State-space representations of (q, r)-d hybrid systems

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Abstract. Different types of multidimensional continuous-discrete systems are considered and their state-space representations are given. For the Attasi-type hybrid model the general response formula is obtained by solving a multiple differential-difference equation. The asymptotic stability of this model is analyzed and a generalized Liapunov equation is presented to provide a stability criterion.

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1 Introduction

The study of the multidimensional (nD) systems has developed in the past three decades, due to the importance of their applications in various domains, such as control and image processing, computer tomography, geophysics, seismology, etc.

The beginning of this domain is represented by the study of 2D systems, introduced by Roesser [15], Fornasini-Marchesini [4] , Attasi [1], Eising [3] et al. Lately, several papers have been published, which have been devoted to the study of hybrid systems, whose dynamic depends on two variables, one continuous and the other discrete [6], [8], [9], [12], [13], [14], [11]. They were used as models in the study of linear repetitive processes [2], [5], [16] with practical applications in long-wall coal cutting and metal rolling or in iterative learning control synthesis [10].

The aim of this paper is to extend this study to multidimensional hybrid systems, having $q \geq 1$ continuous variables and $r \geq 1$ discrete ones.

Section 2 contains the description of the state space representation of different models of hybrid systems, including systems with multi-time delay and descriptor systems. The paper focusses on the Attasi-type model, whose structure allows us to obtain some results which generalize the corresponding theorems for classical (1D) continuous or discrete systems.

A multiple differential-difference equation is solved and its solution is applied to derive the formulas of the state and of the output of the system.

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In Section 3 the concept of asymptotic stability is extended to these systems. A suitable generalized Liapunov function is provided and necessary and sufficient conditions of asymptotic stability are obtained.

2 State space representations

We shall use the following notations: $q \in \mathbf{N}$ and $r \in \mathbf{N}$ being the number of continuous and discrete variables respectively, a function $x(t_1, \dots, t_q; k_1, \dots, k_r)$, $t_i \in \mathbf{R}$, $k_i \in \mathbf{Z}$ will be sometimes denoted by $x(t;k)$, where $t = (t_1, \dots, t_q)$, $k = (k_1, \dots, k_r)$; $\overline{m}$ with $m \in \mathbf{N}^*$ denotes the set $\{1, 2, \dots, m\}$ and $\mathcal{P}(\overline{m})$ the family of all subsets of $\overline{m}$.

The (q,r)-D *hybrid Roesser type model* has the state space representation

$$\begin{bmatrix} \mathrm{D}_t x^c(t;k) \\ \sigma_k x^d(t;k) \end{bmatrix} = A \begin{bmatrix} x^c(t;k) \\ x^d(t;k) \end{bmatrix} + Bu(t;k) \tag{2.1}$$

$$y(t;k) = C \begin{bmatrix} x^c(t;k) \\ x^d(t;k) \end{bmatrix} + Du(t;k) \tag{2.2}$$

where $t = (t_1, \dots, t_q) \in \mathbf{R}_+^q$, $k = (k_1, \dots, k_r) \in \mathbf{Z}_+^r$,

$$x^c(t;k) = \begin{bmatrix} x_1^c(t;k) \\ \dots \\ x_q^c(t;k) \end{bmatrix}, \ x^d(t;k) = \begin{bmatrix} x_1^d(t;k) \\ \dots \\ x_r^d(t;k) \end{bmatrix}, \ x_i^c(t;k) \in \mathbf{R}^{n_{c_i}}, \ n_{c_i} \in \mathbf{N},$$

$i = \overline{1,q}$, $x_j^d(t;k) \in R^{n_{d_j}}$, $n_{d_j} \in \mathbf{N}$, $j = \overline{1,r}$, $u(t;k) \in \mathbf{R}^m$, $y(t;k) \in \mathbf{R}^p$; the operators D_t and σ_k are defined by

$$\mathrm{D}_t x^c(t;k) = \begin{bmatrix} \frac{\partial}{\partial t_1} x_1^c(t;k) \\ \frac{\partial}{\partial t_2} x_2^c(t;k) \\ \dots \\ \frac{\partial}{\partial t_q} x_q^c(t;k) \end{bmatrix}, \quad \sigma_k x^d(t;k) = \begin{bmatrix} x_1^d(t; k_1+1, k_2, \dots, k_r) \\ x_2^d(t; k_1, k_2+1, \dots, k_r) \\ \dots \\ x_r^d(t; k_1, k_2, \dots, k_r+1) \end{bmatrix}.$$

The real matrices A, B, C, D have the dimensions $n \times n$, $n \times m$, $p \times n$, $p \times m$ respectively, where $n = \sum_{i=1}^{q} n_{c_i} + \sum_{j=1}^{r} n_{d_j}$.

To introduce the (q,r)-D hybrid Fornasini-Marchesini type system we shall use the following notations:

If $\tau = \{i_1, \dots, i_l\}$ is a subset of $\overline{m}$, $|\tau| := l$ and $\tilde{\tau} := \overline{m} \setminus \tau$; for $i \in \overline{m}$, $\tilde{i} := \overline{m} \setminus \{i\}$ and $\tilde{\tilde{i}} := \{i+1, \dots, m\}$. The notation $(\tau, \delta) \subset (\overline{q}, \overline{r})$ means that τ and δ are subsets of $\overline{q}$ and $\overline{r}$ respectively and $(\tau, \delta) \neq (\overline{q}, \overline{r})$. For $\tau = \{i_1, \dots, i_l\}$ and $\delta = \{j_1, \dots, j_h\}$ the operators $\dfrac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial\tau}x(t;k) = \frac{\partial^l}{\partial t_{i_1}\dots\partial t_{i_l}}x(t;k),\ \sigma_\delta x(t;k) = x(t;k+e_\delta)$$

where $e_\delta = e_{j_1} + \ldots + e_{j_h}$, $e_j = (\underbrace{0,\ldots,0}_{j-1},1,0,\ldots,0) \in \mathbf{R}^r$; when $\tau = \overline{q}$ and $\delta = \overline{r}$ we denote $\partial/\partial\tau = \partial/\partial t$ and $\sigma_\delta = \sigma$.

