Notable submanifolds in Berwald-Moór spaces

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. After proving that the Berwald-Moór structures are pseudo-Finsler of Lorentz type, for co-isotropic submanifolds of Berwald-Moór spaces ([4]), the Gauss-Weingarten and Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations of the such submanifolds are explicitly presented and discussed, taking into account the locally-Minkowski character of the Finslerian ambient metric.


Key words: pseudo-Finsler structure, co-isotropic submanifold, induced non-linear connection, induced linear connection, screen distribution.

1 The Berwald-Moór pseudo-Finsler structure

The Berwald-Moór framework was initiated by P.K. Rashevski ([23], [24],[25]) and further fundamented by D. Pavlov, G. Garasko and S. Kokarev ([8], [9], [20], [10], [21], [22]). The main geometric structure is here provided by the structure $\mathcal{H}_n = (\tilde{M} = \mathbb{R}^n, F)$, with

$$F(y) = \begin{cases} \sqrt[\nu]{y^1y^2 \ldots y^n}, & \text{for } n \text{ odd} \\ \sqrt[\nu]{|y^1y^2 \ldots y^n|}, & \text{for } n \text{ even} \end{cases} = \sqrt[\nu]{\varepsilon_y \cdot y^1y^2 \ldots y^n},$$

with $\varepsilon_y = (\text{sign}(y^1 \ldots y^n))^{n+1}$. This is a Finsler metric function of locally Minkowski type, i.e., $F$ does not effectively depend on the points of $\tilde{M}$.

Remarks. The Berwald-Moór structure (as M. Matsumoto remarked in [16]), is peculiar, and has specific features which make it very special. In this respect, we note the following:

- though $F$ is defined and continuous on the whole $TM$, the structure is $C^\infty$ not on the whole slit tangent space $\tilde{TM} = TM \setminus \{0\}$, but on the open subset:

$$\tilde{TM} = \bigcup_{x \in \tilde{M}} T_x\tilde{M} \subset \tilde{TM}, \quad T_x\tilde{M} = \{y \in \mathbb{R}^n \equiv T_xM|y^1 \ldots y^n \neq 0\},$$

i.e. on the intersection of the main open subsets of the slit tangent space which project to the natural open charts of the projectivized space $PTM$;
• for \( n \) even \( F \) is (only) positive homogeneous of first order, while for \( n \) odd, it is (completely) homogeneous;

• in the odd case, \( F \) might have negative values as well.

Regarding the associated Finsler tensor field, we have the following result:

**Proposition 1.1 ([16]).** The fundamental tensor field \( g_{ij} dx^i \otimes dx^j \) of \( H^n \) defined by
\[
g_{ij}(y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}
\]
has the explicit expression
\[
g_{ij} = \frac{F^2}{ny^i y^j} \left( \frac{2}{n} - \delta_{ij} \right).
\]

Its determinant is \( g = \det(g_{ij})_{i,j=1}^{n} = \frac{(-1)^{n+1}}{n^n} \), and its inverse (the dual \((2,0)\)-tensor field \( g^{ij} \) given by \( g^{ij} g_{ij} = \delta^i_j \)) has the components
\[
g^{ij} = \frac{ny^i y^j}{F^2} \left( \frac{2}{n} - \delta^{ij} \right).
\]

The proof is mainly computational, and relies on the properties of the determinant and on the following classical result from linear algebra:

**Lemma 1.2.** If the matrix \([a] = (a_{ij})_{i,j=1}^{\Gamma} \) is invertible, \( b_i \in \mathbb{R}, i \in \Gamma, n \), and \( \varepsilon \in \{\pm 1\} \), then \([\tilde{a}] = (\tilde{a}_{ij} = a_{ij} + \varepsilon b_i b_j)_{i,j=1}^{\Gamma} \) is invertible and

a) \( \det[\tilde{a}] = a_* (1 + B_* \varepsilon) \), where \( a_* = \det[a] \), \( b_* = a^{ij} b_i b_j \), with \( [a]^{-1} = (a^{ij})_{i,j=1}^{\Gamma} \);

b) \( [\tilde{a}]^{-1} \) has the coefficients \( \tilde{a}^{ij} = a^{ij} - \frac{\varepsilon}{1 + \varepsilon b_*} b^i b^j \), where \( b^i = a^{ij} b_j \).

