On the equations of motion in Minkowski space

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Abstract. In this paper, we consider the question of addition, subtraction and transition of velocities as 4-vectors in the Minkowski space. The indeterminacies in the space of 4-velocities are discussed in detail and a new method for transition between 4-velocities using a transition tensor P of rank 2 is proposed. The most important properties as well as the physical and geometrical interpretations of the tensor P are also given. Finally, the tensor P is used in obtaining a nonlinear connection and the corresponding equations of motion. These equations differ from the Einstein-Infeld-Hoffmann equations by Lorentz invariant terms of order c^{-2}.

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1 Introduction

Standard approach in representation of relativistic velocities is based on 3-dimensional vectors where vector components correspond to the change of spatial coordinates with the time. Most texts on relativity physics present the relativistic velocity addition only for parallel velocities for the sake of simplicity. Restricted to parallel velocities, Einstein's addition law

$$\vec{w} = \frac{\vec{u}+\vec{v}}{1+\frac{\vec{u}\cdot\vec{v}}{c^2}}$$

is both, commutative and associative.

The general relativistic velocity addition is derived for 3-velocities [2], [8], [7], [17] and its derivation is based on the Lorentz transformations. However, there is some disagreement between the relativistic law of velocity addition and the Lorentz transformation laws, in spite of the fact that the first law is derived from the second one. Due to the time and distance distortions that must be attributed when referring a frame that is in motion relative to the observer's frame, non-collinear velocities cannot be added with plain arithmetic, but must employ a more complex formula

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$$\vec{w} = \vec{u} \oplus \vec{v} = \frac{1}{1+\frac{\vec{u}\cdot\vec{v}}{c^2}}\Big(\vec{u} + \sqrt{1-\frac{u^2}{c^2}}\vec{v} + \frac{1}{c^2}\frac{1}{1+\sqrt{1-\frac{u^2}{c^2}}}(\vec{u}\cdot\vec{v})\vec{u}\Big) \tag{1.1}$$

For general velocity addition we use the symbol $\oplus$ to make a difference with standard vector addition. This velocity addition law is, generally, non-commutative and non-associative [2]. Beside non-commutative and non-associative nature of relativistic 3-velocity addition, the concept of 3-velocities has a drawback in the fact that the velocities are not in their natural environment - a tangent bundle of a 4-dimensional manifold.

If one considers 4-velocities, the standard interpretation is that they are tangent vectors of 4-dimensional space-time manifold rather then physical velocities. Actually, the 4-velocity is one of the most important concepts in the theory of relativity because they are a bridge between the concept of 3-velocity extensively used in Special Relativity and the concept of tangent vector in 4-dimensional space-time manifold that is an essential concept in General Relativity (GR). The definition of the 4-velocities is made in such a way that they correspond to tangent vectors on geodesics on 4-dimensional manifold and in the same time resemble, as much as possible, the well-known concept of 3-velocity.

In this paper, we propose that, generally, the transition between 4-velocities could be made by a tensor of rank 2 whose components depend only on the components of the involved 4-velocities. Intuitively, this tensor acts as a Lorentz transformation in the space of relativistic 4-velocities. We shall also show that the proposed tensor has all necessary features to be a suitable apparatus for transition between 4-velocity acting in a similar way as 4-velocity subtraction in a vector space.

The tensor P leads to an interesting concept of nonlinear connection influencing equations of motion. Namely, GR uses a linear connection [5]. Generally, the connection should reflect the properties of the space of 4-velocities rather than idealized tangent space on 4-dimensional space-time manifold. So, the 4-velocities could be a ground for introducing a new connection in the space-time manifold. However, although the 4-velocities belong to the tangent space, they do not make a vector space. These observations motivated us to introduce a nonlinear connection based on the tensor P and the corresponding acceleration components.

2 Transition between relativistic 4-velocities using a tensor

In the rest of the paper, we shall denote $x_4 = ict$ instead of $x_4 = ct$. Thus the metric takes the form $g_{ij} = \mathrm{diag}(1,1,1,1)$ and all indices will be lower in the rest of the paper.

