Finsler structures associated to control systems

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Abstract. We show how a control system induces a definite Finsler structure on a subset of the tangent bundle, such that the cost value of a curve, which is solution of the control system, is the length of the curve. In other words, Finslerian geodesics are solutions of the optimal control problem - curves, which decrease the cost. Further, we provide an original example of Finsler structures assigned to the control problem determined by a spacecraft motion in the gravitational field.


Key words: Finsler structure, geodesic, optimal control problem.

1 Introduction

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_xM$ the tangent space of $M$. Each point of $TM$ has the form $(x, y)$, where $x \in M, y \in T_xM$. Denote by $F$ a Finsler structure globally defined on $M$.

Consider a function $f : TM \to \mathbb{R}$ of class $C^\infty$ on $TM \setminus 0$, whose partial function $y \to f(x, y)$ has no critical point. Then, for each point $x \in M$, the set $I_x : f(x, y) = 0$ features a hypersurface of $T_xM$. On the other hand, there exists a function $F : TM \setminus 0 \to \mathbb{R}$, implicitly defined, which is $C^\infty$ and positive homogeneous of degree one with respect to $y$, such that $I_x : F(x, y) = 1$. Therefore $I_x$ becomes an indicatrix (the set of unit vectors). Applying the Okubo method [2], one derives that the function $F$ is a solution of implicit equation $\left( x, \frac{y}{F(x, y)} \right) = 0$.

The algorithm which provides a new Finsler structure was developed in [6] and relies on the following remark: let $V_x$ be a cone in $T_xM$, regarded as a subset of $T_xM$, which is invariant with respect to positive homotheties. If $f(x, y) = \begin{cases} f_1(x, y), & y \in V_x \subset T_xM \\ f_2(x, y), & y \in T_xM \setminus V_x \end{cases}$ has suitable properties, then we get a Finsler function $F = \sqrt{F_1^2 + F_2^2}$, which satisfies $f\left( x, \frac{y}{F(x, y)} \right) = 0, \forall y \in T_xM$ via
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\[ f_1 \left( x, \frac{y}{F_1(x,y)} \right) = 0, \quad \forall y \in V_z, \]
\[ f_2 \left( x, \frac{y}{F_2(x,y)} \right) = 0, \quad \forall y \in T_z M \setminus V_z. \]

2 Finsler structures induced by time optimal problems

Following the algorithm of [6], we will introduce the Finsler structure produced by time optimal problems.

To this aim, let us consider \( M \) an \( n \)-dimensional \( C^\infty \) manifold \((U, \eta, M)\) as control fiber bundle, and \((TM, \tau, M)\) the tangent bundle. \( M \) is called state manifold, and the components \( x^i, i = 1, n \) of the point \( x \in M \) are called state variables. Then \((x^i, u^a), i = 1, n, a = 1, k\) are adapted coordinates in \( U \), and \((x^i, y^i)\) are natural coordinates in \( TM \). The components \( u^a \) of the point \( u \) are called controls.

Let us consider a fibered mapping \( X: U \to TM, X = X^i(x,u) \frac{\partial}{\partial x^i} \) over the identity in the state manifold \( M \), which produces a continuous control system \( \frac{dx^i}{dt} = X^i(x,u), i = 1, n \), where \( t \) is the parameter of evolution. The evolution of the state manifold \( M \) is characterized by the image set \( S = \text{Im}(X) \subseteq TM \), which is described by the control equations \( x^i = x^i, y^i = X^i(x,u), i = 1, n \). Consider also the set \( S_x = \{ y \mid \exists u \in U_x = \eta^{-1}(x), y = f(x,u) \} \).

