An intrinsic link between scalar and volume-valued Lagrangians

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Abstract. The aim of this paper is to provide a natural frame for affine Lagrangians and affine Hamiltonians, the focus being on some Hamiltonians applicable in classical fields and their generalizations. A unitary treatment of scalar and volume-valued Hamiltonians in a special class is obtained using some suitable lifting procedures.

Key words: affine Lagrangian, affine Hamiltonian, jet space.

1 Introduction

A general setting concerning Lagrangians and Hamiltonians on affine bundles is given in [4] and [5]. The most known examples of affine bundle used in differential geometry are the higher order tangent space and the jet space of a fibered manifold. These two classical cases were recently studied in many papers. The higher order spaces are studied from the affine point of view in [4]. The jet spaces are studied in the context of multi-time Lagrangian and Hamiltonian geometry in [7] and in an affine setting in [1]-[3]. The purpose of our paper is to indicate a link between these two cases, and also to give a general setting for Lagrangians and Hamiltonians on affine bundles. An F-Hamiltonian (volume-valued) and an affine Hamiltonian (scalar valued) are defined as sections in certain affine bundles, both naturally lifting to $\tilde{F}$-Hamiltonians. Further investigations are given in [6], where considering a Hamilton-Jacobi variational principle for $\tilde{F}$-Hamiltonians, one obtain some Hamilton-Jacobi equations that extend the classical ones studied in [1]-[3].

2 Affine Lagrangians and Hamiltonians on affine spaces

Let $A$ be an affine space modeled on the real (finite dimensional) vector space $V$. A Lagrangian on $A$ is a differentiable function $L : A \to \mathbb{R}$. An affine Hamiltonian on
A is a differentiable map (non-necessary linear) \( h : V^* \to A^1 \) such that \( \pi \circ h = 1_{V^*} \).

Using local coordinates, the affine Hamiltonian is

\[
(p_i) \xrightarrow{h} (p_i, h_0(p_i)).
\]

If the coordinates change, then

\[
h'_0 (p_i) = h_0(p_i) + p_ia^i.
\]

For example, if \( x_0(\alpha_i) \in A \), then \( (p_i) \xrightarrow{h_{x_0}} (p_i, \alpha^i p_i) \) is an affine Hamiltonian. For more details concerning affine Lagrangians and Hamiltonians on affine spaces and affine bundles see [4] and [5].

If \( h_1 \) and \( h_2 \) are two affine Hamiltonians, then \( h_1 - h_2 \) induces a map \( H : V^* \to \mathbb{R} \), called a vectorial Hamiltonian; we write \( H = h_1 - h_2 \), or \( h_1 = H + h_2 \). In particular, if \( h \) is an affine Hamiltonian and \( x_0 \in A \), then \( H_{x_0} = h - h_{x_0} \) is a vectorial Hamiltonian. Every vectorial Hamiltonian \( H : V^* \to \mathbb{R} \) has this form, using the affine Hamiltonian \( H + h_{x_0} \).

The vertical Hessian of a Lagrangian \( L \) (affine Hamiltonian \( h \)) is defined by \( g_{ij}(y^k) = \frac{\partial^2 L}{\partial y^i \partial y^j}(y^k) \) (by \( h^{ij}(p_k) = \frac{\partial^2 h_0}{\partial p_i \partial p_j}(p_k) \) respectively).

The Legendre map defined by a Lagrangian \( L : A \to \mathbb{R} \) is \( L : A \to V^* \), \( L(y^i) = \frac{\partial L}{\partial y^j}(y^i)e^j \) and the co-Legendre map defined by an affine Hamiltonian \( h : V^* \to A^1 \) of the form (2.1) is \( \mathcal{H} : V^* \to A, \mathcal{L}(p_i) = \left( \frac{\partial h_0}{\partial p_i}(p_j) \right) \).

The Lagrangian \( L \) is regular (hyperregular) if the Legendre map is a local diffeomorphism (global diffeomorphism). Analogous one say that an affine Hamiltonian \( h \) is regular (hyperregular) if its co-Legendre map is a local diffeomorphism (global diffeomorphism). A Lagrangian (affine Hamiltonian) is singular if it is not regular. For example, the image of the co-Legendre map of an affine Hamiltonian of the form \( h_{x_0} \) is \( \{x_0\} \) and its vertical Hessian is null (degenerate; an extreme case).