If A_i, $i = \overline{m}$ is a family of matrices, $\sum_{i\in\varnothing} A_i = 0$ and $\prod_{i\in\varnothing} A_i = I$.

If $\tau = \varnothing$ and $\delta = \varnothing$ then $\frac{\partial}{\partial\tau}x(t;k) = x(t,k)$ and $\sigma_\delta x(t;k) = x(t;k)$. $A_{\varnothing,\varnothing}$, $A_{\tau,\varnothing}$, $A_{\varnothing,\delta}$, $A_{\tau,\delta}$ stand for some matrices $A_0, A_{i_1,\ldots,i_l;0}, A_{0;j_1,\ldots,j_n}, A_{i_1,\ldots,i_l;j_1,\ldots,j_n}$.

The *hybrid Fornasini-Marchesini type model* has the state-space representation

$$\frac{\partial}{\partial t}\sigma x(t;k) = \sum_{(\tau,\delta)\subset(\overline{q},\overline{r})} A_{\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta x(t;k) + \sum_{(\tau,\delta)\subset(\overline{q},\overline{r})} B_{\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta u(t;k) \tag{2.3}$$

$$y(t;k) = Cx(t;k) + Du(t;k) \tag{2.4}$$

where $A_{\tau,\delta}$, $B_{\tau,\delta}, C$ and D are $n\times n$, $n\times m$, $p\times n$ and $p\times m$ real matrices.

The descriptor models for the three types of systems have the state-equations similar to (2.1) and (2.3), but with the left hand member replaced respectively by

$$E_1\begin{bmatrix} \mathrm{D}_t x^c(t,k) \\ \sigma_k x^d(t,k)\end{bmatrix},\quad E_2\frac{\partial}{\partial t}\sigma x(t;k),$$

where E_1 and E_2 are rectangular or square singular matrices of appropriate dimensions. If the matrices E_1 and E_2 are square singular, then these systems are called regular.

Now, let us consider the constant delay times

$$a = (a_1,\ldots,a_q) \in \mathbf{R}^q_+ \quad\text{and}\quad b = (b_1,\ldots,b_q) \in \mathbf{Z}^r_+\ .$$

We denote by $x(t-a;k-b)$ the vector $x(t_1-a_1,\ldots,t_q-a_q;k_1-b_1,\ldots,k_r-b_r)$.

The (q,r)-D hybrid Roesser type model with time delay has the state space representation

$$\begin{aligned} E_1\begin{bmatrix}\frac{\partial}{\partial t}x^c(t;k)\\ \sigma x^d(t;k)\end{bmatrix} &= A_0\begin{bmatrix}x^c(t;k)\\ x^d(t;k)\end{bmatrix} + A_1\begin{bmatrix}x^c(t-a;k-b)\\ x^d(t-a;k-b)\end{bmatrix} + \\ &+\ B_0u(t;k) + B_1u(t-a;k-b)\end{aligned} \tag{2.5}$$

$$\begin{aligned} y(t;k) &= C_0\begin{bmatrix}x^c(t;k)\\ x^d(t;k)\end{bmatrix} + C_1\begin{bmatrix}x^c(t-a;k-b)\\ x^d(t-a;k-b)\end{bmatrix} + \\ &+\ D_0u(t;k) + D_1u(t-a;k-b)\end{aligned} \tag{2.6}$$

where $E_1 = I_n$ for the standard system and E_1 is a singular matrix for the descriptor model.

The $(q,r)-D$ hybrid Fornasini-Marchesini type model with time delays has the representation

$$\begin{aligned} E_2\frac{\partial}{\partial t}\sigma x(t;k) &= \sum_{(\tau,\delta)\subset(\bar q,\bar r)} A_{0,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta x(t;k)+ \\ &+ \sum_{(\tau,\delta)\subset(\bar q,\bar r)} A_{1,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta x(t-a;k-b)+ \\ &+ \sum_{(\tau,\delta)\subset(\bar q,\bar r)} B_{0,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta u(t;k)+ \\ &+ \sum_{(\tau,\delta)\subset(\bar q,\bar r)} B_{1,\tau,\delta}\frac{\partial}{\partial\tau}\sigma_\delta u(t-a;k-b) \end{aligned} \tag{2.7}$$

$$y(t;k) = C_0x(t;k) + C_1x(t-a;k-b) + D_0u(t;k) + D_1u(t-a;k-b). \tag{2.8}$$

Now we shall focuss on the Attasi-type multidimensional system

The time set is $T=\mathbf{R}^q\times\mathbf{Z}^r$, $q,r\in\mathbf{N}$.