As consequence, the metric tensor \( g_{ij} \) is non-degenerate. Regarding its signature we prove the following

**Theorem 1.3.** The Berwald-Moór metric tensor of \( \mathcal{H}_n \) having the coefficients (1.1) is of Minkowski type, hence provides in each fiber of the tangent bundle a pseudo-Riemannian structure.

**Proof.** We further denote \([g] = (g_{ij})_{i,j=1}^{\Gamma} \). Then the spectrum of \([g] \) is given by
\[
\sigma([g]) = \{ \lambda \in \mathbb{R} | \det([g] - \lambda I_n) = 0 \}.
\]

We note that
\[
[g] = \frac{\rho^{2/n}}{n^2} \begin{pmatrix}
\mu(a_1)^2 & 2a_1 a_2 & \ldots & 2a_1 a_n \\
2a_2 a_1 & \mu(a_2)^2 & \ldots & 2a_2 a_n \\
\vdots & \vdots & \ddots & \vdots \\
2a_n a_1 & 2a_n a_2 & \ldots & \mu(a_n)^2
\end{pmatrix},
\]

where \( \rho = y^1 y^2 \cdots y^n = F^n \), \( \mu = 2 - n \), \( a_k = 1/y^k \). We perform a change of coordinates given by the matrix \( C = \text{diag} \left( \frac{n}{\rho^{1/n}} \sqrt{2} \right) (y^1, \ldots, y^n) \), and then the
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matrix of the bilinear symmetric form changes to

\[
[g]' = C^t [g] C = \begin{pmatrix}
\alpha & 1 & \ldots & 1 \\
1 & \alpha & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & \alpha
\end{pmatrix}, \text{ where } \alpha = \frac{\mu}{2} = 1 - \frac{n}{2}.
\]

Hence \([g]' = U - \frac{n}{2} I_n\), where \(U\) is the \(n \times n\) matrix with all entries equal to 1. Then by Sylvester’s Theorem, \(\sigma([g]) = \sigma([g]')\) and the characteristic polynomial is

\[
P_{[g]'}(\lambda) = \det([g]' - \lambda I_n) = \det \left( U - (\lambda + \frac{n}{2}) I_n \right) = \det(U - \lambda' I_n).
\]

But the eigenvalues \(\lambda'\) of \(U\) are \(n\) (simple eigenvalue), and 0 (with multiplicity \(n - 1\)). Hence \(\lambda = \lambda' - \frac{n}{2} \in \left\{ \frac{n}{2}, -\frac{n}{2}, \ldots, -\frac{n}{2} \right\} = \sigma([g]')\).

We conclude that \([g]\) is not positive definite, its signature is \((+, \ldots, -)\), and the space \((\mathbb{R}^n, [g])\) is of Minkowski type. Hence the Berwald-Moór structure is pseudo-Finslerian of Lorentz type.

There exist numerous notable submanifolds which naturally emerge from the algebraic and geometric richness of the Berwald-Moór structure of \(H_n\). We can mention the indicatrix of \(H_n\), \(\Sigma_{H_n} : y^1 \cdot \ldots \cdot y^n = 1\), which is a Tzitzeica hypersurface (i.e., the ratio between the curvature at a point \(P \in \Sigma_{H_n}\) and the \((n + 1)\)–th power of the distance from origin to the tangent plane to \(\Sigma_{H_n}\) at \(P\), is constant, briefly \(K(P) \cdot d(O, T_P \Sigma_{H_n})^{-(n+1)} = \text{const.},\) see [26]), and whose study is essential, due to the fact that its isometries which commute with homotheties prove to be isometries of \(H_n\). The Tzitzeica hypersurface is tightly related the geometry of angles on \(H_n\) ([21], [22]). As well, it leads to natural PDEs of soliton type. The study of the symmetries of the PDEs which admit \(\Sigma_{H_n}\) as solution produces mappings which preserve the hypersurface, and which may contain subclasses of isometries of \(H_n\).