Then \( L \) (or \( h \)) is regular iff the vertical Hessian is non-degenerate in every point (as a bilinear form).

Let \( L : A \to \mathbb{R} \) be a hyperregular Lagrangian. Then let us denote by \( \mathcal{L}^{-1} : V^* \to A \) the inverse of the Legendre map; using coordinates, \( \mathcal{L}^{-1}(p_i) = (\mathcal{L}^j(p_i)) \).

Then \( h : V^* \to A^1, \mathcal{L}(p_i) = (p_i, h_0(p_i)), h_0(p_i) = p_j L^j(p_i) - L(\mathcal{L}^j(p_i)), \) is an affine Hamiltonian.

Conversely, let \( h : V^* \to A^1 \) be a hyperregular affine Hamiltonian and \( \mathcal{H}^{-1} : A \to V^* \) the inverse of the co-Legendre map; using coordinates, \( \mathcal{H}^{-1}(y^i) = (\mathcal{H}_j(y^i)) \). Then \( L : A \to \mathbb{R}, L(y^i) = y^i \mathcal{H}_j(y^i) - h_0(\mathcal{H}_j(y^i)) \), is an affine Lagrangian.

A surjective submersion \( E \xrightarrow{\pi} M \) is usually called a fibered manifold; the manifold \( E \) is called the total space, \( M \) is the base space and \( \pi \) is the (canonical) projection. If the projection is understood, the fibered manifold is denoted by \( E \). If \( x \in M \), the submanifold \( E_x = \pi^{-1}(x) \subset E \) is the fiber of \( \pi \) at \( x \). In general, the fibers need not to be all homeomorphic; for example the fibered manifold \( \pi_1 : E = \mathbb{R}^2 \setminus \{(0,0)\} = M, \pi_1(x,y) = y \) has not all the fibers connected.

A fibered manifold map (fmm) sends fibers in fibers: if \( \pi : E \to M \) and \( \pi' : E' \to M' \), then \( f : E \to E' \) is a fmm if \( \pi = \pi' \circ f \); if \( \pi_1 : E \to M \) and \( \pi' : E' \to M' \), then \( f : E \to E' \) is an fmm if there is an induced \( f_0 : M \to M' \) such that \( f_0 \circ \pi = \pi' \circ f \).
The local coordinates on M and E adapted to the submersion π are

\[
(x^i, y^a) \quad \text{on } M \quad \text{and} \quad (x^i, y^a) \quad \text{on } E,
\]

such that π has the local form \((x^i, y^a) \rightarrow (x^i)\).

A case when all the fibers are homeomorphic is that of a \textit{locally trivial fibration} \( E \xrightarrow{\pi} M \) with the \textit{fiber type} a manifold \( F \). In this case there is an open cover of \( M \) with sets \( U \) such that for every \( U \) there is a locally diffeomorphism \( \psi : \pi^{-1}(U) \rightarrow U \times F \). For example, a vector bundle is a locally trivial fibration with the fiber type a vector space and each of its fibers has an intrinsic structure of a vector bundle. In particular, the tangent and the contangent bundles of a manifold are vector bundles.

A locally trivial fibration \( A \xrightarrow{\pi} M \) is an \textit{affine bundle} if its fiber is modeled by a (real) affine space \( A_0 \) and the structural functions are affine transformations of \( A_0 \). A vector bundle is a particular case of an affine bundle. An affine bundle \( \pi : A \rightarrow M \) gives rise to the vector bundles \( \bar{\pi} : \bar{A} \rightarrow M \) (given by the director vector space in every point) and its dual vector bundle \( \bar{\pi}^* : \bar{A}^* \rightarrow M \), called the \textit{dual vector bundle} of the given affine bundle and usually denoted by \( \pi^* : A^* \rightarrow M \), or \( A^* \) for shortness.