Let us consider three points: O, A and B in the Minkowski space. The point O corresponds to our inertial coordinate system where we are observers. The point A rests in the inertial system S_0 which moves with a 4-vector of velocity

$$U = \left(\frac{\frac{u_x}{ic}}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{\frac{u_y}{ic}}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{\frac{u_z}{ic}}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

with respect to O, and assume that the point B rests in a coordinate system which moves with velocity

$$V = \left(\frac{\frac{v_x}{ic}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\frac{v_y}{ic}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\frac{v_z}{ic}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \right)$$

also with respect to O. The problem of finding 4-vector of velocity which would represent the velocity of B with respect to A is a problem of subtraction of the velocities V and U and we shall denote it symbolically by $V-U$. Analogously, we can require to find addition of the velocities U and V, symbolically denoted by $U+V$.

If we denote with $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$ the corresponding 3-velocities to U and V, the coordinates of subtraction and addition of 4-velocities for $u, v \ll c$ should be approximately $(\frac{v_x-u_x}{ic}, \frac{v_y-u_y}{ic}, \frac{v_z-u_z}{ic}, 1)$ and $(\frac{u_x+v_x}{ic}, \frac{u_y+v_y}{ic}, \frac{u_z+v_z}{ic}, 1)$ respectively, if such exist.

According to (1.1), the sum of 4-velocities in each coordinate system is naturally defined by the following 4-components of velocity

$$U + V = \left(\frac{\frac{\vec{u} \oplus \vec{v}}{ic}}{\sqrt{1 - \frac{(\vec{u}\oplus\vec{v})^2}{c^2}}} \;,\; \frac{1}{\sqrt{1 - \frac{(\vec{u}\oplus\vec{v})^2}{c^2}}} \right). \tag{2.1}$$

We will prove the following

Proposition 1. *The 4-components defined by (2.1) do not transform as a 4-vector in the Minkowskian space.*

Proof. Similar statement is also true for the 4-components of $V-U$, which can be defined analogously as (2.1). We will prove the proposition for the 4-components of $V-U$ and the proof is analogous for $V+U$.

Assume that U and V are different 4-vectors, and $V-U$ transforms also as a 4-vector. Let in a chosen coordinate system the unit 4-vector $V-U$ has coordinates (a_1, a_2, a_3, a_4). It is always possible to find a Lorentz transformation, such that in the new coordinate system it has coordinates $(0,0,0,1)$. It means that $V-U$ leads to a zero 3-vector of velocity, i.e. $\vec{v}\oplus(-\vec{u}) = (0,0,0)$. Two cases are possible: 1. $\vec{u}$ and $\vec{v}$ are not collinear, and 2. $\vec{u}$ and $\vec{v}$ are collinear. In the first case from (1.1) it follows that $\vec{u} = \vec{v} = 0$, while in the second case it follows that $\vec{u} = \vec{v}$. Hence in any case $\vec{u} = \vec{v}$, which means that V and U are equal. This is in a contradiction with the assumption. □

Notice that U_iV_i is a scalar, and moreover

$$U_iV_i = \frac{1}{\sqrt{1-\frac{w^2}{c^2}}} \tag{2.2}$$

where w is the velocity between the two bodies, which move with 4-vector of velocities U and V. Indeed, it is easily to be seen this conclusion in the system, where $U = (0,0,0,1)$, and so (2.2) is true in each coordinate system. So we conclude that although the velocity w exists and it is the same in all coordinate systems, its direction is "undetermined". The non-commutativity and non-associativity of relativistic velocity addition makes an apparent indetermination in the Minkowski space. Later we shall see that in the Minkowski space this indetermination disappears by providing a tensor of rank 2 that enables transition among 4-velocities.

As 3-velocity addition $\oplus$ is non-commutative and non-associative, the 4-velocity addition (2.1) is also non-commutative and non-associative, and moreover $U + (V - U) \neq V$, as well as $(V - U) + U \neq V$.