Another significant submanifold is the implicitly defined simultaneity hypersurface, with relevance in the applications of the Berwald-Moór structures in Relativity theory (see e.g., [15]).

Since the geometry of pseudo-Finsler spaces is a very recent field of research and \(H_n\) is an illustrative example of such space, we shall describe further a specific class of pseudo-Finsler submanifolds (the lightlike, or co-isotropic submanifolds), and point out the differences to the proper Finsler approach, illustrated for the case of particular ambient space \(H_n\).

2 Co-isotropic submanifolds in Berwald-Moór spaces

The pseudo-Riemann lightlike (co-isotropic) submanifolds are intensive subject of recent research, since they produce mathematical horizons in General Relativity (e.g., Cauchy or Kruskal; see [1], [6], [11], [7], [14], [13]) and appear in the theory of electromagnetism ([12]). Very recent advances regarding the ruled lightlike surfaces in
Minkowski 3-space have been obtained in [12]. The essential tool in studying the induced objects on a lightlike submanifold is the construction of a certain transversal bundle (see e.g. [6]). There exist at present few works which address such submanifolds in pseudo-Finsler spaces (e.g., [4]).

For convenience, we shall further replace the pseudo-Finslerian metric \( g_{ij} \), whose signature is \((+, -, \ldots, -)\) with the opposite metric, \( g_{ij}^* \), of signature \((-+, +, \ldots, +)\). Then we note that the metric has index 1, is non-degenerate and has constant signature \((+, \ldots, +)\). (2.1) Then we canonically obtain the shape extrinsic objects. This is performed by constructing induced objects on a lightlike submanifold is the construction of a certain transversal bundle (see e.g. [6]). There exist at present few works which address such submanifolds induced objects on a lightlike submanifold is the construction of a certain transversal bundle (see e.g. [6]). There exist at present few works which address such submanifolds.

Consider now a submanifold \( M \subset \tilde{M} \equiv \mathbb{R}^n \), \( \dim M = m \) \((2 \leq m < n)\), locally parametrized by \( x^i = x^i(u), u = (u^1, \ldots, u^m), i = 1, n \). We denote the local coordinates of \( TM \) by \( (u, v) \), where \( v = (v^1, \ldots, v^m) \in \mathbb{R}^m \equiv T_u \tilde{M} \). Then \( TM \) is immersed in \( \tilde{M} \), by \( 2n \)-uples of the form \( x(u), y(u, v) \in \mathbb{R}^{2n} \equiv TM \), with ((17), [4]) \( y^i = B^i_u(u), \) where \( B^i_u = \frac{\partial x^i}{\partial u} \) and by \(*\) we denote transvection with \( v \) (i.e., \( B^i_u = B^i_u v^\alpha \), and we use the indices \( \alpha, \beta, \ldots \) within the submanifold index range \( \overline{1, m} \). The vector fields tangent to \( TM \) have their canonical basis related to the one of the ambient space \( \tilde{M} \) via

\[
\frac{\partial}{\partial u^\alpha} = B^i_u(u) \frac{\partial}{\partial x^i} + B^i_u(u) \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial v^\alpha} = B^i_u(u) \frac{\partial}{\partial y^i},
\]

where \( B^i_{\alpha, \beta}(u) du^\alpha \otimes du^\beta \) is the flat Hessian of \( x^i(u) \), \( B^i_{\alpha, \beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} \). We note that \( g_{ij} \) induces on the submanifold \( M \) the \((0, 2)\)-symmetric tensor field

\[
g^M_{\alpha, \beta} = B^i_u(u) B^j_u(u) g_{ij}(x(u), y(u, v)).
\]

Let now \( TM^0 \) be the normal space to \( M \) in \( \tilde{M} \),

\[
TM^0 = \bigcup_{p \in TM} TM^0_p, \quad \text{where } TM^0_p = \{ X \in VTM_p \mid X \perp Y, \forall Y \in VTM_p \}.
\]

In the proper Finslerian case (i.e., for \( g_{ij} \) positive definite), we have \( TM^0 \cap VTM = \{0\} \). But in the pseudo-Finsler space \( \mathcal{H}_0 \) this property might fail, and we distinguish the typical case of co-isotropic submanifolds ([6], [4]):

**Definition 2.1.** The submanifold \( M \subset \tilde{M} \) is called *co-isotropic* (or *lightlike, null*) if \( TM^0 \subset VTM \).