Definition 2.1. A (q,r)-D *hybrid (continuous-discrete) system* is a set $\Sigma = (A_{ci}, A_{dj}, B, C, D)$ with $A_{ci}(t;k)$, $i\in\bar q$ and $A_{dj}(t;k)$, $j\in\bar r$ commuting $n\times n$ matrices $\forall t\in\mathbf{R}^q$ and $\forall k\in\mathbf{Z}^r$, $B(t;k)$, $C(t;k)$, $D(t;k)$ are respectively $n\times m$, $p\times n$ and $p\times m$ real matrices, all these matrices being continuous with respect to $t\in\mathbf{R}^q$ for any $k\in\mathbf{Z}^r$; the state equation is

$$\begin{gathered} \frac{\partial}{\partial t}\sigma x(t;k) = \sum_{(\tau,\delta)\subset(\bar q,\bar r)} (-1)^{q+r-|\tau|-|\delta|-1}\times \\ \times\left(\prod_{i\in\tilde\tau} A_{ci}(t;k)\right)\left(\prod_{j\in\tilde\delta} A_{dj}(t;k)\right)\frac{\partial}{\partial\tau}\sigma_\delta x(t;k) + B(t;k)u(t;k) \end{gathered} \tag{2.9}$$

and the output equation is

$$y(t;k) = C(t;k)x(t;k) + D(t;k)u(t;k) \tag{2.10}$$

where

$$x(t;k) = x(t_1,\dots,t_q;k_1,\dots,k_r)\in\mathbf{R}^n$$

is the *state*, $u(t;k)\in\mathbf{R}^m$ is the *input* and $y(t;k)\in\mathbf{R}^p$ is the *output.*

For $\tau=\{i_1,\dots,i_l\}\subset\bar q$, $\delta=\{j_1,\dots,j_h\}\subset\bar r$ and $t_i\in\mathbf{R}$, $i\in\tau$, $t_i^0\in\mathbf{R}$, $i\in\tilde\tau$, $k_j\in\mathbf{Z}$, $j\in\delta$, $k_j^0\in\mathbf{Z}$, $j\in\tilde\delta$ we use the notation

$$\begin{gathered} x(t_\tau, t^0_{\tilde\tau}; k_\delta, k^0_{\tilde\delta}) := x(t_1^0,\dots,t^0_{i_1-1}, t_{i_1}, t^0_{i_1+1},\dots,t^0_{i_l-1}, t_{i_l}, t^0_{i_l+1},\dots,t^0_q; \\ k_1^0,\dots,k^0_{j_1-1}, k_{j_1}, k^0_{j_1+1},\dots,k^0_{j_h-1}, k_{j_h}, k^0_{j_h+1},\dots,k^0_{j_r}). \end{gathered}$$

Let $\Phi_i(t_i,t_i^0;t_{\bar i};k)$ be the (continuous) fundamental matrix of $A_{ci}(t;k)$ with respect to the variables t_i,t_i^0, $i\in\bar q$, i.e. the unique matrix solution of the system $\dfrac{\partial Y}{\partial t_i}(t;k) =$

$A_{ci}(t;k)Y(t;k), Y(t_{\tilde{i}}, t_i^0; k) = I$ for any $t_l \in \mathbf{R}$, $l \in \tilde{i}$ and $k \in \mathbf{R}^r$. If A_{ci} is a constant matrix then $\Phi_i(t_i, t_i^0; t_{\tilde{i}}, k) = e^{A_{ci}(t_i - t_i^0)}$.

The *discrete fundamental matrix* $F_j(t; k_j, k_j^0; k_{\tilde{j}})$ of the matrix $A_{dj}(t;k)$ is defined by

$$F_j(t; k_j, k_j^0; k_{\tilde{j}}) =$$

$$= \begin{cases} A_{dj}(t; k_j - 1, k_{\tilde{j}}) A_{dj}(t; k_j - 2, k_{\tilde{j}}) \dots A_{dj}(t; k_j^0, k_{\tilde{j}}) & \text{for} \quad k_j > k_j^0 \\ I_n & \text{for} \quad k = k_j^0, \end{cases}$$

for any $k_h \in \mathbf{Z}$, $h \in \tilde{j}$ and $t \in \mathbf{R}^q$.

$F_j(t; k_j, k_j^0; k_{\tilde{j}})$ is the unique matrix solution of the difference system

$$Y(t; k_j + 1, k_{\tilde{j}}) = A_{dj}(t;k) Y(t; k_j, k_{\tilde{j}}),\ Y(t; k_j^0; k_{\tilde{j}}) = I.$$

If A_{dj} is a constant matrix then $F_j(t; k_j, k_j^0; k_{\tilde{j}}) = A_{dj}^{k_j - k_j^0}$.

Definition 2.2. The vector $x_0 \in \mathbf{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau, t_{\tilde{\tau}}^0; k_\delta, k_{\tilde{\delta}}^0) = \left(\prod_{i \in \delta} \Phi_i(t_i, t_i^0; t_{\tilde{i}}; k) \right) \left(\prod_{j \in \delta} F_j(t; k_j, k_j^0; k_{\tilde{j}}) \right) x_0 \tag{2.11}$$

for any $(\tau, \delta) \subset (\overline{q}, \overline{r})$; equalities (2.11) are called initial conditions of Σ.