Let \( \pi_1 : F \rightarrow M \) be an affine bundle with the affine line \( \mathbb{R} \) as typical fiber (i.e. with a one-dimensional fiber). The local coordinates on \( F \) change according to the rules

\[
\begin{align*}
(x^i) & \quad \text{on } M \\
(y^a) & \quad \text{on } E,
\end{align*}
\]

(2.3)

If \( \sigma = 1 \) and \( \tau = 0 \), then \( \pi_1 : F \rightarrow F = M \times \mathbb{R} \rightarrow M \) is the projection on the first factor, thus it is the trivial vector bundle. If only \( \sigma(x^i) = 1 \) (for every local chart), then the affine bundle is associated with the trivial vector bundle \( M \times \mathbb{R} \rightarrow M \); we say that the affine bundle \( F \) has \textit{structural translations}.

Let \( \pi : A \rightarrow M \) be an affine bundle and \( \pi_1 : F \rightarrow M \) be an affine bundle with a one-dimensional fiber. The \( F \)-dual of \( A \) is \( L(\bar{A}, \bar{F}) \), denoted by \( A^F \). The local coordinates on \( A^F \) change according to the rules

\[
\begin{align*}
x^i & = x^i(x^i) \\
y^a & = y^a(y^a) + \tau(x^i),
\end{align*}
\]

(2.4)

Let us consider \( \bar{F}_* \subset \bar{F} \), the fibered submanifold of the vector bundle \( \bar{\pi}_0 : \bar{F} \rightarrow M \), consisting in non-null vectors. Denote \( A = A \times_M \bar{E} \). The natural projection \( \bar{\pi} : \bar{A} \rightarrow \bar{F}_* \) is the canonical projection of an affine bundle. Let us denote also by \( \bar{F} = F \times_M \bar{F}_* \) and by \( \bar{\pi}_0 : \bar{F} \rightarrow \bar{F} \) the canonical projection.

**Proposition 1.** The projection \( \bar{\pi}_0 : \bar{F} \rightarrow \bar{F}_* \) is the canonical projection of an affine bundle with structural translation (i.e. the associated vector bundle is the trivial vector bundle \( M \times \mathbb{R} \rightarrow M \)).

**Proof.** The local coordinates on \( F \) change according to the rules (2.4). Let us denote by \((x^i, \bar{y})\) the local coordinates on \( \bar{F}_* \) and by \((x^i, \bar{y}, y)\) the local coordinates on \( \bar{F} \); such that \( \bar{\pi}_1 \) has the local form \((x^i, \bar{y}, y) \rightarrow (x^i, \bar{y})\). The local coordinates \( \bar{y} \) change according to the rule
\[ \dot{y}' = \sigma(x') \dot{y}. \]

Since on \( F \) one have \( y'' = \sigma' y' + \tau' = \sigma'(\sigma y + \tau) + \tau' = \sigma' \sigma y + (\sigma' \tau + \tau') \) and \( y'' = \sigma'' y + \tau'' \), it follows \( \sigma'' = \sigma' \sigma \) and \( \tau'' = \sigma' \tau + \tau' \). Also \( \dot{y}'' = \sigma' \dot{y}' = \sigma'' \dot{y} \). Consequently
\[
\frac{\dot{\tau}'}{\sigma' \dot{y}'} + \frac{\tau'}{\sigma \dot{y}} = \frac{\tau' + \sigma' \tau}{\sigma' \sigma \dot{y}} = \frac{\tau''}{\sigma' \sigma \dot{y}}. \]
Also denoting \( z = \frac{y}{y} \) on \( \tilde{F} = F \times_M \tilde{F}_* \), one have \( z'' = z' + \frac{\dot{\tau}}{\sigma \dot{y}} \), thus the conclusion follows. \( \square \)

If \( \pi : E \rightarrow M \) is a fibered manifold, its first jet space \( J^1 \pi \) can be regarded as an affine bundle \( J^1 \pi \rightarrow E \). Using local coordinates (2.3), adapted to the submersion, the coordinates on \( J^1 \pi \) have the form \((x^i, y^\alpha, \bar{y}^\alpha)\) and change according to the rules:

\[
\begin{align*}
\frac{dx^i}{dx^j} &= x'^i(x^j) \\
\frac{dy^\alpha}{dx^j} &= y'^\alpha(x^j, y^\alpha) \\
\frac{\bar{y}^\alpha}{\bar{x}^j} &= \bar{y}'^\alpha \frac{\partial y^\alpha}{\partial x^j} + \frac{\partial \bar{y}^\alpha}{\partial x^j}.
\end{align*}
\]

If \( s : M \rightarrow E \) is a section (it can be a local one), then it lifts to a section \( s' : M \rightarrow J^1 \pi \rightarrow E \). Using local coordinates, if \( s \) has the local form \((x^i) \rightarrow (x^i, s^\alpha(x^i))\), then \( s' \) is \((x^i) \rightarrow (x^i, s^\alpha(x^i), \frac{\partial s^\alpha}{\partial x^i})\).

