant and  $B$  is a positive constant.

If  $c$  is positive, the anti-Kähler structure from the second case is defined on the all  $TM$ , and if  $c$  is strictly negative, the structure is defined only on the tube  $t < -\frac{B}{2c}$  around the null section in  $TM$ .

#### 4 Conformally anti-Kähler structures of general natural lift type on the tangent bundle

The conformally anti-Kähler structures or  $\omega_1$ -structures on the tangent bundle of a Riemannian manifold  $(M, g)$  are characterized in [8] by the relation

$$(4.1) \quad \begin{aligned} 2nF(X, Y, Z) &= G(X, JY)\Phi(JZ) + G(X, JZ)\Phi(JY) \\ &+ G(X, Y)\Phi(Z) + G(X, Z)\Phi(Y), \quad \forall X, Y, Z \in \mathcal{T}_0^1(TM), \end{aligned}$$

where  $F$  is the usual tensor field of type  $(0,3)$ , given by (3.1), and  $\Phi$  is a 1-form associated with  $F$ , which in the case of the tangent bundles with general natural almost anti-Hermitian structures, is given by

$$(4.2) \quad \begin{aligned} \Phi(X) &= H_{(1)}^{ij} F(\delta_i, \delta_j, X) + H_{(3)}^{ij} F(\delta_i, \partial_j, X) \\ &+ H_{(3)}^{ij} F(\partial_i, \delta_j, X) + H_{(2)}^{ij} F(\partial_i, \partial_j, X), \end{aligned}$$

$\forall X \in \mathcal{T}_0^1(TM), \forall i, j = \overline{1, n}$ ,  $H$  being the inverse matrix of  $G$  (see [5]).

In the following theorem we shall characterize the tangent bundles endowed with conformally anti-Kähler structures of general natural lift type.

**Theorem 4.1.** *The almost anti-Hermitian manifold  $(TM, G, J)$ , is general natural conformally anti-Kähler, if and only if the almost complex structure  $J$  is integrable (the base manifold is of constant sectional curvature  $c$ , and  $b_1, b_2, b_3$  have the expressions (2.5)), and  $d_1, d_3$  are of the forms*

$$(4.3) \quad d_1 = c \frac{2a_1 a_3 c_3 - c_1(1+a_3^2)}{a_1^2}, \quad d_3 = \frac{a'_3 c_1 - a'_1 c_3}{a_1}.$$

*Proof.* The vertical and horizontal components of the 1-form  $\Phi$  are

$$\begin{aligned} \Phi \partial_k &= H_{(1)}^{ij} F X X Y_{ijk} + H_{(3)}^{ij} F X Y Y_{ijk} + H_{(3)}^{ij} F Y X Y_{ijk} + H_{(2)}^{ij} F Y Y Y_{ijk}, \\ \Phi \delta_k &= H_{(1)}^{ij} F X X X_{ijk} + H_{(3)}^{ij} F X Y X_{ijk} + H_{(3)}^{ij} F Y X X_{ijk} + H_{(2)}^{ij} F Y Y X_{ijk}. \end{aligned}$$

If we replace  $X, Y, Z$  by  $\partial_i, \partial_j, \delta_k$ , we obtain from (4.1) the relation

$$(4.4) \quad \begin{aligned} 2n F Y Y X_{ijk} &= [G_{il}^{(2)}(J_1)_k^l - G_{il}^{(3)}(J_3)_k^l][(J_3)_j^m \Phi \partial_m - (J_2)_j^m \Phi \delta_m] \\ &+ [G_{il}^{(2)}(J_3)_j^l - G_{il}^{(3)}(J_2)_j^l][(J_1)_k^m \Phi \partial_m - (J_3)_k^m \Phi \delta_m] + G_{ij}^{(2)} \Phi \delta_k + G_{ik}^{(3)} \Phi \partial_j, \end{aligned}$$

which becomes quite complicated after replacing the components of  $J, G$ , and  $\Phi$ . Multiplying the expression by  $g^{il} g^{jm}$ , and taking into account that the curvature of the base manifold do not depend on the tangential coordinates, we compute the derivative of the final expression with respect to  $y^k$ . Taking the value in  $y = 0$ , we obtain an expression which depends on the components of the curvature of the base manifold, and on those of the Ricci tensor, which leads to the Einstein condition for the base manifold. Using this condition and the first Bianchi identity, we obtain that the base manifold is a space form, i.e. its curvature has the expression (3.2), where  $c$  depends on the values in  $t = 0$  of  $a_1, a'_1, a_3, a'_3, c_1, c_3, b_1, b_3$ .