Proposition 2.3. *The solution of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} \sigma x(t;k) = & \sum_{(\tau,\delta) \subset (\overline{q}, \overline{r})} (-1)^{q+r-|\tau|-|\delta|-1} \times \\ & \times \left(\prod_{i \in \tilde{\tau}} (\sigma_\delta A_{ci}(t;k)) \right) \left(\prod_{j \in \tilde{\delta}} A_{dj}(t,k) \right) \frac{\partial}{\partial \tau} \sigma_\delta x(t;k) + f(t;k) \end{aligned} \tag{2.12}$$

with the initial conditions (2.11) *is*

$$\begin{aligned} x(t;k) = & \left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t^0_{\overline{i-1}}, t_{\tilde{\tilde{i}}}; k) \right) \left(\prod_{j=1}^{r} F_j(t^0; k_j, k_j^0; k^0_{\overline{j-1}}, k_{\tilde{\tilde{j}}}) \right) x_0 + \\ & + \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1 = k_1^0}^{k_1 - 1} \dots \sum_{l_r = k_r^0}^{k_r - 1} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\tilde{i}}}; k) \right) \times \\ & \times \left(\prod_{j=1}^{r} F_j(s; k_j, l_j + 1; l_{\overline{j-1}}, k_{\tilde{\tilde{j}}}) \right) f(s;l) ds_1 \dots ds_q \end{aligned} \tag{2.13;}$$

here $s = (s_1, \dots, s_q)$, $l = (l_1, \dots, l_r)$ *and if for instance* $i = 1$, *then the corresponding variable* $t_{\overline{i-1}} = t^0_{\varnothing}$ *lacks;* $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{R}^n$ *is a continuous function.*

Proof. We shall prove (2.13) firstly for the case $q = r = 1$, that is for the equation

$$\begin{aligned}\frac{\partial}{\partial t}x(t;k+1) &= A_c(t;k+1)x(t;k+1)+A_d(t;k)\frac{\partial}{\partial t}x(t;k)- \\ &- A_c(t;k)A_d(t;k)x(t;k)+f(t;k)\end{aligned} \tag{2.14}$$

with the initial conditions

$$x(t;k_0)=\Phi(t,t_0;k_0)x_0, \quad x(t_0;k)=F(t_0;k,k_0)x_0 \tag{2.15}$$

where $\Phi(t,t_0;k_0)$ is the fundamental matrix of $A_c(t;k)$ and $F(t;k,k_0)$ is the discrete fundamental matrix of $A_d(t;k)$.

Let us consider the vector

$$z(t;k)=\frac{\partial}{\partial t}x(t;k)-A_c(t;k)x(t;k). \tag{2.16}$$

From (2.15) and the first property of the fundamental matrix we have

$$z(t;k_0)=\frac{\partial}{\partial t}x(t;k_0)-A_c(t;k_0)x(t;k_0)=$$

$$=\frac{\partial}{\partial t}\Phi(t;t_0,k_0)x_0-A_c(t;k_0)\Phi(t;t_0,k_0)x_0=0,$$

hence $z(t;k_0)=0$, $\forall t\geq t_0$. Equation (2.14) can be written as the 1D difference equation $z(t;k+1)=A_j(t;k)z(t;k)+f(t;k)$ whose solution, given by the discrete-time variation of parameters formula, is $z(t;k)=F(t;k,k_0)z(t,k_0)+\sum_{l=k_0}^{k-1}F(t;k,l+1)f(t,l)$ hence

$$z(t,k)=\sum_{l=k_0}^{k-1}F(t;k,l+1)f(t,l) \tag{2.17}$$

since $z(t,k_0)=0$. But the equation (2.16) can be written as the differential equation $\frac{\partial}{\partial t}x(t;k)=A_c(t;k)x(t;k)+z(t;k)$ and by the variation of parameters formula its solution is

$$x(t;k)=\Phi(t,t_0;k)x(t_0;k)+\int_{t_0}^{t}\Phi(t,s;k)z(s;k)ds.$$

By replacing $x(t_0;k)$ and $z(s;k)$ by their expressions (2.15) and (2.17) we get

$$x(t;k)=\Phi(t,t_0;k)F(t_0;k,k_0)x_0+\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t,s;k)F(s;k,l+1)f(s,l) \tag{2.18}$$

hence formula (2.13) is true for $q=1$, $r=1$.

Similarly, we can prove (by applying two times the discrete variation of parameters formula) that in the case $q=0$, $r=2$, the solution of (2.12) and (2.11) is

$$\begin{aligned}&x(k_1,k_2)=F_1(k_1,k_1^0;k_2)F_2(k_1^0;k_2,k_2^0)x_0+ \\ &+\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1}F_1(k_1,l_1+1;k_2)F_2(l_1;k_2,l_2+1)f(l_1,l_2).\end{aligned} \tag{2.19}$$

Now, let us consider the case $(q,r)=(1,2)$; equation (2.9) becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}x(t;k_1+1,k_2+1)=A_{c1}(t;k_1+1,k_2+1)x(t;k_1+1,k_2+1)+\\
&+A_{d1}(t;k_1,k_2+1)\frac{\partial}{\partial t}x(t;k_1,k_2+1)+A_{d2}(t;k_1+1,k_2)\frac{\partial}{\partial t}x(t;k_1+1,k_2)-\\
&-A_{c1}(t;k_1,k_2+1)A_{d1}(t;k_1,k_2+1)x(t;k_1,k_2+1)-\\
&-A_{c1}(t;k_1+1,k_2)A_{d2}(t;k_1+1,k_2)x(t;k_1+1,k_2)-\\
&-A_{d1}(t;k_1,k_2)A_{d2}(t;k_1,k_2)\frac{\partial}{\partial t}x(t;k_1,k_2)+\\
&+A_{c1}(t;k_1,k_2)A_{d1}(t;k_1,k_2)A_{d2}(t;k_1,k_2)x(t;k_1,k_2)+f(t;k_1,k_2).
\end{aligned}\tag{2.20}$$