**Example 2.2.** The hyperquadrics

\[
K_y = \{ w \in \mathbb{R}^n \mid w^i [g]_y w = 0 \} \subset \mathbb{R}^n, \quad \forall y \in \mathbb{R}^n
\]

and their submanifolds are co-isotropic. We have denoted \([g]_y = (g_{ij}(y))_{i,j=1,n}\).

In the following, in order to study the induced geometry of the co-isotropic submanifolds \( M \) of \( \mathcal{H}_0 \), we remark the need of a proper transversal to \( M \) bundle, for canonically obtaining the shape extrinsic objects. This is performed by constructing several total spaces for subbundles in \( VTM \), determined by the following decompositions

\[
\begin{align*}
\begin{cases}
VTM = S(VTM) \perp TM^0 \\
VTM = S(VTM) \perp S(VTM)^+ \\
S(VTM)^+ = TM^0 \oplus tr(VTM),
\end{cases}
\end{align*}
\]
where $\perp$ and $\oplus$ denote the orthogonal direct, and simple direct sums, respectively. Then (2.1) lead to the decompositions ([4]):

\[(2.2) \quad \tilde{V}T\mathcal{M} = S(VT\mathcal{M}) \perp (TM^0 \oplus tr(VT\mathcal{M})) = VT\mathcal{M} \oplus tr(VT\mathcal{M}).\]

The defined by (2.1) subspaces are

- $S(VT\mathcal{M})$ - called the screen distribution of $VT\mathcal{M}$;
- $S(VT\mathcal{M})^\perp$ - the orthogonal complement of the screen distribution;
- $tr(VT\mathcal{M})$ - called the transversal vector bundle of $\mathcal{M}$.

![Fig.1. Building the transversal bundle in co-isotropic submanifolds.](image)

The proof of existence of a local basis $B' = \{N_a | a \in [m+1, n]\}$ of the $F(TM)$-module of sections $\Gamma(tr(VT\mathcal{M}))$ relies on imposing the requirements:

\[g(N_a, \xi_b) = \delta_{ab}, \quad B' \perp g, \quad B' \perp g S(VT\mathcal{M}),\]

i.e., the new basis has to be dual to a pre-existing basis $B = \{\xi_a | a \in [m+1, n]\}$ of $TM^0$, has to be $g$-orthogonal and isotropic, and $g$-orthogonal to the screen distribution. The new local basis is still lightlike, but slant, transversal to $\mathcal{M}$, hence replacing the tangent to $\mathcal{M}$ orthogonal basis $B$. Then one may consider the classic Gauss-Weingarten and Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations of the co-isotropic submanifold $\mathcal{M}$, as follows.

### 3 Nonlinear connections on co-isotropic submanifolds of $\mathcal{H}_n$

We further use the common for general Finsler spaces abbreviations:

\[\partial_i = \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}, \quad \delta_i = \frac{\delta}{\delta x^i} = \partial_i - N_i^j \dot{\partial}_j,\]

where $N_i^j = \dot{\partial}_j \gamma_{00}^i$ are the coefficients of the nonlinear Cartan canonic connection of $\tilde{\mathcal{M}}$, the null index denotes transvection with $\gamma$ and $\gamma_i^j_k = \frac{1}{2} g^{xy}(\partial_j g_{yk} + \partial_k g_{sj} - \partial_s g_{jk})$ are the usual Christoffel symbols of second kind. We note that since $\mathcal{H}_n$ is locally Minkowski, the nonlinear connection has locally null coefficients. Assuming that $\mathcal{H}_n$ admits an affine atlas, the null nonlinear connection can be regarded as a global
geometric object. This connection still locally induces a generally non-trivial non-linear connection $\tilde{N}_\alpha^\beta$ on $M \subset \mathcal{H}_n$, given by