Using the theorems 2.4, 2.2, and the relation (3.2), the condition (4.4) becomes

$$(4.5) \quad A g_{ij} g_{0k} + \frac{2na_3}{a_1} (a'_1 c_3 - a'_3 c_1 + a_1 d_3) g_{jk} g_{0i} + B g_{ik} g_{0j} + C g_{0k} g_{0i} g_{0j} = 0,$$

where  $A, B, C$  are some functions depending on the coefficients of  $J$  and  $G$ .

Multiplying (4.5) by  $y^k, y^i, y^j$  successively, and using lemma 2.1, we obtain that (4.5) is equivalent to the vanishing condition for all the coefficients involved. Thus,  $d_3$  must have the expression from (4.3), given in the theorem.

Writing the relation (4.1) for  $\delta_i, \partial_j, \partial_k$  instead of  $X, Y, Z$ , respectively, we get the value of  $d_1$ , in a similar way as  $d_3$  was obtained.

Replacing (4.3) into the vanishing condition for the coefficient  $A$ , we get

$$b_3 = \frac{a_1 a'_3 [a_1^2 - 2ct(1 + 3a_3^2)] + 2a_3 c (a_1 - 2a'_1 t)(1 + a_3^2)}{(a_1 - 2a'_1 t)[a_1^2 - 2ct(1 + a_3^2)] - 8a_1 a_3 a'_3 ct^2}.$$

Similarly, from (4.1), written for  $F Y X X_{ijk}$ , we have

$$b_1 = \frac{a_1 a'_1 [a_1^2 - 2ct(1 + a_3^2)] - a_1^2 c [1 - a_3(3a_3 + 4a'_3 t)] + 2c^2 t (1 + a_3^2)^2}{(a_1 - 2a'_1 t)[a_1^2 - 2ct(1 + a_3^2)] - 8a_1 a_3 a'_3 ct^2}.$$

We may easily verify, that the above values and of  $b_1$  and  $b_3$ , and the value of  $b_2$  from (2.4) coincide with those given in the integrability conditions (2.5), in which we replace  $a_2$  from (2.4). Thus the theorem is proved.  $\square$

**Remark 4.1** Particularizing the result from the theorem 4.1 to the diagonal case, we obtain the characterization of the conformally anti-Kähler structures of natural diagonal lift type on  $TM$ , given in [24, theorem 14].

## 5 General natural complex anti-Hermitian structures on $TM$

The complex anti-Hermitian structures, or  $\omega_1 \oplus \omega_2$ - structures on the tangent bundle of a Riemannian manifold  $(M, g)$ , are characterized by the condition

$$(5.1) \quad F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \forall X, Y, Z \in \mathcal{T}_0^1(TM),$$

where  $F$  is the usual tensor field of type  $(0,3)$ , given by (3.1), and  $J$  is the almost complex structure of general natural lifted type, defined by (2.3).

In this section we shall prove the characterization theorem for the complex anti-Hermitian manifolds  $(TM, G, J)$  of general natural lift type.

**Theorem 5.1.** *The general natural anti-Hermitian manifold  $(TM, G, J)$ , is complex anti-Hermitian, if and only if the almost complex structure  $J$  is integrable.*

*Proof.* Since the curvature of the base manifold satisfy the Bianchi identity, the relation (5.1) is verified when we replace  $X, Y, Z$  by  $\partial_i, \partial_j, \partial_k$ , or by  $\delta_i, \delta_j, \delta_k$ , respectively. Thus, the tangent bundle endowed with general natural complex anti-Hermitian structure will be characterized by the following essential relations only:

$$(5.2) \quad F(\partial_i, \partial_j, J\delta_k) + F(\partial_j, \delta_k, J\partial_i) + F(\delta_k, \partial_i, J\partial_j) = 0,$$

$$(5.3) \quad F(\partial_i, \delta_j, J\delta_k) + F(\delta_j, \delta_k, J\partial_i) + F(\delta_k, \partial_i, J\delta_j) = 0.$$