By denoting

$$z(t;k_1,k_2)=\frac{\partial}{\partial t}x(t;k_1,k_2)-A_{c1}x(t;k_1,k_2)x(t;k_1,k_2)\tag{2.21}$$

(2.20) becomes an equation of the type $q=0$, $r=2$ with $z(t;k_1,k_2)$ instead of $x(k_1,k_2)$ and by the initial conditions of the form (2.11) we have $z(t;k_1^0,k_2)=z(t;k_1,k_2^0)=0$. By applying (2.19) one obtains

$$z(t;k_1,k_2)=\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1}F_1(t,k_1,l_1+1;k_2)F_2(t;l_1;k_2,l_2+1)f(t;l_1,l_2)\tag{2.22}$$

and, by the variation of parameters formula applied to (2.21) one obtains the solution

$$x(t;k_1,k_2)=\Phi_1(t,t^0;k_1,k_2)x(t^0;k_1,k_2)+\int_{t_0}^{t}\Phi_1(t,s;k_1,k_2)z(s;k_1,k_2)$$

hence by (2.22) and (2.11)

$$\begin{aligned}
x(t;k_1,k_2)&=\Phi_1(t,t^0;k_1,k_2)F_1(t^0;k_1,k_1^0;k_2)F_2(t^0;k_1^0;k_2,k_2^0)x_0+\\
&+\int_{t_0}^{t}\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1}\Phi_1(t,s;k_1,k_2)F_1(s;k_1,l_1+1;k_2)F_2(s;l_1;k_2,l_2+1)f(s;l_1,l_2)ds
\end{aligned}$$

hence formula (2.13) is true for $(q,r)=(1,2)$.

Similarly we can verify (2.13) for $(q,r)=(2,1)$. Using the same ideas we shall prove that (2.13) is true for the cases $(q+1,r)$ and $(q,r+1)$ if it is true for (q,r) hence, by induction (2.13) is true for any $(q,r)\in\mathbf{N}\times\mathbf{N}$.

Assume that the formula (2.13) is true for some $q,r\in\mathbf{N}$ and consider the equation (2.12) with $q+1$ instead of q. We introduce the function

$$\begin{aligned}
z(t_1;t_2,\dots,t_{q+1};k_1,\dots,k_r)&=\frac{\partial x}{\partial t_1}(t_1,t_2,\dots,t_{q+1};k_1,\dots,k_r)-\\
&-A_{c1}(t_1,t_2,\dots,t_{q+1};k_1,\dots,k_r)x(t_1,t_2,\dots,t_{q+1};k_1,\dots,k_r).
\end{aligned}\tag{2.23}$$

Then the equation (2.12) of order $(q+1, r)$ can be written as an equation (q, r) of the same type with z instead of x and the initial conditions (2.11) give null initial conditions for z.

By the induction assumption the corresponding solution is

$$\begin{aligned} & z(t_1; t_2, \dots, t_{q+1}; k_1, \dots, k_r) = \\ & = \int_{t_2^0}^{t_2} \dots \int_{t_{q+1}^0}^{t_{q+1}} \sum_{l_1=k_1^0}^{k_1-1} \dots \sum_{l_r=k_r^0}^{k_r-1} \left(\prod_{i=2}^{q+1} \Phi_i(t_i, s_i; t_1; s_{\overline{i-1}\setminus\{1\}}, t_{\tilde{\overline{i}}}; k \right) \times \\ & \times \left(\prod_{j=1}^{r} F_j(t_1; s_2, \dots, s_{q+1}; k_j, l_j+1; l_{\overline{j-1}}; k_{\tilde{\overline{j}}} \right) f(s; l) ds_2 \dots ds_{q+1}. \end{aligned} \tag{2.24}$$

But the variation of parameters formula give the solution of (2.19)

$$\begin{aligned} & x(t_1, t_2, \dots, t_{q+1}; k_1, \dots, k_r) = \\ & = \Phi_1(t_1, t_1^0; t_2, \dots, t_{q+1}; k_1, \dots, k_r) x(t_1^0; t_2, \dots, t_{q+1}; k_1, \dots, k_r) + \\ & + \int_{t_1^0}^{t_1} \Phi_1(t_1, s_1; t_2, \dots, t_{q+1}; k_1, \dots, k_r) z(s_1; t_2, \dots, t_{q+1}; k_1, \dots, k_r) ds_1 \end{aligned}$$

and, by replacing z (2.24) one obtains formula (2.13) for the case $(q+1, r)$.

Similarly one can derive the case $(q, r+1)$ from (q, r), hence we proved by induction (2.13) for any $q, r \in \mathbf{N}$:

PROPOSITION 2.4. *The state of the system* Σ (2.9) *determined by the initial state* $x_0 \in \mathbf{R}^n$ *and the control* u *is*

$$\begin{aligned} x(t; k) = & \left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t_{\overline{i-1}}^0, t_{\tilde{\overline{i}}}; k) \right) \left(\prod_{j=1}^{r} F_j(t^0; k_j, k_j^0; k_{\overline{j-1}}^0, k_{\tilde{\overline{j}}}) \right) x_0 + \\ & + \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \dots \sum_{l_r=k_r^0}^{k_r-1} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\overline{i}}}; k) \right) \times \\ & \times \left(\prod_{j=1}^{r} F_j(s; k_j, l_j+1; l_{\overline{j-1}}, k_{\tilde{\overline{j}}}) \right) B(s; l) u(s; l) ds_1 \dots ds_q. \end{aligned} \tag{2.25}$$

Proof. Equation (2.9) has the form (2.12) with $f(t; k) = B(u; k)u(t; k)$ and (2.25) results from (2.13) by replacing $f(t; k)$.