The manifold \( J^1 \pi \tau = V^* E \otimes \pi^* TM \) is the total space of a vector bundle over \( E \). The local coordinates on \( J^1 \pi \tau \) have the form \((x^i, y^\alpha, p^i_\alpha)\). The change rule of \((x^i)\) and \((y^\alpha)\) is given by relations (2.6), while

\[
\begin{align*}
p^i_\alpha \frac{\partial y^\alpha}{\partial x^j} &= p^i_\alpha \frac{\partial x'^i}{\partial x^j}.
\end{align*}
\]

If \( E = M \times T \), where \( T \) is a manifold, then \( x'^i = x^i(x^i) \), \( y'^\alpha = y'^\alpha(y^\alpha) \) and the coordinates \((y^\alpha)\) on \( J^1 \pi \tau \) change in a tensor manner, thus \( J^1 \pi \tau = V^* E \otimes \pi^* TM \) is a vector bundle and \( J^1 \pi \tau \) is its dual vector bundle. This vector bundle is used in a systematic way in the study of multi-time Lagrangians and Hamiltonians (see [8] and the references therein). Another particular case, considered below, is when \( \pi_1 : F \rightarrow M \) is an affine bundle with a one dimensional fiber. In this case the formulas (2.6) have the form:

\[
\begin{align*}
x'^i &= x^i(x^i) \\
y' &= y\sigma(x^i) + \tau(x^i) \\
y_i \frac{\partial x'^i}{\partial x^j} &= y_i \sigma(x^i) + y \frac{\partial \sigma}{\partial x^i} + \frac{\partial \tau}{\partial x^i}.
\end{align*}
\]

If \( \pi_1 : F \rightarrow M \) is a vector bundle, then \( \tau = 0 \).

Let us suppose that \( \pi_1 : F \rightarrow M \) is an affine bundle with structural translations. If \((x^i)\) and \((x^i, y)\) are local coordinates on \( M \) and on \( F \) respectively, then the coordinates change according to the formula

\[ x'^i = x^i(x^i), \quad y' = y + f(x^i). \]

The first jet bundle \( J^1 \pi_1 \) has as coordinates \((x^i, y, u_i)\), where the coordinates \((u_i)\) change following the rule: \( u'_i = u_i + \frac{\partial f}{\partial x^i} \). There is an affine bundle \( \nu : F_1 \rightarrow M \)
such that $F_1$ has as coordinates $(x^i, u_i)$ and the affine bundle $J^1 \pi_1$ is canonically isomorphic with the induced bundle $\pi_1^* \nu$; we write $J^1 \pi_1 = \pi_1^* \nu$.

A section $s \in \Gamma(\pi_1)$ lifts naturally to a section $s' \in \Gamma(J^1 F_1 \to M)$, given locally by $(x^i) \to (x^i, s(x^i), \frac{\partial s}{\partial x^i})$. It induces a section $s'' \in \Gamma(\nu)$ and implicitly an affine section $s^J \in \Gamma(J^1 F \to F)$ having the local form $(x^i, y) \to (x^i, y, \frac{\partial s}{\partial x^i})$. The section $s^J$ defines a null curvature connection on the bundle $\pi : E \to M$. If a connection on $\pi : E \to M$ is defined by a section $\xi \in \Gamma(J^1 F \to F)$, $(x^i, y) \to (x^i, y, \xi_i(x^i, y^a))$, its curvature is locally given by $R_{ij} = \frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i}$. The curvature vanishes iff locally $\xi$ has the form $\xi = s^J$, i.e. it is a lift of a local section $s \in \Gamma(\pi_1)$.

We are going to prove that one can associate an affine bundle with a one-dimensional fiber and structural translations with every affine bundle with a one-dimensional fiber.