The value in  $y = 0$  of the derivative of (5.2) with respect to the tangential coordinates  $y^k$ , leads to the condition for the base manifold to be a space form. Then, taking (3.2) into account, we may write the relations (5.2) and (5.3) in the forms  $f_1(g_{jk}g_{0i} - g_{ik}g_{0j}) = 0$ ,  $f_2(g_{ik}g_{0j} - g_{ij}g_{0k}) = 0$ , where  $f_1$  and  $f_2$  are rational functions depending on  $a_1, b_1, c_1, a_3, b_3, c_3, a'_1, a'_3$ , on the constant sectional curvature  $c$ , and on the energy density,  $t$ . By using lemma 2.1, we have that  $f_1$  and  $f_2$  must vanish. Taking into account these vanishing conditions, and considering the expression (2.4) of  $b_2$ , we obtain the integrability conditions from theorem 2.3.  $\square$

**Remark 5.1** If in the theorem 5.1 we consider the anti-Hermitian structure  $(G, J)$  of diagonal lift type on the tangent bundle  $TM$ , we obtain the result from [24, theorem 9], which characterize the natural diagonal complex anti-Hermitian tangent bundles.

## 6 Tangent bundles with general natural quasi-anti-Kählerian structures

In this section we shall obtain the conditions under which the tangent bundle  $TM$ , with a general natural anti-Hermitian structure  $(G, J)$  is quasi-anti-Kähler.

The quasi-anti-Kählerian manifolds, or  $\omega_3$ - manifolds, are characterized in [8] by the following vanishing condition:

$$(6.1) \quad F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0, \forall X, Y, Z \in \mathcal{T}_0^1(TM).$$

where  $F$  is the (0,3)-tensor field, given by (3.1).

If in (6.1) we take  $\partial_i, \delta_j, \delta_k$ , instead of  $X, Y, Z$ , respectively, we have

$$F Y X X_{ijk} + F X X Y_{jki} + F X Y X_{kij} = 0,$$

relation which depends on the curvature of the base manifold. Differentiating the final form of the above relation, with respect to the tangential coordinates  $y^h$ , and taking the value in  $y = 0$ , we obtain by standard calculation that the base manifold is a space form, i.e  $R_{kij}^h$  has the form (3.2). Then, the condition

$$F X X X_{ijk} + F X X X_{jki} + F X X X_{kij} = 0,$$

becomes of the form

$$(6.2) \quad (a_1 c'_1 + 2a_1 d_1 + 2b_1 c'_1 t + 4b_3 c c_3 t)(g_{jk} g_{0i} + g_{ik} g_{0j} + g_{ij} g_{0k}) \\ - 3(2b_3 c c_3 - 2b_1 d_1 - a_1 d'_1 - 2b_1 d'_1 t) g_{0i} g_{0j} g_{0k} = 0.$$

Using lemma 2.1, we obtain from (6.2) the expressions of  $c'_1$  and  $d'_1$ :

$$(6.3) \quad c'_1 = -\frac{2(a_1 d_1 + 2b_3 c c_3 t)}{a_1 + 2b_1 t}, \quad d'_1 = \frac{2(b_3 c c_3 - b_1 d_1)}{a_1 + 2b_1 t}.$$

Replacing (6.3) into the other components of the sum (6.1), we obtain by similar computations, the values of  $a'_1, a'_3, c'_3, d'_3$ , and we may formulate:

**Theorem 6.1.** *The tangent bundle  $TM$  endowed with the almost anti-Hermitian structure  $(G, J)$  of general natural lift type, is a quasi-anti-Kähler manifold, if and only if the base manifold is of constant sectional curvature  $c$ , the first order derivatives of  $c_1$  and  $d_1$  have the expressions (6.3), and the other coefficients verify*