It has been shown (see for example [15], [16]) that $\vec{u} \oplus \vec{v} = T(\vec{u}, \vec{v})\vec{v} \oplus \vec{u}$ and $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus T(\vec{u}, \vec{v})\vec{w}$ where $T(\cdot, \cdot)$ is a rotation known as the Thomas precession [3]. So, the Thomas precession obstructs commutativity and associativity in the relativistic velocity addition.

Although (2.1) does not transform as a 4-vector, in a chosen coordinate system we can examine the algebraic structures of the operation + defined in (2.1). As first, the groupoid of 4-velocities is not a semigroup because of non-associativity. To check if the groupoid of 4-velocities is a quasigroup we should solve the following equations by the unknown 4-velocity vectors X and Y

$$X + U = V \quad \text{and} \quad U + Y = V.$$

As we already pointed out, the 4-velocity $V - U$ is not a solution of these equations. For the corresponding 3-velocities, the solution for the first equation $\vec{x} \oplus \vec{u} = \vec{v}$ is $\vec{x} = \vec{v} \oplus (-T(\vec{v}, \vec{u})\vec{u})$. The solution of the second equation $\vec{u} \oplus \vec{y} = \vec{v}$ is simply $\vec{y} = -\vec{u} \oplus \vec{v}$. Thus, the solutions of 4-vector equations can be directly obtained by transferring the 3-vectors $\vec{x}$ and $\vec{y}$ to the corresponding 4-vectors X and Y.

As a common unit element exists (it is (0,0,0,1)), the space of 4-velocities is a quasigroup with unit, i.e. a loop. Note that in this case the existence of a unique inverse for each element follows, if one considers the definition of 4-velocity.

Addition of 4-velocity is not a group operation, so the space of 4-velocities is not a vector space. Thus, there is not a natural way to make transition from 4-velocity U to 4-velocity V, although, as we already discussed, such a transformation exists (the space of 4-velocities is a loop). The transformation that would enable to cross from one vector representing 4-velocity U to another vector representing 4-velocity V is essential because enables connection in the space of 4-velocities. So, our goal is to find a transformation that will replace the expected operation $(V - U)$ in the space of 4-velocities enabling transition from the 4-velocity U to 4-velocity V via an observer placed at the point O. This transformation should act in the space of 4-velocities in a similar way as a Lorentz transformation acts for space-time events.

Parallel to the existence of the inner product $U_i V_i$, there exists a tensor of rank 2 [9],[13], which will be denoted by $P(U, V)$ and defined by

$$P(U, V)_{ij} = \delta_{ij} - \frac{V_i V_j + V_i U_j + U_i V_j + U_i U_j}{1 + U_s V_s} + 2V_i U_j. \tag{2.3}$$

This tensor solves our problem of transition between 4-velocities because of the following three properties:

1. $P(U, V)_{ij} U_j = V_i$, i.e. $P(U, V)U = V$.
2. $P(U, V)$ belongs to the group of Poincare (with the notation $x_4 = ict$);
3. If $(U_i) = (0, 0, 0, 1)$ in the system S_0, then $P(U, V)$ is just a Lorentz transformation which corresponds to a motion with 3-velocity $-\vec{v}$.

The proofs of these and the next properties are straight, and we omit them. The tensor $P(U, V)$ has also the following properties:

4. $P(U,V)^{-1} = P(V,U)$.

5. $(P(U,V)_{i4}) \approx \left(\frac{v_x-u_x}{ic}, \frac{v_y-u_y}{ic}, \frac{v_z-u_z}{ic}, 1\right)$, which is very convenient for $\vec{u}, \vec{v} \ll c$, since the ordinary subtraction is consisted in the 4-th column $P(U,V)_{i4}$.

6. $P(U,V)$ $(U \neq V)$ can not represent only a space rotation. Namely, if $P(U,V)$ is a space rotation, then $P_{44} = 1$, and $P_{41} = P_{42} = P_{43} = P_{14} = P_{24} = P_{34} = 0$, but one can prove that they imply $U = V$.

