Subnormality and moment problems

Anca Manuela Zlătescu and Luminiţa Lemnete-Ninulescu

Abstract. In this note, in Section 1, we give a necessary and sufficient condition on a commuting multioperator to admit a commuting normal extension having the support in a prescribed set. In Section 2, we give a necessary and sufficient condition for an operator-valued sequence to be an operator valued moment sequence.

M.S.C. 2000: 49M15, 26A09.
Key words: commuting subnormals, commuting normal extension, the support of a measure, positive operator-valued measure, self-adjoint operator.

1 Introduction and preliminaries

The connection between the moment problem and subnormality was the subject of many papers [2, 5]. In this note, in Section 1, we refine some result from [5].In Section 2, we give a necessary and sufficient condition on an operator-valued sequence to be a moment sequence.The condition obtained in operators is the same with the Choquet's condition in scalar case.

Definition. Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of bounded, linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *subnormal* if and only if, there exists a normal operator N on some Hilbert space $\mathcal{K} \supset \mathcal{H}$, such that:

$$N(\mathcal{H}) \subseteq \mathcal{H} \quad \text{and} \quad N|_{\mathcal{H}} = T.$$

For n commuting subnormals $(T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})$ (with $T_iT_j = T_jT_i$, $\forall 1 \leq i, j \leq n$), the operator tuple $(T_1, \dots, T_n)$ is said *to have commuting normal extension* if and only if there exist commuting normals $(N_1, \dots, N_n)$ all defined on some $\mathcal{K} \supset \mathcal{H}$ such that:

$$N_i|_{\mathcal{H}} = T_i, \quad 1 \leq i, j \leq n \quad \text{and} \quad N_i(\mathcal{H}) \subseteq \mathcal{H}.$$

Let

$$\mathcal{P} = \left\{ \sum_{\alpha \in H \text{ finite} \subset \mathbf{R}_+^n} a_\alpha x^\alpha, \ a_\alpha \in \mathbf{R}, \ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \right\}$$

Proceedings of The 4-th International Colloquium "Mathematics in Engineering and Numerical Physics" October 6-8 , 2006, Bucharest, Romania, pp. 209-213.

be the *real vector space of polynomials* in n-real variables with real coefficients.

For $g \in \mathcal{P}$, $g = \sum_{\alpha \in H} a_\alpha x^\alpha$, $a_\alpha \in \mathbf{R}$ we associate the complex polynom:

$$\tilde{g}(z,w) = \sum_{\alpha \in H \subset \mathbf{R}_+^n} a_\alpha (zw)^\alpha, \quad \text{with} \quad (z,w) \in \mathbf{C}^n \times \mathbf{C}^n.$$

It is clear, that for all $z \in \mathbf{C}^n$ we have:

$$\tilde{g}(z,\bar{z}) = \sum_{\alpha \in H} a_\alpha |z|^{2\alpha} \quad \text{so that} \quad \tilde{g}(z,\bar{z}) \in \mathbf{R}.$$

Now, let $g = (g_1, \ldots, g_n), g_i \in \mathcal{P}, \forall 1 \le i \le n$ and the sets

$$W_g = \{x \in \mathbf{R}^n,\ \sum a_\alpha^i x^{2\alpha} \ge 0,\ 1 \le i \le n,\ \text{when}\ g_i(x) = \sum_{\alpha \in H} a_\alpha^i x^\alpha\}$$

and

$$\tilde{W}_{\tilde{g}} = \{x \in \mathbf{C}^n,\ g_i(z,\bar{z}) \ge 0,\ \forall 1 \le i \le n\}$$

and we consider also $||T_i|| \le 1$, $\forall 1 \le i \le n$.

2 Existence of normal exension with prescribed support

With the above notation we have the following theorem.

Theorem. *The commuting multioperator $T = (T_1, \ldots, T_n)$ admits commuting normal extension $N = (N_1, N_2, \ldots, N_n) \in (\mathcal{B}(\mathcal{K}))^n$ with the joint spectrum supported on $\tilde{W}_{\tilde{g}} \cap \mathbf{D}^n$ if and only if there exists a positive operator-valued measure ρ with the support on $(\mathbf{D}^n \cap \mathbf{R}_+^n) \cap \tilde{W}_{\tilde{g}} = [0,1]^n \cap W_g$ such that*

$$T^{*J} T^J = \int_{[0,1]^n \cap W_g} t^{2J} d\rho(t)$$

for any multiindices $J = (j_1, \ldots, j_n) \in \mathbf{Z}_+^n$.