$$a'_1 = a_1 \frac{b_1 c_1 (1 + a_3^2 + 2a_3 a'_3 t) + 2(1 + a_3^2) c c_3 (a_3 - b_3 t)}{[c_1 (1 + a_3^2) - a_1 a_3 c_3] (a_1 + 2b_1 t)} - \frac{(1 + a_3^2)^2 c c_1 + a_1^3 [(a'_3 - b_3) c_3 + a_3 c'_3]}{[c_1 (1 + a_3^2) - a_1 a_3 c_3] (a_1 + 2b_1 t)} \\ - a_1^2 \frac{2d_1 + a_3 (2b_1 c_3 + 2a_3 d_1 - a'_3 c_1) + 2b_1 (a'_3 c_3 + a_3 c'_3) t}{[c_1 (1 + a_3^2) - a_1 a_3 c_3] (a_1 + 2b_1 t)}, \\ a'_3 = -a_1 (1 + a_3^2) \frac{b_3 (c_1^2 + 2c c_3^2 t + 4c_1 d_1 t) + 2c_1 [a_3 d_1 - b_1 (c_3 + 2d_3 t)]}{[(1 + a_3^2) c_1^2 - a_1 c_3 (2a_3 c_1 + a_1 c_3)] (a_1 + 2b_1 t)} \\ + 2a_1^2 \frac{c_1 d_3 - c_3 d_1 + a_3^2 (c_3 d_1 + c_1 d_3) - a_3 c_3 [b_1 (c_3 + 2d_3 t) - 2b_3 d_1 t]}{[(1 + a_3^2) c_1^2 - a_1 c_3 (2a_3 c_1 + a_1 c_3)] (a_1 + 2b_1 t)} \\ + \frac{a_1^4 c_3 (b_3 c_3 - 2a_3 d_3) - 2(1 + a_3^2) b_3 c c_1 t [c_1 (1 + a_3^2) + 2a_1 a_3 c_3]}{a_1 [(1 + a_3^2) c_1^2 - a_1 c_3 (2a_3 c_1 + a_1 c_3)] (a_1 + 2b_1 t)}, \\ c'_3 = \frac{2(1 + a_3^2) b_3 c c_1 t}{a_1^2 (a_1 + 2b_1 t)} + \frac{a_1 (b_3 c_1 - b_1 c_3 - 2a_3 d_1) + c c_3 (1 + a_3^2 - 4a_3 b_3 t)}{a_1 (a_1 + 2b_1 t)},$$

and  $d'_3$  has a longer and more complicated expression of the same type.

The result is true only when the non-vanishing conditions for the denominators of  $a'_1$  and  $a'_3$  is satisfied.

**Remark 6.1** Considering the case of the natural diagonal quasi-anti-Kähler structures on the tangent bundle, from the above theorem we obtain the characterization given in [24, theorem 8].

## 7 Tangent bundles with semi-anti-Kähler structures of general natural lift type

The semi-anti-Kähler manifolds, or  $\omega_2 \oplus \omega_3$ - manifolds, are defined in [8] as being the manifolds for which the 1-form  $\Phi$ , given by (4.2), vanishes:

$$(7.1) \quad \Phi = 0.$$

We shall prove the following proposition, in which we give some necessary conditions for the anti-Hermitian manifold  $(TM, G, J)$  of general natural lift type to be semi-anti-Kähler.

**Proposition 7.1.** *If the general natural almost anti-Hermitian structure  $(G, J)$  on the tangent bundle  $TM$  is semi-anti-Kähler, then the base manifold is Einstein, and the first order derivative of the coefficient  $c_3$  from the definition of  $G$  is given by*

$$c'_3 = \frac{2a'_1(1+a_3^2)b_3c_1^2t - a_1^2c_3\{c'_1(1-a_3^2) - 2a'_3b_3c_1t + a_3[a'_3c_1 + a'_1c_3 - 2b_3(c_1+c'_1t)] - a_1c_3(a'_3-b_3)\}}{a_1^2[a_1c_3(a_3+2b_3t) - c_1(1+a_3^2+2a_3b_3t)]} \\ - \frac{c_1(1+a_3^2)(b_3c_1+a_3c'_1 - a'_1c_3) + 2b_3(c'_1+a_3(a'_3c_1+a_3c'_1+a'_1c_3))t}{a_1c_3(a_3+2b_3t) - c_1(1+a_3^2+2a_3b_3t)},$$

when  $a_1c_3(a_3 + 2b_3t) \neq c_1(1 + a_3^2 + 2a_3b_3t)$ .