In general case, $P(U,V)$ does not represent a single Lorentz transformation as a motion with a 3-vector of velocity, because the 3×3 submatrix $P(U,V)_{ij}$ for $i, j \in \{1,2,3\}$ is not a symmetric matrix. This space rotation can be exactly calculated via the method given in [14]. Neglecting the expressions of order c^{-4} we obtain that $P(U,V)$ contains also a space rotation for angle

$$\varphi \approx \left|\frac{\vec{v} \times \vec{u}}{c^2}\right|. \tag{2.4}$$

Namely, the slight space rotation determined by the matrix $P(U,V)$ is given by the following 3-vector:

$$\begin{aligned} &\frac{1}{2}(P_{32} - P_{23}, P_{13} - P_{31}, P_{21} - P_{12}) = \\ &= (U_2V_3 - U_3V_2, U_3V_1 - U_1V_3, U_1V_2 - U_2V_1) = \\ &= \frac{1}{c^2}(v_y u_z - v_z u_y, v_z u_x - v_x u_z, v_x u_y - v_y u_x) = \frac{\vec{v} \times \vec{u}}{c^2}. \end{aligned}$$

So, we come to the conclusion that $\left|\frac{\vec{u} \times \vec{v}}{c^2}\right|$ is a magnitude of indetermination of the vector $V - U$ as a tool for transition between the velocity vectors U and V.

3 Application of the tensor *P* to the equations of motion

In the previous section we presented the tensor $P(U,V)$ and some properties, but we did not devote attention of its physical and geometrical meaning. So, we give it now.

Let us denote by $L(\vec{v})$ a Lorentz transformation which represents a motion with velocity $\vec{v}$ and assume that the points O, A, and B are the same points as in the previous section. Then $L(\vec{u})$ gives connection between our coordinate system and S_0, while $L(\vec{v})$ gives a connection between our coordinate system and the system in which B rests. Assume that, observed from A, the point B moves with 3-vector of velocity $\vec{w}$. If the vectors $\vec{u}$ and $\vec{v}$ are collinear, then the composition of the transformations $L(\vec{u})$ and $L(\vec{w})$ yields $L(\vec{v})$, i.e.

$$L(\vec{w}) \circ L(\vec{u}) = L(\vec{v}). \tag{3.1}$$

If the vectors $\vec{u}$ and $\vec{v}$ are not collinear, then the equation (3.1) will not be true, because the composition on the left side of (3.1) contains also a slight space rotation, while on the right side it does not. In this general case when the vectors $\vec{u}$ and $\vec{v}$ are not collinear, the transformation $L(\vec{v}) \circ L(\vec{u})^{-1}$ seems to solve our problem. However, it is not good because of the following reasons: $L(\vec{v}) \circ L(\vec{u})^{-1}$ is not a tensor of rank 2, and this corresponds to the observer at A, but not at O. So the tensor $P(U,V)$

(or more precisely $P(U,V)^T$) successfully avoids all possible perturbations, and thus we conclude that $P(U,V)$ is a tensor which represents the transformation between the local coordinate systems of the points A and B which move from O with the 4-vector velocities U and V. Since $P(U,V)$ is a tensor, the tensor components (2.3) are attributed to the observer O. $P(U,V)$ will be called *transition tensor*.

While the transformation $L(U,V) = L(\vec{v})\circ L(\vec{u})^{-1}$ satisfies the following transition equality

$$L(V,W)\circ L(U,V) = L(U,W)$$

for arbitrary vectors U, V and W, like the Jacobi matrices, the tensor P does not posses this property, i.e.

$$P(U,V)\cdot P(V,W) \neq P(U,W) \tag{3.2}$$

where " $\cdot$ " denotes a matrix multiplication. Specially, $P(U,V) \neq P(U,0)\cdot P(0,V) = L(\vec{u})\cdot L(\vec{v})^{-1}$, where $P(U,0) = L(\vec{u})$ and $P(V,0) = L(\vec{v})$, i.e. $P(U,V)^T \neq L(\vec{v})\circ L(\vec{u})^{-1} = L(U,V)$. Later we will consider the consequences from (3.2).