$$\tilde{N}_\alpha^\beta(u, v) = \tilde{B}_i^\beta (B_{\alpha}^a N_j^a + B_{\alpha}^a t_{v}^a) = \tilde{B}_i^\beta B_{\alpha}^a t_{v}^a,$$

which is practically provided by the flat Hessian of the immersion components, where we denoted by $[\tilde{B}_i^\alpha, \tilde{N}_j^\alpha]$ the inverse matrix of $[B_{\alpha}^a, N_j^a]$, and assume that the indices $a, b, \ldots$ take the complementary index values $m + 1, n$. The components of this inverse matrix practically link the fields of frames $\{\partial_{\hat{a}}, \ldots, \partial_{\hat{n}}\}$ to the local adapted frame $\{\frac{\partial}{\partial x^a}, \ldots, \frac{\partial}{\partial x^n}, \xi_{m+1}, \ldots, \xi_n\} \subset \Gamma_{loc}(VT\tilde{M})$. As well, for $M$ co-isotropic, $M \subset \tilde{M} = \mathcal{H}_n$ admits a local adapted basis of the sections of $TTM = HTM \oplus VTM$, with the horizontal fields

$$\delta\frac{\partial}{\partial u^a} = \delta\frac{\partial}{\partial v^a} - \tilde{N}_\alpha^a \delta\frac{\partial}{\partial \xi^a} =$$

$$= B_{\alpha}^a \delta\frac{\partial}{\partial x^a} + \tilde{N}_\alpha^a (B_{\alpha}^a N_j^a + B_{\alpha}^a) B_{\alpha}^a \delta\frac{\partial}{\partial y^a} =$$

$$= B_{\alpha}^a \delta\frac{\partial}{\partial x^a} + \tilde{N}_\alpha^a B_{\alpha}^a, N_a \in \Gamma(HTM \oplus tr(VTM)).$$

We note that $\{\delta\frac{\partial}{\partial u^a}\}$ locally span $\Gamma(HTM)$, where $HTM$ is a subbundle of $HT\tilde{M} \oplus tr(VTM)$.

4 The co-isotropic Gauss and Weingarten equations in $\mathcal{H}_n$

For a given Finsler linear connection $\tilde{\nabla}$ on $\tilde{M}$, we infer the Gauss formula

$$\nabla_{\tilde{\nabla}}(X, Y) = B(X, Y),$$

for all $X \in \Gamma(TTM), Y \in \Gamma(VTM)$. Here $B$ is the second fundamental form on $M$ and $\nabla$ is the induced by $\tilde{\nabla}$ on $M$ linear connection; locally we have $B(X, Y) = B^a(X, Y)lera$. Then, due to a result in [4, Prop. 2.1, p. 65], we have $B^a(X, Y) = g(\nabla_X Y, \xi_a)$ and hence the second fundamental form does not depend on the choice of the screen distribution; this form vanishes for $X, Y \in \Gamma(TM^0)$. Moreover, the induced linear connection is metrical, if and only if it vanishes for $Y \in \Gamma(S(VTM))$. Hence we can state the following

**Theorem 4.1.** The induced tangent connection on a co-isotropic submanifold $M \subset \mathcal{H}_n$ is metrical if and only if the second fundamental form is zero outside the sections

$$\Gamma(S(VTM)) \times \Gamma(TM^0) \ni (X, Y).$$

The Weingarten formula has the form

$$\nabla_{\tilde{\nabla}}(X, V) = -A_V(X) + \nabla_{\tilde{\nabla}}V,$$

for all $X \in \Gamma(TTM), V \in \Gamma(tr(VTM))$. Here $\nabla_{tr}$ is the vertical linear connection on $tr(VTM)$ and $A_V$ is the Weingarten (shape) morphism of $\mathcal{F}(TM)$ modules $A_V$: 
\[ \Gamma(TTM) \to \Gamma(VTM). \]