*Proof.* The tangent bundle  $TM$ , endowed with the general natural almost anti-Hermitian structure  $(G, J)$  is a semi-anti-Kähler manifold if and only if the tensor  $\Phi$  vanishes, i. e. if and only if  $\Phi\delta_k = \Phi\partial_k = 0$ . The final expression of  $\Phi\delta_k$ , is

$$(7.2) \quad \Phi\delta_k = f(t)g_{0k} - \frac{1+a_3^2}{a_1}y^h Ric_{hk},$$

where  $f$  is a rational function depending on the coefficients of  $G$  and  $J$ , on the energy density,  $t$ , and on the dimension  $n$  of  $M$ . Since the Ricci tensor of the base manifold,  $Ric$  do not depend on the tangential coordinates, the value in  $y = 0$  for the derivative of  $\Phi\delta_k$  with respect to  $y^h$  leads to the Einstein condition for the base manifold:

$$Ric_{hk} = \rho g_{hk},$$

where  $\rho = f(0)\frac{a_1(0)}{1+a_3(0)^2}$ .

The vertical component,  $\Phi\partial_k$  has the form  $(u(t) + nv(t))g_{0k} = 0$ ,  $u$  and  $v$  being two rational functions depending on the same parameters as  $f$ .

Since  $\Phi\partial_k$  must vanish for every dimension  $n$  of the base manifold, we have that  $u = v = 0$ , and solving the equation  $v = 0$  with respect to  $c'_3$ , we obtain the expression from the proposition.  $\square$

**Remark 7.1** The sufficient conditions for the general natural anti-Hermitian manifold  $(TM, G, J)$  to be semi-anti-Kähler, are given by the Einstein property of the base manifold, by the value of  $c'_3$ , and by other two more complicated relations between the coefficients of  $G$  and  $J$ , which may not be presented here.

**Remark 7.2** If the base manifold is Ricci flat and the general natural almost anti-Hermitian manifold  $(TM, G, J)$  is semi-anti-Kähler, then  $c'_3$  has the expression from proposition 7.1, and when  $c_1(1 + a_3^2) \neq a_1a_3c_3$ ,  $a'_1$  has the form:

$$a'_1 = a_1^2 \frac{a_3[c_1(a'_3-b_3) - b_1c_3] + c'_1 - 2a'_3b_1c_3t + c_3(b_3-a'_3)}{[c_1(1+a_3^2) - a_1a_3c_3](a_1+2b_1t)} + b_1 \frac{2c'_1t + c_1(1+a_3^2+2a_3a'_3t)}{[c_1(1+a_3^2) - a_1a_3c_3](a_1+2b_1t)}.$$

**Remark 7.3** For the natural diagonal semi-anti-Kähler structure  $(G, J)$  on  $TM$ , the expressions which we have to study become simpler, and the relation between the coefficients of  $G$  and  $J$  is that obtained in [24, theorem 10], which has an interesting consequence, furnishing a simple example of semi-anti-Kähler structure, presented in the same paper.

## 8 Special complex anti-Hermitian structures of general natural lift type on the tangent bundle

The complex anti-Hermitian manifolds which are at the same time semi-anti-Kählerian manifolds are called in [8], special complex anti-Hermitian manifolds, or  $\omega_2$ -manifolds.

The special complex anti-Hermitian manifolds  $(TM, G, J)$ , of general natural lift type are characterized by the conditions

$$\Phi = 0, \quad F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \quad \forall X, Y, Z \in \mathcal{T}_0^1(TM).$$

where  $F$  is the usual tensor field of type  $(0,3)$ , defined by (3.1),  $\Phi$  is the 1-form associated with  $F$ , given by (4.2), and  $J$  is the almost complex structure of general natural lifted type with the expression (2.3).

In this section we shall find some necessary conditions under which the tangent bundle  $TM$  endowed with a general natural anti-Hermitian structure is special complex anti-Hermitian.