### 3 Lagrangians and Hamiltonians on affine bundles

Let $\pi : A \to M$ be an affine bundle and $\pi_1 : F \to M$ be an affine bundle with a one-dimensional fiber. An $F$-Lagrangian on $E$ is a fibered manifold map $L : A \to F$ (i.e. $\pi_1 \circ L = \pi$). Since every affine map induces a linear map on the director vector space, then there is a canonical projection $\Pi : Aff(A, F^*) \to A^*F$. An $F$-Hamiltonian on $E$ is a fibered manifold map $h : A^*F \to Aff(A, F^*)$ such that $\Pi \circ h = 1_{A^*F}$. For example, let us consider $F = M \times \mathbb{R}$ and $p_1 : M \times \mathbb{R} \to M$ be the projection on the first factor. The $F$-dual of $A$ is just $A^*$. An $F$-Lagrangian has the form $L(e) = (\pi(e), L_0(e))$, where $L_0 : A \to \mathbb{R}$ is usually called a Lagrangian. An $F$-Hamiltonian on $A$ has the form $h : A^* \to Aff(A, M \times \mathbb{R})$. This case was considered in the study of affine Hamiltonians of higher order (see [3]). Another more elaborated example, using jet spaces, is given in [6].

Then an $F$-Lagrangian $L$ has the local form $(x^i, y^a) \xrightarrow{L_0} (x^i, L_0(x^i, y^a))$ and the local functions $L_0$ change according to the rules given by (2.4):

\begin{equation}
L_0'(x^i, y^a') = L_0(x^i, y^a)\sigma(x^i) + \tau(x^i).
\end{equation}

Since

\[ \frac{\partial L_0}{\partial y^a} = \sigma \frac{\partial L_0}{\partial y^a'} \frac{\partial y^a'}{\partial y^a} = \sigma \frac{\partial L_0}{\partial y^a} \delta_{a a'} , \]

the formula $(x^i, y^a) \to (x^i, \frac{\partial L_0}{\partial y^a})$ defines a Legendre map $\mathcal{L} : A \to A^*F$ of $L$. The local form of a map $\Omega \in Aff(A, F^*)$ is $(y^a) \xrightarrow{\Omega} (y^a, p_a)$; then $\Pi(\Omega)$ has the local form $(y^a) \xrightarrow{\Pi(\Omega)} (y^a p_a)$.

There are also local forms of $\Pi : Aff(A, F^*) \to A^*F$ and of an $F$-Hamiltonian $h : A^*F \to Aff(A, F^*)$ given by $(p_a, p) \xrightarrow{h} (p_a, p_0)$ and by $(x^i, p_0) \xrightarrow{h} (x^i, p_0, h_0(x^i, p_0))$, respectively. The change rules of local coordinates are:

\[ \begin{pmatrix} p_a \\ p' \end{pmatrix} = \sigma \cdot \begin{pmatrix} p_0 \\ p' \end{pmatrix} \begin{pmatrix} a_{a'}^{a'} \\ 0 \\ 1 \end{pmatrix} , \]
or \((p_{a\alpha}, p') = \sigma' \cdot (p_{a\alpha}, p)\left( \begin{array}{cc} a_{\alpha}^\alpha & a^\alpha \\ 0 & 1 \end{array} \right)\), where \((a_{\alpha}^\alpha)^{-1}, \sigma = (\sigma')^{-1}\) and \(a^\alpha = -a^\epsilon a_{\alpha}^\epsilon\). Thus \(h_0(x^i, p_{a\alpha}) = \sigma^{-1}(x^i) \cdot (p_{a\alpha}(x^i) + h_0(x^i, p_{a\alpha}))\). Since

\[
p_{a\alpha} = \sigma p_{a\alpha} a_{\alpha}^\alpha,
\]

\[
\frac{\partial h_0}{\partial p_{a'}} = \frac{\partial h_0}{\partial y^\alpha} a^\alpha - a^\alpha = \frac{\partial h_0}{\partial p_{a'}} a_{\alpha}^\alpha - a^\alpha,
\]

it follows that \((x^i, p_{a\alpha}) \rightarrow (x^i, -\frac{\partial h_0}{\partial p_{a'}})\) defines a co-Legendre map \(H^* : A^{*F} \rightarrow A\) of \(h\).

A Lagrangian \(L : A \rightarrow F\) is \textit{regular} if its Legendre map is a local diffeomorphism; it is equivalent to say that the \textit{vertical hessian}, given by the local matrix

\[
\left( g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right)
\]

is non-singular. The Lagrangian is \textit{hyperregular} if its Legendre map is a (global) diffeomorphism.