COROLLARY 2.5. *If* A_{ci}, A_{dj} *and* B *are constant matrices, then the state of* Σ *determined by the control* u *and the initial state* x_0 *with initial moments* $t_i^0 = 0$, $i \in \overline{q}$, $k_j^0 = 0$, $j \in \overline{r}$ *is*

$$x(t;k) = \left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x_0 + \int_0^{t_1} \dots \int_0^{t_q} \left(\prod_{i=1}^{q} e^{A_{ci}(t_i - s_i)}\right) \times$$
$$\times \sum_{l_1=0}^{k_1-1} \dots \sum_{l_r=0}^{k_r-1} \left(\prod_{j=1}^{r} A_{dj}^{k_j - l_j - 1}\right) B(s;l)u(s,l)ds_1 \dots ds_q. \tag{2.26}$$

By replacing the state $x(t;k)$ (2.25) in the output equation (2.10) we obtain

PROPOSITION 2.6. *The input-output map of the* (q, r)-D *continuous-discrete system* Σ (2.9), (2.10) *is*

$$y(t;k) = C(t;k)\left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t^0_{\overline{i-1}}, t_{\tilde{\bar{i}}}, k)\right)\left(\prod_{j=1}^{r} F_j(t^0; k_j; k^0_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) x_0 +$$
$$+ \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \dots \sum_{l_r=k_r^0}^{k_r-1} C(t;k)\left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\bar{i}}}; k)\right) \times \tag{2.27}$$
$$\times \left(\prod_{j=1}^{r} F_j(s; k_j, l_j; l_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) B(s;l)u(s;l)ds_1 \dots ds_q + D(t;k)u(t;k).$$

Similarly with Corollary 2.5 we can derive from (2.27) the input-output map of a time-invariant system Σ.

3 Stability of (q, r)-d continuous-discrete systems

We consider the time-invariant (q, r)-D continuous-discrete system Σ with the state equation (2.9). For the control $u(t;k) \equiv 0$ one obtains from (2.26) the "free" state of Σ determined by the initial state $x_0 \in \mathbf{R}^n$:

$$x(t;k) = \left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x_0. \tag{3.1}$$

We shall extend the results concerning the stability of 1D continuous-time and discrete-time systems to (q, r)-D systems. If $t_i \geq 0$, $\forall i \in \overline{q}$ we write $t \geq 0$.

If P is a positive definite matrix, one writes $P > 0$.

Definition 3.1. i) The system $\Sigma = ((A_{ci})_{i\in\overline{q}}, (A_{dj})_{j\in\overline{r}})$ is *stable* if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that for any $x_0 \in \mathbf{R}^n$ with $||x_0|| < \delta$ it results $||x(t;k)|| < \varepsilon$, $\forall t \geq 0$ and $\forall k \geq 0$.

ii) The system Σ is *unstable* if it is not stable.

iii) Σ is *asymptotically stable* if it is stable and for any $x_0 \in \mathbf{R}^n$ the corresponding solution $x(t;k)$ verifies

$$\lim_{t_i\to\infty} x(t_1, \dots, t_i, \dots, t_q; k_1, \dots, k_j, \dots, k_r) = 0$$

and

$$\lim_{k_j \to \infty} x(t_1, \dots, t_i, \dots, t_q; k_1, \dots, k_j, \dots, k_r) = 0$$

$\forall i \in \overline{q}$ and $\forall j \in \overline{r}$.

We denote by $\sigma(A)$ the spectrum of a matrix $A \in \mathbf{R}^{n \times n}$, by $m_a(\lambda)$ and $m_g(\lambda)$ respectively the algebraic and the geometric multiplicity of an eigenvalue $\lambda \in \sigma(A)$, by $\mathbf{C}^-$, $\mathbf{C}^0$ and $\mathbf{C}^+$ the set of the complex numbers with respectively negative, null and positive real parts and by $\mathbf{D}_1^-$, $\mathbf{D}_1^0$ and $\mathbf{D}_1^+$ the set of the complex numbers with the modulus less than, equal to or greater than 1.

Theorem 3.2. i) *The system* Σ *is asymptotically stable if and only if*

$$\bigcup_{i \in \overline{q}} \sigma(A_{ci}) \subset \mathbf{C}^- \quad \text{and} \quad \bigcup_{j \in \overline{r}} \sigma(A_{dj}) \subset \mathbf{D}_1^- \ .$$

ii) *If*

$$\left((\bigcup_{i \in \overline{q}} \sigma(A_{ci})) \cap \mathbf{C}^+ \right) \cup \left((\bigcup_{j \in \overline{r}} \sigma(A_{dj})) \cap \mathbf{D}_1^+ \right) \neq \varnothing \ .$$

then Σ *is unstable.*

iii) *If* $\bigcup_{i \in \overline{q}} \sigma(A_{ci}) \subset \mathbf{C}^- \cup \mathbf{C}^0$, $\bigcup_{j \in \overline{r}} \sigma(A_{dj}) \subset \mathbf{D}_1^- \cup \mathbf{D}_1^0$ *and for any*

$$\lambda \in \left((\bigcup_{i \in \overline{q}} \sigma(A_{ci})) \cap \mathbf{C}^0 \right) \cup \left((\bigcup_{j \in \overline{r}} \sigma(A_{dj})) \cap \mathbf{D}_1^0 \right)$$