We note that in \( \mathcal{H}_n \) the components \( (L^i_{jk}, C^i_{jk}) \) of the Cartan linear connection are given by
\[
L^i_{jk} = \frac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{sj} - \delta_s g_{jk}) \equiv 0
\]
\[ (4.3) \]
\[ C^i_{jk} = g^{is} C_{jsk} = \frac{g^{is}}{g^{ys}} \left(-\frac{2}{n} + \frac{\delta_{jk} + \delta_{sj} + \delta_{sk}}{n} - \delta_j \delta_k \right). \]

where \( \tilde{\nabla} \) is the h-covariant derivation associated to the Cartan connection, i.e.,
\[
\tilde{\nabla}_i = \frac{1}{2} \delta_j \partial_i \log g^{ij} + \frac{1}{2} \delta_i \partial_j \log g^{ij} + \frac{1}{2} \delta_j \partial_i \log g^{ij}.
\]

5 The co-isotropic Gauss, Codazzi and Ricci equations in \( \mathcal{H}_n \)

The three curvature tensors of a Finsler space have the coefficients:
\[
\begin{cases}
\delta_i [L^i_{jk}] + L^i_{jk}[L^i_{kl}] + C^i_{jk} R^i_{kl} \\
P^i_{jk} = \tilde{\nabla}_i C^i_{jk} - C^i_{jc} P^c_{jk} \\
S^i_{bcd} = \tilde{\nabla}_i C^i_{bc} + C^i_{bc} C^i_{rd}.
\end{cases}
\]

(5.1)

where \( \delta_k \) denotes the h-covariant derivation associated to the Cartan connection, i.e.,
\[
C^i_{jc} = \delta_k C^i_{jc} + C^i_{jc} L^i_{jk} - C^i_{jc} L^i_{jk} - C^i_{jc} L^i_{jk},
\]

and we considered the torsion fields, \( R^i_{kl} = \delta_i [N^i_{jk}] \) \( P^i_{kc} = \tilde{\nabla}_i N^i_{jk} - L^i_{jk} \) (which vanish for Berwald-Moór spaces), the Cartan tensor \( C_{jk} = \frac{1}{2} \tilde{\nabla}_j \tilde{\nabla}_k - \tilde{\nabla}_l \tilde{\nabla}^l \), for \( H^n \) has the form ([16]):
\[
C_{jk} = \frac{F^2}{ny^y y^y} \left( \frac{2}{n^2} - \delta_{ij} + \delta_{jk} + \delta_{ki} + \delta_{ij} \delta_{jk} \delta_{ki} \right),
\]

and we denoted by square braces the skew-symmetrization (e.g., \( \tau_{[i \ldots j]} = \tau i \ldots j - \tau j \ldots i \)). The covariant curvature tensors are given by
\[
R^i_{jk} = g_{jk} R^i_{kl}, \quad P^i_{jk} = g_{jk} P^i_{kl}, \quad S^i_{jk} = g_{jk} S^i_{kl}.
\]

Since \( H^n \) is locally Minkowski, its horizontal and mixed curvatures identically vanish; moreover, it is known that ([16])

**Theorem 5.1.** The Berwald-Moór space \( \mathcal{H}_n \) has the following properties:

a) \( A_i \equiv F \cdot \tilde{\nabla}_i (\log g^{1/2}) = 0; \)

b) The covariant vertical curvature satisfies the equality
\[
S_{ijkl} = F^{-2} S \cdot (g_{ij} k g_{iji}),
\]

with \( S = -1 = \text{const}, \) i.e., \( \mathcal{H}_n \) is \( S^3 \)-like.

c) The \( T \)-tensor \( T_{ijkl} = FC_{ijkl} + \Sigma_{ijkl} l^1 C_{ijkl} \) identically vanishes, where \( l_i = \frac{F}{ny} \), and \( \Sigma_{ijkl} \) denotes the cyclic sum over the lower indices.
It is known as well ([16]), that $S_{ijkl} \equiv 0$ for $n = 2$, and that the $S3$–property always holds true for 3-dimensional Finsler spaces. The $T$–property shows that the Finsler space is locally symmetric, like in (pseudo-)Riemannian geometry, and that the space becomes (pseudo-)Riemannian under some weak assumptions. The conformally deformed space $\tilde{F}(x, y) = e^{s(x)} \sqrt{g_1^1y^2\cdots y^n}$ satisfies the Theorem as well, and $P_{ijkl} \equiv 0$.