If \(L : A \rightarrow F\) is an \(F\)-Lagrangian, then \(\tilde{L} : \tilde{A} \rightarrow \tilde{F}\) defined locally by \(\tilde{L}(x^i, y^\alpha, \tilde{y}) = \frac{L(x^i, y^\alpha)}{\tilde{y}}\) is an \(\tilde{F}\)-Lagrangian on \(\tilde{A}\). We say that \(\tilde{L}\) is the \textit{lift} of \(L\) from \(A\) to \(\tilde{A}\). It is easy to see that the following statement is true.

**Proposition 2.** The lift \(\tilde{L}\) is regular (hyperregular) iff \(L\) is regular (hyperregular).

Analogously, if \(h : A^{*F} \rightarrow Aff(A, F)\) is an \(F\)-Hamiltonian, then one can consider an \(\tilde{F}\)-Hamiltonian \(\tilde{h} : \tilde{A}^{*F} \rightarrow Aff(\tilde{A}, \tilde{F})\) defined by \(\tilde{h}(x^i, \tilde{y}, \tilde{p}_{a\alpha}) = \frac{1}{\tilde{y}} h(x^i, \frac{1}{\tilde{y}} \tilde{p}_{a\alpha})\). We say that \(\tilde{h}\) is the lift of \(h\) from \(A^{*F}\) to \(\tilde{A}\). It is easy to see that the following statement is also true.

**Proposition 3.** The lift \(\tilde{h}\) is regular (hyperregular) iff \(h\) is regular (hyperregular).

There are natural maps \(\Phi : A^* \times_{\bar{M}} \bar{F} = \bar{A}^* \rightarrow A^{*F}\) and \(\Psi : Aff(A, \bar{R}) \times_{\bar{M}} \bar{F} = Aff(\bar{A}, \bar{R}) \rightarrow Aff(A, F)\) given in local coordinates by

\[
\Phi : (x^i, \tilde{p}_{a\alpha}, \tilde{z}) \rightarrow (x^i, p_{a\alpha} = \tilde{z}^{-1} \tilde{p}_{a\alpha}),
\]

\[
\Psi : (x^i, \bar{p}_{a\alpha}, \bar{z}, \bar{p}) \rightarrow (x^i, p_{a\alpha} = \bar{z}^{-1} \bar{p}_{a\alpha}, \bar{z}^{-1} \bar{p}).
\]

One can see that considering the natural maps

\[
\tilde{\Pi} : Aff(\tilde{A}, \bar{R}) \rightarrow \tilde{A}^*,
\]

\[
\Pi : Aff(A, F) \rightarrow A^{*F},
\]

the following diagram

\[
\begin{array}{ccc}
Aff(\tilde{A}, \bar{R}) & \xrightarrow{\tilde{\Pi}} & \tilde{A}^* \\
\Psi \downarrow & & \downarrow \Phi \\
Aff(A, F) & \xrightarrow{\Pi} & A^{*F}
\end{array}
\]

is commutative. If \(\tilde{h}\) is the lift an \(F\)-Hamiltonian \(h\), then the diagram
we consider our purpose we consider also the induced vector bundles with one dimensional fibers $\pi$. We say that $\tilde{\pi}$ is commutative. 

Analogously, if $\tilde{h} : A^* \to Aff(A, \mathcal{F})$ is an affine Hamiltonian, then one can consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{A}^* \to Aff(\tilde{A}, \tilde{F}^*)$ defined by $\tilde{h}(x^i, \tilde{y}, \tilde{p}_\alpha) = \tilde{h}(x^i, \tilde{p}_\alpha)$. We say that $\tilde{h}$ is the lift of $\tilde{h}$ (from $A^*$ to $A$).

We consider below some examples.