$m_a(\lambda) = m_g(\lambda)$ *then* Σ *is stable (but not asymptotically stable).*

iv) *If* $\exists \lambda \in \left((\bigcup_{i \in \overline{q}} \sigma(A_{ci})) \cap \mathbf{C}^0 \right) \cup \left((\bigcup_{j \in \overline{r}} \sigma(A_{dj})) \cap \mathbf{D}_1^0 \right)$ *with* $m_a(\lambda) > m_g(\lambda)$ *then* Σ *is unstable.*

Proof. By employing the Jordan canonical form of the matrices A_{ci} and A_{dj} it results that the elements of the matrix $\left(\prod_{i=1}^{q} e^{A_{ci} t_i} \right) \left(\prod_{j=1}^{r} A_{dj}^{kj} \right)$ and hence by (3.1) those of the vector solution $x(t;k)$ are linear combinations of product functions of the form $t_i^{\alpha_i} e^{\beta_i t_i}$ and $k_j^{\gamma_j} \mu^{k_j}$ where $\lambda_i = \alpha_i + i\beta_i \in \sigma(A_{ci})$ and $\mu \in \sigma(A_{dj})$. If $Re\, \lambda < 0$ and $|\mu| < 1$ both functions tend to 0 as $t_i \to \infty$ and $k_j \to \infty$, hence $\lim_{\substack{t_i \to \infty \\ k_j \to \infty}} x(t;k) = 0$, i.e. Σ is asymptotically stable.

If for some eigenvalues $Re\, \lambda > 0$ or $|\mu| > 1$ then these functions approach ∞, hence Σ is unstable. In the case iii) $m_a(\lambda) = m_g(\lambda)$ implies that these functions have the form $e^{i\beta t}$ or μ^{k_j}, $|\mu| = 1$, hence $1 = |e^{i\beta t}| \not\to 0$ and $|\mu^{k_j}| = 1$. It results that all these functions are bounded, hence Σ is stable.

In the case (iv) the elements of $x(t;k)$ contain functions of the form $t^{\alpha_l} e^{i\beta_l t_l}$ or $k_j^{\gamma_j} \mu^{k_j}$ with $\alpha_l \geq 1$, $\gamma_j \geq 1$; therefore these functions have the limit ∞, i.e. Σ is unstable.

Remark 3.3. From Theorem 3.2i it results that it is possible to check the asymptotic stability of Σ by applying the Routh-Hurwitz criterion to each matrix A_{ci} and the same criterion, via a linear fractional map, to the matrices A_{dj}.

A necessary condition of asymptotic stability can be obtained by introducing a generalized Liapunov equation.

Theorem 3.4. *If the system $\Sigma = ((A_{ci})_{i\in\overline{q}}, (A_{dj})_{j\in\overline{r}})$ is asymptotically stable, then for any positive definite matrix Q the following equation*

$$\sum_{\tau\in\mathcal{P}(\overline{q})}\sum_{\delta\in\mathcal{P}(\overline{r})}(-1)^{q-|\delta|-1}\left(\prod_{j\in\delta}A_{dj}^T\right)\left(\left(\prod_{i\in\tau}A_{ci}^T\right)P+\right.$$

$$\left.+P\left(\prod_{i\in\tau}A_{ci}\right)\right)\left(\prod_{j\in\delta}A_{dj}\right)=-Q \tag{3.2}$$

has a unique positive solution P.

Proof. Equation (3.2) can be written as

$$A_{dr}^T P_1 A_{dr} - P_1 = -Q \tag{3.3}$$

where

$$-P_1 = \sum_{\tau\in\mathcal{P}(\overline{q})}\sum_{\delta\in\mathcal{P}(\overline{r-1})}(-1)^{q-|\delta|-1}\left(\prod_{j\in\delta}A_{dj}^T\right)\left(\left(\prod_{i\in\tau}A_{ci}^T\right)P+\right.$$

$$\left.+P\left(\prod_{i\in\tau}A_{ci}\right)\right)\left(\prod_{j\in\delta}A_{dj}\right). \tag{3.4}$$

Since $\sigma(A_{dr}) \subset \mathbf{D}_1^-$, for any $Q > 0$ the $1D$ discrete Liapunov equation (3.3) has the unique solution $P_1 > 0$

$$P_1 = \sum_{k_r=0}^{\infty}(A_{dr}^T)^{k_r} Q (A_{dr})^{k_r}. \tag{3.5}$$

By repeating this procedure for equation (3.4) (which is similar to (3.2)) we obtain a chain of 1D discrete Liapunov equations

$$A_{d,r-j}^T P_{j+1} A_{d,r-j} - P_{j+1} = -P_j,\ P_j > 0,\ j \in \overline{r-1}$$

and since $\sigma(A_{d,r-j}) \subset \mathbf{D}_1^-$ their positive definite solutions P_j are similar to (3.5).