**Proposition 8.1.** *If the anti-Hermitian manifold  $(TM, G, J)$  of general natural lift type is special complex anti-Hermitian, then the almost complex structure  $J$  is integrable (i.e. the base manifold is a space form, and  $b_1, b_2, b_3$  have the values (2.5)),  $c'_3$  has the expression from proposition 7.1, and  $c'_1$  is given by*

$$c'_1 = 2c \frac{a_1^3 [2a_1 c_3 (a_3 + a'_3 t) - c_1 (1 + a_3^2) - 4a'_1 a_3 c_3 t] - 4a'_1 (1 + a_3^2)^2 c c_1 t^2}{a_1^4 + 4a_1^3 (a_3^2 - 1) ct + 4a_1 (1 + a_3^2)^2 c^2 t^2} - 4ct \frac{(1 + a_3^2) [c c_1 (1 + a_3^2 + 4a_3 a'_3 t) + a_1 (a'_1 c_1 - 2a'_3 c c_3 t)]}{a_1^4 + 4a_1^3 (a_3^2 - 1) ct + 4(1 + a_3^2)^2 c^2 t^2}.$$

For  $c \geq 0$  the result is always valid, and if  $c > 0$  the coefficients must verify the relation  $[a_1 + 2ct(1 + a_3^2)]^2 \neq 8a_1 ct$ .

*Proof.* Since the special complex anti-Hermitian manifold  $(TM, G, J)$  is complex anti-Hermitian and semi-anti-Kähler, we have from theorem 5.1 that the base manifold  $M$  has constant sectional curvature  $c$ , and from proposition 7.1 we obtain that  $M$  is Einstein, the constant  $\rho$  being in this case equal to  $c(n - 1)$ . Thus we obtain that  $\Phi \delta_k = (\alpha(t) + \beta(t)n)g_{0k} = 0$ , where  $\alpha$  and  $\beta$  are rational functions depending on the energy density and on the coefficients of the metric  $G$  and of the almost complex structure  $J$ . After replacing the value of  $c'_3$  and the expressions (2.5) and (2.4) into the relation  $\beta = 0$ , we obtain the value of  $c'_1$  from the proposition.  $\square$

**Remark 8.1** The sufficient conditions for the general natural anti-Hermitian manifold  $(TM, G, J)$  to be special complex anti-Hermitian, are given by the integrability of the almost complex structure  $J$ , by the expressions of  $c'_1$  and  $c'_3$  from the propositions 8.1, 7.1, and by two complicated relations between the coefficients  $G$  and  $J$ , which may not be presented here.

**Remark 8.2** When the almost anti-Hermitian manifold  $(TM, G, J)$  is of natural diagonal lift type, the sufficient conditions for  $(TM, G, J)$  to be special complex anti-Hermitian become simpler, the complete characterization of this structures being given in [24, theorem 12].

## 9 $\omega_1 \oplus \omega_3$ structures on $TM$

An almost anti-Hermitian manifold  $(TM, G, J)$  is an  $\omega_1 \oplus \omega_3$ - manifold, if it satisfies the relation

$$(9.1) \quad \begin{aligned} & n[F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y)] = \\ & = G(X, Y)\Phi(Z) + G(Z, X)\Phi(Y) + G(Y, Z)\Phi(X) \\ & + G(X, JY)\Phi(JZ) + G(Y, JZ)\Phi(JX) + G(Z, JX)\Phi(JY), \end{aligned}$$

where  $X, Y, Z$  are any vector fields on  $TM$ ,  $F$  is the usual tensor field of type  $(0,3)$ , with the expression (3.1),  $\Phi$  is the associated 1-form, defined by (4.2),  $G$  and  $J$  are the general natural semi-Riemannian metric and almost complex structure, given by (2.6) and (2.3), respectively.

In the following proposition we shall present some necessary conditions for the almost anti-Hermitian manifold  $(TM, G, J)$  of general natural lift type to be an  $\omega_1 \oplus \omega_3$ - manifold.