Having a cost functional \( \int X^0(x(t), u(t)) \, dt \), the optimal control theory requires allowed curves of the control system complying some boundary conditions and minimizing the cost functional. In this way, the Pontryagin maximum principle offers a set of necessary conditions for a curve \((x(t), u(t))\) to be optimal, using a Hamiltonian \( H(x, p_0, p, u) = p_0 X^0(x, u) + p_i X^i(x, u) \), where \((p_0, p_i)\) are called moment coordinates. The optimal curves \((x(t), u^*(t))\) have to satisfy:

- the control system, \( \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = X^i(x(t), u^*(t)); \)
- the adjoint differential equations,
  \[ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = \left( -p_0 \frac{\partial X^0}{\partial x^i} - p_j \frac{\partial X^j}{\partial x^i} \right) (x(t), u^*(t)); \]
- the maximally condition \( H(x, p_0, p, u^*) \geq H(x, p_0, p, u), \forall u \in U_z. \)

For a time optimal problem, we shall minimize the final time \( T \) (taking \( t = 0 \) as initial value) such that \( X^0(x, u) = 1 \) and \( H(x, p, u) = p_i X^i(x, u) - 1 \) (for the normal case \( p_0 = -1 \)). Therefore, the optimal curves take values on the subset \( H = 0 \) of \( W = U \times MT^*M \).

In order to define a Finsler structure \( F = \sqrt{F_1^2 + F_2^2} \), it is required to have a set of allowed directions, a cone bundle \( V \subset TM \), an indicatrix included in \( V_x \subset T_x M \), a homogeneous function \( F_1 \) on \( V_x \), an indicatrix included in \( T_x M \setminus V_x \) and a homogeneous function \( F_2 \) on \( T_x M \setminus V_x \).

Therefore, it is considered:
- the subset of possible optimal controls,
  \[ U_\ast^x = \{ (x, u_0), \exists (x, p) \in T^\ast M, \ H(x, p, u_0) \geq H(x, p, u), \ \forall u \in U_x \}; \]

- the indicatrix
  \[ S_{1x} = \{ y \in S_x, \text{ with } ty \not\in S_x, \ \forall t > 1 \}; \]

- the cone
  \[ V_x = \{ y \in T_xM, \ \exists y_0 \in S_{1x} \text{ such that } y = \lambda y_0, \ \lambda > 0 \}; \]

- the positive homogeneous functions of degree one
  \[ F_{1x}: V_x \to \mathbb{R}_+; \quad F_{1x}(y) = \lambda, \quad \text{where } y = \lambda y_0, \ y_0 \in S_{1x} \]

(\lambda is the factor between \( y \) with \( y \in V_x \) and \( y_0 \) with \( y_0 \in S_{1x} \) in the same ray).

If there exists a suitable function \( f_{1x}: V_x \to \mathbb{R} \) with \( S_{1x} = f_{1x}^{-1}(0) \), then \( F_1 \) is a solution of \( f_1 \left( x, \frac{y}{F_1(x, y)} \right) = 0 \). In this context, an optimal curve has the cost

\[
T = \int_0^T dt = \int_{\tau_1}^{\tau_2} \frac{dt}{d\tau} d\tau = \int_{\tau_1}^{\tau_2} \lambda(\tau)d\tau = \int_{\tau_1}^{\tau_2} f(x(\tau), y(\tau)) d\tau
\]

(Finslerian length of the curve);

- an arbitrary indicatrix \( S_{2x}: f_2(x, y) = 0 \) in \( T_xM \setminus V_x \) and the positive homogeneous function of degree one \( F_2(x, y) \) produced via the equation

\[
f_2 \left( x, \frac{y}{F_2(x, y)} \right) = 0.
\]

### 3 Finsler structures associated to an optimal control problem

In the following we give an original example of getting a Finsler structure assigned to an optimal control problem.

We consider a spacecraft whose motion is influenced by central gravity field. Its motion equations are:

\[
\ddot{\vec{r}} = -\frac{\mu}{r^3} \vec{r} + \frac{\vec{E}}{m},
\]

where:

- \( \vec{r} \) is the position vector of the spacecraft from the center and \( r = |\vec{r}| \) is the distance of the spacecraft from the center of gravity;
- \( \mu = GM \) is the gravitational parameter of the central body, where \( G \) is the universal gravity constant and \( M \) is the mass of the central body;
- \( \vec{E} \) is the non-gravitational control force;
- \( m \) is the mass of the spacecraft, which is assumed constant.