For deriving the Gauss, Codazzi and Ricci equations, we build the linear connection $\nabla'$ on $TM$ ([4]) via

$$\nabla'_X Y = \tilde{\nabla}_X V(Y) + \varphi(\tilde{\nabla}_X \varphi(h Y)), \forall X, Y \in \Gamma(TM),$$

where $h$ and $v$ are the horizontal, respectively vertical projectors for the decomposition $\Gamma(TM) = \Gamma(TM) \oplus \Gamma(VTM)$, and $\varphi \in \text{End}(\Gamma(TM))$ is the almost product $\mathcal{F}(TM)$–module endomorphism given on the local basis via

$$\varphi \left( \frac{\partial}{\partial u^\alpha} \right) = \frac{\partial}{\partial v^\alpha}, \quad \varphi \left( \frac{\partial}{\partial v^\alpha} \right) = \frac{\partial}{\partial u^\alpha}, \forall \alpha \in \overline{1,m}.$$  

Denoting by $\tilde{R}, R, R^{tr}$ the curvatures of $\tilde{\nabla}, \nabla, \nabla^{tr}$ one can explicitly derive the Gauss, Codazzi and Ricci equations of $M \subset \tilde{M} = \mathcal{H}_n$. Namely, denoting by $+(X/Y)$ the addition of the term which has $X$ and $Y$ interchanged, and by $-(X/Y)$ its substraction, $A(V, X) = A_V(X)$, considering the torsion $T'$ of $\nabla'$ and the objects:

$$\begin{cases} 
(\nabla'_X B)(Y, Z) = \nabla'_X (B(Y, Z)) - [B(\nabla'_X Y) + (X/Y)] + \\
(\nabla_X A)(V, Y) = \nabla_X (A(V, Y)) - [A(\nabla'_X V) + (X/Y)],
\end{cases}$$

we infer the announced three sets of equations ([4, Sec. 3.6]):

\begin{align}
\begin{cases}
\tilde{R}(X, Y)Z & = R(X, Y)Z + [A(B(X, Z), Y) + (\nabla'_X B)(Y, Z) - (X/Y)] + \\
& + B(T'(X, Y) Z)
\end{cases} &  \quad (5.2) \\
\begin{cases}
\tilde{R}(X, Y)V & = R^{tr}(X, Y)V + [B(Y, A_V(X)) + (\nabla'_Y A)(V, X) - (X/Y)] + \\
& + A_V(T'(X, Y),)
\end{cases} \\
\begin{cases}
\tilde{g}R((X, Y) Z, U) & = g(R(X, Y)Z, U) + [\delta_{ab} B^{ab}(X, Z) g(A_{N_a}(Y), U) - (X/Y)] \\
g(\tilde{R}(X, Y) Z, \xi) & = [g(\nabla'_X B)(Y, Z, \xi) - (X/Y)] + g(B(T'(X, Y), Z), \xi) \\
g(\tilde{R}(X, Y) Z, N) & = g(R(X, Y)Z, N),
\end{cases}
\end{align} \quad (5.3)

for all $X, Y \in \Gamma(TM), \ Z \in \Gamma(VTM), \ V \in \Gamma(tr(VTM)), \ U \in \Gamma(S(VTM)), \ \xi \in \Gamma(TM^0), \ N \in \Gamma(tr(VTM))$. In $M = \mathcal{H}_n$, the relations (5.3) are nontrivial only
when $X, Y \in \Gamma(VTM)$, due to the splitting $\Gamma(VTM) = \Gamma(TM^0) \perp \Gamma(S(VTM))$ and considering the skew-symmetry in $X, Y$, these are refined to $3(2+3+3)$ equations.

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