Proof. Let $N = (N_1, \ldots, N_n)$ the commuting normal extension of the multioperator $T = (T_1, \ldots, T_n)$, $N_i : \mathcal{K} \to \mathcal{K}$, $\mathcal{K} \supset \mathcal{H}$ and $N_i|_{\mathcal{H}} = T_i$, $\forall 1 \le i \le n$ and its spectral joint measure E with the support lies on $\mathbf{D}^n \cap \tilde{W}_{\tilde{g}}$.

We define a spectral measure F on $\mathbf{R}^n$ by:

$$F(S) = E(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, \ldots, r_n e^{i\theta_n}),$$

with $(r_1, r_2, \ldots, r_n) \in S$ and $(\theta_1, \theta_2, \ldots, \theta_n) \in [0, 2\pi)^n$.

If $\tau : \mathbf{C}^n \to \mathbf{R}^n$, $\tau(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|)$ then $F(S) = E(\tau^{-1}(S))$.

In this case, supp $F \subset [0,1]^n \cap W_g$ and according to the theorem of changing the variable in an operator-valued integral, we have the equalities:

$$N^{*J} N^J = \int_{\mathbf{D}^n \cap \tilde{W}_{\tilde{g}}} |z|^{2J} dE(z) = \int_{[0,1]^n \cap W_g} r^{2J} dF(r), \quad \text{for any} \quad J \in \mathbf{Z}_+^n\ .$$

Let P be the projection operator from $\mathcal{K}$ on $\mathcal{H} \subset \mathcal{K}$. We have for all $x \in \mathcal{H}$:

$$\langle T^{*J}T^J x, x\rangle_{\mathcal{H}} = \langle PN^{*J}N^J x, x\rangle_{\mathcal{H}} = \int_{[0,1]^n \cap W_g} r^{2J} d(PF(r)x, x).$$

Let ρ be the positive operator-valued measure on $\mathcal{H}$ defined by $\rho(S) = PF(S)$. By a standard polarization argument we have:

$$T^{*J}T^J = \int_{[0,1]^n \cap W_g} r^{2J} d\rho(r).$$

Conversely, we suppose there exists a positive operator valued measure $\rho : \text{Bor}\,([0,1]^n \cap W_g) \to \mathcal{A}(\mathcal{H})$ with the support on $[0,1]^n \cap W_g$, such that

$$T^{*J}T^J = \int_{W_g \cap [0,1]^n} r^{2J} d\rho(r), \quad \forall J \in \mathbf{Z}_+^n .$$

Using Ito's criteria for subnormality, we have:

$$\sum_{I,J} \langle T^{I+J}x_I, T^{I+J}x_J\rangle_{\mathcal{H}} = \sum_{I,J} \langle T^{*I+J}, T^{I+J}x_I, x_J\rangle_{\mathcal{H}} =$$

$$= \int_{[0,1]^n \cap W_g} d\left(\sum_{I,J} \rho(t)t^I x_I, t^J x_J\right) =$$

$$= \int_{[0,1]^n \cap W_g} d\left(\langle \sum_I \rho^{1/2}(t)t^I x_I, \sum_J \rho^{1/2}(t)t^J x_j\rangle\right) \geq 0$$

Thus $T = (T_1, \dots, T_n)$ admits commuting normal extension.

In this case it exists $\mathcal{K} \supset \mathcal{H}$ and normals $N_i : \mathcal{K} \to \mathcal{K}$, $N_i\mathcal{H} \subseteq \mathcal{H}$ and $N_i|_{\mathcal{H}} = T_i$, with

$$N_i^{*J}N_i^J = \int_{D^n} t^{2J} dE(t), \quad \forall J \in \mathbf{Z}_+^n .$$

Following the same steps as in the direct implication proof and constructing

$$\tilde{F}(S) = E \circ \tau^{-1}(S),$$

we have also

$$T^{*J}T^J = \int_{[0,1]^n} t^{2J} d\rho_1(t),$$

with $\rho_1(t) = P\tilde{F}$.

From both representations with respect to spectral measures,

$$T^{*J}T^J = \int_{[0,1]^n} t^{2J} d\rho_1(t) \quad \text{and} \quad T^{*J}T^J = \int_{[0,1]^n \cap W_g} t^{2J} d\rho(t)$$

and from the uniqueness of the integral representation of a self-adjoint operator with respect to a spectral measure, we have $\rho_1 = \rho$ and $\operatorname{supp} E \subset D^n \cap \tilde{W}\tilde{g}$.