Let us consider the application to the equations of motion. The equations of motion were formerly given in [9] and later [13]. In [9] were considered both equations of motion in gravitational field and also motion under the inertial forces. In [13] was considered the motion of a test body according to an observer from a uniformly rotational system. In case of gravitation we must emphasize that the equations are presented only according to the so-called orthonormal coordinates, yielded after normalization with respect to the actual metric from the GR. Namely, at each point we consider an orthonormal tetrahedron (X,Y,Z,T). The vector fields X, Y, Z, and T can not be unified locally in a single coordinate system, but we can consider them as non-holonomic coordinates. The influence of the non-holonomic coordinates is studied in [10], [11], [12], and it is in agreement with the experiments concerning the Pioneer anomaly and change of the orbital periods of the binary pulsars. But we shall not consider them now.

Further, we assume existence of orthonormal coordinates at each point, and it means that at each point the tangent space is the Minkowskian space. These coordinate systems are not used in the GR, and we give two resons for the using of the orthonormal coordinate systems.

1. GR uses the classical theory of manifolds which accepts existence of a coordinate transformation between any two coordinate neighborhoods with common interior points. In that case the Jacobi matrix at each point would belong to the general linear group, so it may not be a Lorentz transformation. This means that the tangent space is not Minkowskian space at each point. If the space is Minkowskian at each point, then the coordinates are orthonormal and it leads to non-holonomic connection with the old coordinates [10], [12].

2. GR uses a linear connection, such that the parallel displacement of an arbitrary vector A^i along a given curve is given by

$$\frac{dA^i}{ds} = \Gamma^i_j A^j. \tag{3.3}$$

The linearity means that

$$\Gamma^i_j = \Gamma^i_{jk}\frac{dx^k}{ds} \tag{3.4}$$

where dx^k/ds is the tangent vector of the curve of parallel displacement, and Γ^i_{jk} depend only on the local coordinates, but not on the tangent vector dx^k/ds. On the other hand the 4-velocity dx^k/ds does not belong to a vector space according to the previous section, and should be considered as a linear (Lorentz) transformation.

Further we are going to avoid the linearization (3.4). In order to involve the velocity into the connection, we rewrite the equality of parallel transport (3.3) for the flat metric $g_{ij} = \delta_{ij}$ in the following manner

$$\frac{dA_i}{ds} = -\Gamma_{ij} A_j. \tag{3.5}$$

Since the equation (3.5) should be Lorentz invariant, i.e. it preserves the metric δ_{ij}, the matrix Γ in (3.5) must be antisymmetric. It has the following geometrical/physical meaning. If $A = (1, 0, 0, 0)$, then $dA_1/ds = 0$ because of the orthonormality, dA_2/ds represents the rotation of the vector A in the (x, y) plane, dA_3/ds represents the rotation in the (z, x) plane (these two cases are angular velocities), while dA_4/ds represents the rotation in the (x, t) plane, which has the meaning of acceleration along the x-axis. Analogously, one can consider the parallel displacement of the vectors $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. The conclusion is that

$$(\Gamma_{ij}) = -\begin{bmatrix} 0 & -iw_z/c & iw_y/c & -a_x/c^2 \\ iw_z/c & 0 & -iw_x/c & -a_y/c^2 \\ -iw_y/c & iw_x/c & 0 & -a_z/c^2 \\ a_x/c^2 & a_y/c^2 & a_z/c^2 & 0 \end{bmatrix}, \tag{3.6}$$

where $\vec{a} = (a_x, a_y, a_z)$ is the 3-vector of acceleration and $\vec{w} = (w_x, w_y, w_z)$ is the 3-vector of angular velocity.