Let $\pi : E \to M$ be a fibered manifold (or a bundle). The vector bundle $\Lambda^m(TM) \to M$, $m = \dim M$, has a one-dimensional fiber; it has as sections the top forms (or volume densities) on $M$. It is easy to see that $\Lambda^m(TM)^* = \Lambda^m(T^*M)$. For our purpose we consider also the induced vector bundles with one dimensional fibers $\pi_1 : F = \pi^*\Lambda^m(TM) \to E$, $\pi_1^* : F^* = \pi^*\Lambda^m(T^*M) \to E$. In this particular case, our $F$-Hamiltonian on $E$ is just a Hamiltonian considered in [1, 2, 3] as a section $h : J^1\pi^* F \to J^1\pi^+ F$ having the local form

$$\tilde{h}(x^i, y^\alpha, p_\alpha, \omega(x^i, y^\alpha, p_\alpha)).$$

The local coordinates $(p^i_\alpha)$ and the local functions $h$ change according to the rules

$$p^i_\alpha \phi' = \phi \frac{\partial y^i}{\partial y^\alpha}, \quad h' = \phi^{-1} \left( h + \phi \frac{\partial y^i}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^i} \right).$$

An affine Hamiltonian on $J^1\pi^+$ is a section $\tilde{h} : J^1\pi^+ \to J^1\pi^+$ and it has the local form

$$\tilde{h}(x^i, y^\alpha, p_\alpha) = (x^i, y^\alpha, p_\alpha, \tilde{h}(x^i, y^\alpha, p_\alpha)).$$

The local coordinates $(\tilde{p}_\alpha)$ and the local functions $\tilde{h}$ change according to the rules

$$p^i_\alpha \phi = \tilde{p}_\alpha \frac{\partial x^i}{\partial y^\alpha}, \quad h' = \tilde{h} + \tilde{p}_\alpha \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^i}.$$

We are going to put together $F$-Hamiltonians and affine Hamiltonians. In order to do this we consider $\tilde{F}$-Hamiltonians. In order to simplify notations and the exposition, we consider $F^*_\omega$ instead of $F_\omega$ previously. We denote by $\tilde{F} = F \times_M \tilde{F}^*_\omega$ and we use the canonical projection $\tilde{\pi}_0 : \tilde{F} \to \tilde{F}^*_\omega$. Also, $\tilde{J} = J^1\pi^+ F \times_M \tilde{F}^*_\omega$ and $\tilde{\pi} : \tilde{J} \to \tilde{E} = E \times_M \tilde{F}^*_\omega$ (a canonical projection of a fibered manifold). An $\tilde{F}$-Hamiltonian on $E$ is a section $h : \tilde{J}^* \to \tilde{J}$ that has the local form

$$\tilde{h}(x^i, y^\alpha, \omega, \tilde{p}_\alpha) = (x^i, y^\alpha, \omega, \tilde{p}_\alpha, \tilde{h}(x^i, y^\alpha, \omega, \tilde{p}_\alpha)).$$

The local functions $\tilde{h}$ change according to the rules $\tilde{h}' = \tilde{h} + \tilde{p}_\alpha \frac{\partial y^\alpha}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^i}$. As we have already seen, an $F$-Hamiltonian, as well as an affine Hamiltonian, lift both to an $\tilde{F}$-Hamiltonian. More specifically,

- if $h$ is an $F$-Hamiltonian that has the local form (3.2), then its lift $\tilde{h}$ is an $\tilde{F}$-Hamiltonian that has the local form (3.4), with $\tilde{h}(x^i, y^\alpha, \omega, \tilde{p}_\alpha) = \frac{1}{\omega} h(x^i, y^\alpha, \omega \tilde{p}_\alpha)$;
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– if \( \tilde{h} \) is an affine Hamiltonian that has the local form (3.3), then its lift \( \tilde{\tilde{h}} \) is an \( \tilde{F} \)-Hamiltonian that has the local form (3.4), with \( \tilde{\tilde{h}}(x^i, y^\alpha, \omega, \tilde{p}^\alpha_i) = \tilde{h}(x^i, y^\alpha, \tilde{p}^\alpha_i) \).

An important tool in the study of \( F \)-Hamiltonians (Hamiltonians in the classical terminology) can be found in the multi-symplectic formalism developed in [1, 2, 3] (see also the bibliography therein). In [1] one define the action of an \( F \)-Hamiltonian \( h \) on sections on \( E \rightarrow M \) and one deduce the equation of a critical section of this action, using a Hamilton-Jacobi principle. In [6] one defines an action for an \( \tilde{F} \)-Hamiltonian, in order to recover the same action for the lift of an \( F \)-Hamiltonian.

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