So, the proof is complete.

3 Operator-valued moment sequence

G. Choquet [1] has given a solution of the 1-dimensional moment problem on $\mathbf{R}$ (or on a compact sets in $\mathbf{R}$) using the notation of adapted spaces.

This result is famous:

Theorem. *Let $\{a_n\}_n$ be a sequence of real numbers. Then, there exists a positive Radon measure on $\mathbf{R}$ (or on compacts sets $K \subset \mathbf{R}$) such that:*

$$a_n = \int_{\mathbf{R}} t^n d\mu(t), \quad \forall n \in \mathbf{N}$$

if and only if the sequence $\{a_0, \dots, a_n\}$ satisfies the conditions

$$\sum_{i,j=0} \alpha_i \alpha_j a_{i+j} \geq 0,$$

for any finite family of reals $\{\alpha_0, \dots, \alpha_n\} \subset \mathbf{R}$.

Using this result, we state the following theorem:

Theorem. *Let $\{A_\alpha\}_\alpha$ be a sequence of bounded self adjoint operators on a complex Hilbert space with $A_0 = Id$. There exists an operator-valued measure $E : Bor(\mathcal{K}) \to \mathcal{A}(\mathcal{H})$ such that $A_\alpha = \int_K t^\alpha dE(t)$, $\forall \alpha \in \mathbf{N}$ with K compact $\subset \mathbf{R}$ if and only if we have $\sum_{i,j=0}^{n} \alpha_i \alpha_j A_{i+j} \geq 0$, for any finite reals $\{\alpha_0, \dots, \alpha_n\}$.*

Proof. If $\sum_{i,j=0}^{n} \alpha_i \alpha_j A_{i+j} \geq 0$ for any $\{\alpha_0, \dots, \alpha_n\} \in \mathbf{R}$, then for any $x \in \mathcal{H}$, we have

$$\sum_{i,j=0}^{n} \alpha_i \alpha_j \langle A_{i+j}, x, x\rangle \geq 0.$$

If we denote with $a_i^x = \langle A_i, x, x\rangle$ from Choquet theorem, for all $n \in \mathbf{N}$, there exists a positive scalar Radon measure

$$\mu_x : \text{Bor}\,(\mathcal{K}) \to \mathbf{R}_+$$

such as:

$$\langle A_i, x, x\rangle = a_i^x = \int_K t^i d\mu_x(t).$$

Let $\langle \rho(A)x, x\rangle_{\mathcal{H}} = \mu_x(A)$ for any $A \in \text{Bor}\,(\mathcal{K})$. By a standard polarization results:

$$\langle \rho(A)x, y\rangle = \frac{1}{4} \sum i^k \mu_{x+i^k y}(A) = \langle x, \rho(A)y\rangle \Rightarrow \rho(A) \in A(\mathcal{H}).$$

Owing to of Riesz representation theorem ($C^*(\Omega) \simeq M(\Omega)$) and the uniqueness of the representation measure, $\rho(S_1 \cap S_2) = \rho(S_1)\rho(S_2)$ and $\rho(K) = Id = A_0$, and thus $\rho : \text{Bor}\,(\mathcal{K}) \to \mathcal{A}(\mathcal{H})$ is a spectral measure and we have the representation

$$A_n = \int_K t^n d\rho(t), \quad \forall n \in \mathbf{N}.$$

The converse is straightforward.

References

[1] N.I.Akhiezer, *The Classical Moment Problem*, Oliver&Boyd, Edinburgh, 1965 .

[2] Ameer Athavale, Steen Pedersen, *Moment Problems and subnormality*, Journal of Math. Analysis and App. 146 (1990), 431-441.

[3] Lemnete Luminiţa, *Multidimensional Moment Problems in complex spaces*, Rev. Roumaine Math. Pures et Appl., 39 (1994), 905-906.

[4] Lemnete Luminiţa, *A Multidimensional Moment Problem on the unit polydisc*, Rev. Rev. Roumaine Math. Pures et Appl., 39 (1994), 9, 911-914.

[5] Arthur Lubin, *Weighted shifts and commuting normal extension*, Journal Austral. Math. Soc. 27 (1979), 17-26.

Authors' address:

Anca Manuela Zlătescu and Luminiţa Lemnete-Ninulescu
University "Politehnica" of Bucharest, Splaiul Independentei 313,
RO-060042, Bucharest, Romania.
e-mail: ancazlatescu@easynet.ro, luminita_lemnete@yahoo.com