Notice that the matrix (3.6) was found in order to determine the sligh change of the orthonormal frame $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. Thus at the moment when the 4-vectors U and V are equal, and the orthonormal frame is chosen as previously, then (3.5) is true, where $\Gamma_{ij} = -\phi_{ij}$ and the matrix ϕ is given by

$$(\phi_{ij}) = \begin{bmatrix} 0 & -iw_z/c & iw_y/c & -a_x/c^2 \\ iw_z/c & 0 & -iw_x/c & -a_y/c^2 \\ -iw_y/c & iw_x/c & 0 & -a_z/c^2 \\ a_x/c^2 & a_y/c^2 & a_z/c^2 & 0 \end{bmatrix}. \tag{3.7}$$

Obviously ϕ is a tensor in the Minkowskian space, analogous to the tensor of electromagnetic field. Now if the vectors U and V are different, then we apply the transition tensor in order to determine the matrix Γ in (3.5), i.e. we detemine

$$\Gamma_{ij} = -P(U, V)^T_{ir} \phi_{rk} P(U, V)_{kj} \tag{3.8}$$

or in matrix form

$$\Gamma = -P^T \phi P. \tag{3.9}$$

Now the following question appears. What happens if we choose another coordinate system instead of S_0, for example a system determined by a 4-vector of velocity

U' instead of U? Instead of the tensor ϕ we would have a tensor ϕ' and then (3.8) yields

$$P(U,V)^T_{ir}\phi_{rk}P(U,V)_{kj} = P(U',V)^T_{ir}\phi'_{rk}P(U',V)_{kj},$$
$$\phi' = [P(U,V)P(U',V)^{-1}]^T\phi[P(U,V)P(U',V)^{-1}].$$

The matrix $P(U,V)P(U',V)^{-1}$ is close to $P(U,U')$, but different from $P(U,U')$ according to (3.2). Since ϕ is a tensor field, which should not depend on the velocity of the test body, we conclude that the system S_0 may not be arbitrarily chosen. Accepting that there does not exist any privileged coordinate system, the system S_0 must be the system in which the gravitational body rests and it is unique up to a space rotation.

The connection (3.8) together with (3.7) was found in [9] and applied in special cases assuming that $U = (0,0,0,1)$. The results yielded the known and experimentally verified effects about motion in a gravitational field: deflection of the light ray near the Sun and the displacement of the perihelion of the planetary orbits. Recently, the same formulae about the geodetic precession and frame dragging of inertia as in GR were also obtained (submitted for publication). In [13] this connection was studied for rotating systems, and it is deduced that according to the observer in the rotating system, the free test body moves with velocity of constant magnitude, i.e. $U_iV_i = const$. This is natural to expect, because the free test body moves with a constant velocity according to any inertial system. But an observer from a non-inertial system is also able to conclude that the magnitude is constant, although it is not able to find any relative vector as discussed in the previous section.

The construction of the tensor ϕ via the mass of the gravitational body and the distance to the test body is not presented here, but it can be done analogous to the electromagnetic tensor field from the electrodynamics, in the same way as the tensor field is determined via the Lienard-Wichert potentials. Specially, if the test body rests, then the tensor ϕ contains only the Newtonian acceleration, while the angular velocity disappears. Indeed, the study the motion in orthonormal coordinates reduces to the following three steps:

1. Determining the tensor ϕ analogous to the tensor of electromagnetic field,
2. Introducing the tensor P, and
3. Influence of the mass and energy to the equations of motion.

We skip the step 1. because the tensor of electromagnetic field is well known, and analogously we obtain that the tensor ϕ is well determined, where the charge of a particle is replaced by mass and so on. We also omit the step 3., which will be done in a future paper. The biggest attention in this paper we devoted to the step 2, trying to explain the reason and the validity of this step. A long study shows that considering 1PN approximation, the equations obtained in this way are the following