First we fix the origin of the inertial frame at the center of gravity and introduce another coordinate frame, which is rotating along a circular orbit at a constant angular velocity:
\(- \delta \vec{r} = (xyz)^T\) is the position vector of the spacecraft whose components represent radial, tangential and normal displacements from the origin of the rotating frame, respectively;

\(- \vec{R} = \vec{R}_i\) is the position vector of the origin of the rotating frame from the origin of the inertial frame;

\(- \vec{\omega} = \omega \hat{k}\) is the constant angular velocity equal to the mean motion of the circular orbit;

\(- \vec{u} = \frac{\vec{F}}{m} = u_x i + u_y j + u_z k\) is the control acceleration whose components are given in the rotating frame.

The position vector has the expression

\[ \vec{r} = \vec{R} + \delta \vec{r} = (R + x)i + yj + zk. \]

Using the formula ([1])

\[ \dot{\vec{r}} = (\dot{\vec{R}} + \dot{x})i + \dot{y}j + \dot{z}k + \vec{\omega} \times \vec{r}, \]

we find the velocity vector expression

\[ \dot{\vec{r}} = (\dot{x} - \omega y)i + (\dot{y} + \omega (R + x))j + \dot{z}k. \]

Using the formula ([1])

\[ \ddot{\vec{r}} = (\ddot{x} - \omega y - \omega^2 (R + x))i + (\ddot{y} + \omega (R + x))j + \dot{z}k + \vec{\omega} \times \vec{\omega} \times \vec{r} + \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \]

we derive that the acceleration vector has the expression

\[ \ddot{\vec{r}} = (x - 2 \omega \dot{y} - \omega^2 (R + x))i + (\dot{y} + 2 \omega \dot{x} - \omega^2 y)j + \dot{z}k. \]

From Newton’s equation we get the following component-wise equations of motion in the rotating frame:

\[ \ddot{x} - 2 \omega \dot{y} - \omega^2 (R + x) = -\frac{\mu}{r^3} (R + x) + u_x \]
\[ \ddot{y} + 2 \omega \dot{x} - \omega^2 y = -\frac{\mu}{r^3} y + u_y \]
\[ \ddot{z} = -\frac{\mu}{r^3} z + u_z, \]

where \(r = \sqrt{(R + x)^2 + y^2 + z^2}. \)

If non-dimensionalized with reference length \(R\) and reference time \(\frac{1}{\omega}\), we get the simplified formula

\[ \ddot{x} - 2 \dot{y} + (1 + x) \left( \frac{1}{r^2} - 1 \right) = u_x \]
\[ \ddot{y} + 2 \dot{x} + y \left( \frac{1}{r^2} - 1 \right) = u_y \]
\[ \ddot{z} + \frac{1}{r^2} z = u_z, \]

where, this time, \(r = \sqrt{(1 + x)^2 + y^2 + z^2}. \)
We further define the states in the following way: \( \vec{x} = (x_1, x_2, x_3, x_4)^T = (xy\dot{y})^T \) and get the equations of planar motion in the state space, of the form

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 
\end{pmatrix} = \begin{pmatrix}
x_3 \\
x_4 \\
2x_2 - (1 + x_1) \left( \frac{1}{r_3} - 1 \right) + u_x \\
-2x_3 - x_2 \left( \frac{1}{r_3} - 1 \right) + u_y 
\end{pmatrix},
\]

where \( r = \sqrt{(x_1 + 1)^2 + x_2^2} \).

We specify that in [4] the Taylor series expansion was employed, in order to define a dynamic system from gravity center field against which a cost quadratic function is minimized.

Further on, let us consider the planar motion such that \( u_x = u_1 \cos u_2 \), \( u_y = u_1 \sin u_2 \), \( u_1 \in [0, 1] \), \( u_2 \in [0, 2\pi] \).