Finally (3.2) becomes an equation of the type $q > 0$, $r = 0$:

$$\sum_{\tau\in\mathcal{P}(\overline{q})}(-1)^{q-1}\left(\left(\prod_{i\in\tau}A_{ci}^T\right)P + P\left(\prod_{i\in\tau}A_{ci}\right)\right) = -P_r$$

which can be written as an 1D Liapunov equation

$$A_{cq}^T W_1 + W_1 A_{cq} = -P_r \tag{3.6}$$

where

$$-W_1 = \sum_{\tau\in\mathcal{P}(\overline{q-1})}(-1)^{q-2}\left(\left(\prod_{i\in\tau}A_{ci}^T\right)P + P\left(\prod_{i\in\tau}A_{ci}\right)\right). \tag{3.7}$$

Since $\sigma(A_{cq}^T) \subset \mathbf{C}^-$, (3.6) has the unique solution $W_1 > 0$,

$$W_1 = \int_0^\infty e^{A_{cq}^T t_q} P_r e^{A_{cq} t_q} dt_q.$$

By recurrently applying this approach to (3.7) and to its followers, we get finally the equation

$$A_{c1}^T P + P A_{c1} = -W_{q-1}$$

whose solution is

$$P = \int_0^\infty e^{A_{c1}^T t_1} W_{q-1} e^{A_{c1} t_1} dt_1.$$

By summarizing this approach, we obtain the unique solution $P > 0$ of (3.2)

$$P = \int_0^\infty \dots \int_0^\infty \sum_{k_1=0}^\infty \dots \sum_{k_r=0}^\infty e^{\sum_{i=1}^q A_{ci}^T t_i} \left(\prod_{j=1}^r (A_{dj}^T)^{k_j} \right) Q \left(\prod_{j=1}^r A_{dj}^{k_j} \right) \times \\ \times e^{\sum_{i=1}^q A_{ci} t_i} dt_1 \dots dt_q. \tag{3.8}$$

Remark 3.5. The above necessary condition is not sufficient, as the following counterexample shows. Consider $n = 1$, $q = 1$, $r = 1$, $A_c > 0$, $A_d > 1$, hence Σ is unstable. The equation (3.2) has the form

$$A_c^T A_d^T P A_d + A_d^T P A_d A_c - A_c^T P - P A_c = Q$$

i.e. (since $n = 1$) $2A_c(A_d^2 - 1)P = Q$. For $Q > 0$ this equation has the solution $P = \dfrac{Q}{2A_c(A_d^2 - 1)} > 0$ but Σ is unstable.

If for some $Q > 0$ the generalized Liapunov equation (3.2) has a solution $P > 0$ we can show only the relation (3.9) below concerning the eigenvalues of the matrices of Σ. Since $\Sigma = ((A_{ci})_{i\in\overline{q}}, (A_{dj})_{j\in\overline{r}})$ is a commuting family then there is a vector $v \in \mathbf{C}^n$ which is an eigenvector of every matrix in the family (see [7, Lemma 1.3.17]). Let λ_i, $i \in \overline{q}$ and μ_j, $j \in \overline{r}$ be the eigenvalues (associated to v) of A_{ci} and A_{dj} respectively.

Proposition 3.6. *Assume that equation* (3.2) *has a solution $P > 0$ for some $Q > 0$. If λ_i, $i \in \overline{q}$ and μ_j, $j \in \overline{r}$ are the eigenvalues of A_{ci} and A_{dj} respectively, associated to a common eigenvector v, then*

$$\left(\prod_{i\in\overline{q}} Re\, \lambda_i \right) \left(\prod_{j\in\overline{r}} (|\mu_j|^2 - 1) \right) > 0 \tag{3.9}$$

Proof. We premultiply and postmultiply (3.2) respectively with v^* and v. Since $A_{ci}v = \lambda_i v$, $A_{dj}v = \mu_j v$, $v^* A_{ci}^T = \overline{\lambda}_i v^*$, $v^* A_{dj}^T = \overline{\mu}_j v^*$, $\lambda_i \overline{\lambda}_i = 2Re\, \lambda_i$, $\mu_j \overline{\mu}_j = |\mu_j|^2$ we obtain from (3.2)

$$\left(\prod_{i\in\overline{q}} (\lambda_i + \overline{\lambda}_i) \right) \left(\prod_{j\in\overline{r}} (\mu_j \overline{\mu}_j - 1) \right) v^* P v = v^* Q v,$$

which implies (3.9) since $v^*Pv > 0$ and $v^*Qv > 0$.

Definition 3.7. Let P be a positive definite matrix of order n. The quadratic form $V(x) = x^T Px$ is a *generalized Liapunov function* for the system Σ if, for any initial state $x_0 \in \mathbf{R}^n$, the corresponding solution $x(t; k)$ of (2.1) (with $u(t, k) \equiv 0$) verifies the following conditions, for any $t \geq 0$, $k \geq 0$:

$$i)\ V(x(t; k'_j, k_{\tilde{j}})) < V(x(t; k''_j, k_{\tilde{j}})),\ \forall k''_j > k'_j,\ \forall j \in \overline{r} \tag{3.10}$$

$$ii)\ \frac{\partial}{\partial t_i} V(x(t_i, t_{\tilde{i}}; k)) < 0,\ \forall i \in \overline{q} \tag{3.11}$$

.

Theorem 3.8. *The system Σ is asymptotically stable iff there exists a generalized Liapunov function for Σ.*

Proof. Assume that Σ is asymptotically stable. Let Q be a positive definite matrix and let P be the solution (3.8) of the Liapunov equation (3.2). It can be shown that the quadratic form $V(x) = x^T Px$ is a Liapunov function for Σ.

Conversely, if there exists a Liapunov function $V(x) = x^T Px$ for Σ, from (3.10) and (3.11) it results that $\sigma(A_{ci}) \subset \mathbf{C}^-$, $\forall i \in \overline{q}$ and $\sigma(A_{dj}) \subset \mathbf{D}_1^-$, $\forall j \in \overline{r}$, hence Σ is asymptotically stable.

Conclusion. The state space representations of some classes of time-varying (q, r)-D continuous-discrete systems was studied and some results concerning stability of these systems were obtained. This study can be continued by analysing for these models other important concepts such as controllability, observability, realizations etc.

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