$$\frac{d^2\vec{r}_j}{dt^2} = \sum_{i\neq j}\Big\{-\frac{(\vec{r}_j-\vec{r}_i)Gm_i}{r_{ij}^3}\Big[1-\frac{3}{2}\frac{[\vec{v}_i\cdot(\vec{r}_j-\vec{r}_i)]^2}{r_{ij}^2c^2}+\frac{v_i^2}{c^2}-2\frac{\vec{v}_i\cdot\vec{v}_j}{c^2}+\frac{(\vec{v}_i-\vec{v}_j)^2}{c^2}\Big]-$$
$$+\frac{3}{2}\frac{Gm_i}{r_{ij}^3c^2}(\vec{v}_j-\vec{v}_i)[(\vec{r}_j-\vec{r}_i)\cdot(\vec{v}_j-\vec{v}_i)]+\frac{Gm_i}{r_{ij}^3c^2}(\vec{v}_j-\vec{v}_i)[(\vec{r}_j-\vec{r}_i)\cdot\vec{v}_j]-$$
$$-\frac{Gm_i}{r_{ij}^3c^2}(\vec{r}_j-\vec{r}_i)\times((\vec{r}_j-\vec{r}_i)\times\dot{\vec{v}}_i)\Big\}, \tag{3.10}$$

expressing the motion of the j-th body. Here m_i is the mass of the i-th body, $\vec{r}_i$ and $\vec{v}_i$ is the radius vector and the velocity of the i-th body, and $\vec{r}_{pq} = |\vec{r}_p - \vec{r}_q|$. We can distinguish the terms in (3.10) which are Lorentz invariant, and write (3.10) in the following form

$$\frac{d^2\vec{r}_j}{dt^2} = \sum_{i \neq j} \Big\{ -\frac{(\vec{r}_j - \vec{r}_i)Gm_i}{r_{ij}^3} \Big[1 - \frac{3}{2}\frac{[\vec{v}_i \cdot (\vec{r}_j - \vec{r}_i)]^2}{r_{ij}^2 c^2} + \frac{v_i^2}{c^2} - 2\frac{\vec{v}_i \cdot \vec{v}_j}{c^2} \Big] +$$

$$+\frac{Gm_i}{r_{ij}^3 c^2}(\vec{v}_j - \vec{v}_i)[(\vec{r}_j - \vec{r}_i) \cdot \vec{v}_j] \Big\} + \text{Lorentz invariant terms.} \tag{3.11}$$

Notice that the Einstein-Infeld-Hoffmann equations [5]

$$\frac{d^2\vec{r}_j}{dt^2} = \sum_{i \neq j} \frac{(\vec{r}_i - \vec{r}_j)Gm_i}{r_{ij}^3} \Big(1 - \frac{3}{2c^2}\frac{[\dot{\vec{r}}_i \cdot (\vec{r}_j - \vec{r}_i)]^2}{r_{ij}^2} - \frac{4}{c^2}\sum_{k \neq j}\frac{Gm_k}{r_{jk}} - \frac{1}{c^2}\sum_{k \neq i}\frac{Gm_k}{r_{ik}} +$$

$$+\frac{1}{2c^2}(\vec{r}_i - \vec{r}_j)\frac{d^2\vec{r}_i}{dt^2} - \frac{4}{c^2}\dot{\vec{r}}_i\dot{\vec{r}}_j + \Big(\frac{v_j}{c}\Big)^2 + 2\Big(\frac{v_i}{c}\Big)^2 \Big) +$$

$$+\frac{1}{c^2}\sum_{i \neq j}\frac{Gm_i}{r_{ij}^3}((\vec{r}_j - \vec{r}_i) \cdot (4\dot{\vec{r}}_j - 3\dot{\vec{r}}_i))(\dot{\vec{r}}_j - \dot{\vec{r}}_i) + \frac{7}{2c^2}\sum_{i \neq j}\frac{Gm_i}{r_{ij}}\frac{d^2\vec{r}_i}{dt^2}, \tag{3.12}$$

can also be written in the same form (3.11), and hence the conclusion that *the equations (3.10) differ from the Einstein-Infeld-Hoffmann equations (3.12) by Lorentz invariant terms of order* c^{-2}. Although this conclusion is obtained only by using the previous two steps 1. and 2., the same conclusion should be true after modification with respect to the step 3.

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