This new system may be regarded as a control system whose equations are

\[
\begin{align*}
y^1 &= x_3 \\
y^2 &= x_4 \\
y^3 &= 2x_4 - (1 + x_1) \left( \frac{1}{r_3} - 1 \right) + u_1 \cos u_2 \\
y^4 &= -2x_3 - x_2 \left( \frac{1}{r_3} - 1 \right) + u_1 \sin u_2.
\end{align*}
\]

**Remark.** In the plane \( y^3Oy^4 \) consider the circle with the center at \( C(\lambda, \mu) \) with \( \lambda > 1 \) and radius 1.

As \( \lambda > 1 \), it results that \( \lambda^2 + \mu^2 > 1 \), hence \( O \) is outside the circle.

The equations of the tangents \( d_1 \) and \( d_2 \), carried out from \( O \) to this circle are

\[
d_1: \quad y^4 = \frac{\lambda \mu - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \cdot y^3
\]

\[
d_2: \quad y^4 = \frac{\lambda \mu + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \cdot y^3.
\]

If we denote by \( T_1 \) and \( T_2 \) the crossing points of the tangents \( d_1 \) and \( d_2 \) respectively with the considered circle, then equation of the line \( T_1T_2 \) is

\[
\mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) = 0.
\]

Then the half plane determined by \( T_1T_2 \) which does not contain \( O \), is described by

\[
\mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) \geq 0.
\]

The cone between \( d_1 \) and \( d_2 \) for which \( y^3 > 0 \), is defined by

\[
\begin{align*}
\frac{\lambda \mu - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \leq \frac{y^4}{y^3} \leq \frac{\lambda \mu + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1}, \\
y^3 > 0.
\end{align*}
\]
Comming back to the control system took into consideration, we denote
\[
\lambda = 2x_4 - (1 + x_1) \left( \frac{1}{x_3} - 1 \right) \quad \text{and} \quad \mu = -2x^3 - x_2 \left( \frac{1}{x_3} - 1 \right).
\]

Suppose \( \lambda > 1 \), which is equivalent to
\[
2x_4 - (1 + x_1) \left( \frac{1}{x_3} - 1 \right) > 1.
\]

Consider the sets
\[
S: \quad \{ (y^3 - \lambda)^2 + (y^4 - \mu)^2 \leq 1 \} \quad \lambda > 1
\]
and
\[
S_1: \quad \begin{cases} 
(y^3 - \lambda)^2 + (y^4 - \mu)^2 = 1 \\
\mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) \geq 0 \\
\lambda > 1.
\end{cases}
\]

The cone \( V_x \) is defined as in the remark,
\[
\begin{cases}
\frac{\lambda u - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \leq \frac{y^4}{y^3} \leq \frac{\lambda u + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \\
y^3 > 0 \\
\lambda > 1.
\end{cases}
\]

We look for a positive and homogeneous solution of the equation
\[
\left( \frac{y^3}{F_1} - \lambda \right)^2 + \left( \frac{y^4}{F_1} - \mu \right)^2 = 1,
\]
with respect to \( F_1 \).

Equivalently,
\[
(\lambda^2 + \mu^2 - 1)F_1^2 - 2F_1(\lambda y^3 + \mu y^4) + (y^3)^2 + (y^4)^2 = 0,
\]
whose discriminant \( \Delta = 4(y^3)^2 \left[ \left( \frac{y^4}{y^3} \right)^2 (1 - \lambda^2) + 2\lambda \mu \frac{y^4}{y^3} + 1 - \mu^2 \right] \) is positive for points inside the cone and which additionally check condition \( \lambda > 1 \).

The convenient positive homogeneous solution is
\[
F_1(y^3, y^4) = \frac{(y^3)^2 + (y^4)^2}{\lambda y^3 + \mu y^4 + \sqrt{(y^3)^2 + (y^4)^2} + 2\lambda \mu y^3 y^4 - \lambda^2 (y^4)^2 - \mu^2 (y^4)^2}.
\]

On the set \( T_x M \setminus V_x \) we have built in a general manner a first degree positive and homogeneous function \( F_2(y^3, y^4) \); finally, \( F = \sqrt{F_1^2 + F_2^2} \) is a first degree homogeneous function on \( T_x M \).
References


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