**Proposition 9.1.** *If the almost anti-Hermitian manifold  $(TM, G, J)$  is an  $\omega_1 \oplus \omega_3$ - manifold of general natural lift type, then the base manifold has constant sectional curvature  $c$ ,*

$$\begin{aligned} a'_1 &= \frac{a_1^3(b_3 - a'_3)c_3 - (1 + a_3^2)cc_1(1 + a_3^2 + 2a_3b_3t)}{[c_1(1 + a_3^2) - a_1a_3c_3](a_1 + 2b_1t)} + a_1^2 \frac{a_3[c_1(a'_3 - b_3) - b_1c_3] - 2(d_1 + a'_3b_1c_3t)}{[c_1(1 + a_3^2) - a_1a_3c_3](a_1 + 2b_1t)} \\ &+ a_1 \frac{b_1c_1(1 + a_3^2 + 2a_3a'_3t) + cc_3[a_3(1 + a_3^2) - 2b_3t(1 - a_3^2)]}{[c_1(1 + a_3^2) - a_1a_3c_3](a_1 + 2b_1t)}, \\ a'_3 &= \frac{a_1^4c_3(b_3c_3 - 2a_3d_3) - 2(1 + a_3^2)b_3cc_1t[(1 + a_3^2)c_1 + 4a_1a_3c_3]}{a_1[(1 + a_3^2)c_1^2 - a_1c_3(2a_3c_1 + a_1c_3)](a_1 + 2b_1t)} \\ &- a_1^2(1 + a_3^2) \frac{b_3(c_1^2 + 2cc_3^2t + 4c_1d_1t) + 2c_1[a_3d_1 - b_1(c_3 + 2d_3t)]}{a_1[(1 + a_3^2)c_1^2 - a_1c_3(2a_3c_1 + a_1c_3)](a_1 + 2b_1t)} \\ &+ 2a_1^3 \frac{c_1d_3 - c_3d_1 + a_3^2(c_3d_1 + c_1d_3) - a_3c_3[b_1(c_3 + 2d_3t) - 2b_3d_1t]}{a_1[(1 + a_3^2)c_1^2 - a_1c_3(2a_3c_1 + a_1c_3)](a_1 + 2b_1t)}, \end{aligned}$$

and other more complicated relations between the coefficients are satisfied, for example the first order derivatives of  $d_1$  and  $d_3$  may be expressed as rational functions of the other coefficients, their derivatives,  $c$  and  $t$ .

The proposition is true only when the denominators do not vanish.

*Proof.* If we write the relation (9.1) for the vector fields  $\partial_i, \partial_j, \partial_k$ , we obtain an expression depending on the tangential coordinates  $y^h$ , on the components  $g_{ij}$  of the metric  $g$ , on the entries  $g^{ij}$  of the inverse matrix of  $g$ , and on the Ricci tensor of the base manifold. From the value in  $y = 0$  of the derivative with respect to  $y^h$  of this expression, we obtain, after the multiplication by  $g^{ik}g^{jl}g^{hm}$ , that the base manifold is Einstein.

Writting (9.1) for the vector fields  $\partial_i, \partial_j, \partial_k$ , we get, by taking into account the Einstein condition for the base manifold, that  $M$  has constant sectional curvature,  $c$ .

From (9.1) written for  $\delta_i, \delta_j, \delta_k$ , we obtain an expression of the form

$$(u_1 + u_2n)(g_{jk}g_{0i} + g_{ik}g_{0j} + g_{ij}g_{0k}) + (v_1 + v_2n)g_{0i}g_{0j}g_{0k} = 0,$$

where  $u_1, u_2, v_1, v_2$  are rational functions depending on the coefficients of  $G$  and  $J$ . Since the expression must vanish for every dimension  $n$  of  $M$ ,  $u_1, u_2, v_1, v_2$  must be zero. From this vanishing conditions and from the similar relations obtained from the other components in (9.1), we obtain the expressions from the proposition.  $\square$

**Remark 9.1** The sufficient conditions under which the anti-Hermitian manifold  $(TM, G, J)$  is a of general natural  $\omega_1 \oplus \omega_3$ - manifold, are given by the condition for the base manifold to have constant sectional curvature,  $c$ , by the expressions of  $a'_1, a'_3$ , from proposition 9.1, and by other more complicated relations between the coefficients of  $G$  and  $J$ , such as some quite long expressions of  $d'_1$  and  $d'_3$ .

**Remark 9.2** In the diagonal case it is easier to present all the necessary and sufficient conditions for  $(TM, G, J)$  to be an  $\omega_1 \oplus \omega_3$ - manifold (see [24, theorem 15] and the consequence, which furnishes a simple example of  $\omega_1 \oplus \omega_3$ - manifold).

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