An algorithm for computing the spectra of multiplication systems

Mircea Olteanu, Tania-Luminiţa Costache and Mariana Zamfir

Abstract. In this paper is developed an algorithm for computing the spectra of multiplication systems. In the first part are reviewed the basic theoretical results of the spectra theory of multiplication operator. The general setup is as follows : for every $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ we associate the corresponding multiplication system $M_\Phi\colon L^2(\mathbb{R}, \mathbb{C}^n) \to L^2(\mathbb{R}, \mathbb{C}^n)$. We finally develop a Mathcad-application which helps us do numerical computation of the spectra.

M.S.C. 2000: 47B38, 45P05.
Key words: multiplication systems, spectrum of a multiplication system.

1 Introduction

Let $\mathbb{R}$ be the aditive group of real numbers, let $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$ be the complex separable Hilbert space and let $(\mathcal{L}(\mathbb{C}^n), \|\|)$ be the Banach algebra of linear continuous operators on $\mathbb{C}^n$. Let $(L^2(\mathbb{R}, \mathbb{C}^n), \|\|_2)$ the usual Hilbert space $\mathbb{C}^n$- valued, square integrable functions on $\mathbb{R}$. Let $\mathcal{C}(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ be the Banach algebra of $\mathcal{L}(\mathbb{C}^n)$- valued continuous functions on $\mathbb{R}$. Let λ be the Lebesgue measure on R, let $\mathrm{Bor}(\mathbb{R})$ be the family of Borelian sets on $\mathbb{R}$ and let $\Phi\colon \mathbb{R} \to \mathcal{L}(\mathbb{C}^n)$ be a measurable function; the essential supremum of Φ is :

$$\|\Phi\|_\infty = \inf\big\{ \sup_{t\in\mathbf{R}\setminus A} \|\Phi(t)\|; A \in \mathrm{Bor}(\mathbb{R}), \lambda(A) = 0\big\}$$

We denote by $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ the Banach algebra of essentialy bounded functions, i. e. $\|\Phi\|_\infty < \infty$.

The (noncommutative) product is $(\Phi\Psi)(t) = \Phi(t)\Psi(t)$, where $\Phi, \Psi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$; the unit is $1(t) = I_{\mathcal{L}(\mathbb{C}^n)}$; a function Φ is invertible in $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ if there is $\Psi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ such that $\Phi\Psi = \Psi\Phi = 1$.

Definition 1.1. A *multiplication system* is an operator

$$M_\Phi\colon L^2(\mathbb{R}, \mathbb{C}^n) \to L^2(\mathbb{R}, \mathbb{C}^n)$$

Proceedings of The 4-th International Colloquium "Mathematics in Engineering and Numerical Physics" October 6-8 , 2006, Bucharest, Romania, pp. 126-130.

defined by

$$M_\Phi x = \Phi x, \forall \Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$$

2 Basic results

Proposition 2.1. *For every $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ and $x \in L^2(\mathbb{R}, \mathbb{C}^n)$, the function Φx defined by $(\Phi x)(t) = \Phi(t)x(t)$ satisfies the inequality*

$$\|\Phi x\|_2 \leq \|\Phi\|_\infty \|x\|_2,$$

hence it is in $L^2(\mathbb{R}, \mathbb{C}^n)$.

Remark 2.1. It results that the multiplication system M_Φ is well defined, linear, continuous and $\|M_\Phi\| \leq \|\Phi\|_\infty$.

Proposition 2.2. *If $\Phi\colon \mathbb{R} \to \mathcal{L}(\mathbb{C}^n)$ is an arbitrary measurable function such that $\Phi x \in L^2(\mathbb{R}, \mathbb{C}^n), \forall x \in L^2(\mathbb{R}, \mathbb{C}^n)$, then $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ and $\|M_\Phi\| = \|\Phi\|_\infty$.*

Proposition 2.3. *Let $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$. Then M_Φ is an invertible system if and only if Φ is an invertible function in the Banach algebra $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$.*

Theorem 2.1. *Let $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n)) \cap \mathcal{C}(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$. Then*

$$spec(M_\Phi) = \bigcup_{t \in \mathbf{R}} spec\Phi(t),$$

where spec denotes the spectrum.

3 The algorithm

According to the results of the previous section (especially theorem 2.1), the steps of computing the spectrum of a multiplication system M_Φ for a fixed $\Phi \in L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{C}^n)) \cap \mathcal{C}(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ are:

1. Read the set of $n \times n$ matrices $(\Phi(t)), t \in \mathbb{R}$.
2. For every $t \in \mathbb{R}$, compute the spectrum (eigenvalues) of the matrix $(\Phi(t))_{t \in \mathbf{R}}$.
3. The spectrum of the multiplication system is $\mathrm{spec}(M_\Phi) = \bigcup_{t \in \mathbf{R}} \mathrm{spec}\Phi(t)$.
4. Draw the graphics of the eigenvalues in the same coordinates system.

4 A Mathcad- application

Pressing CTRL+M, we can insert a matrix by giving the number of rows and columns.
$\lambda(t)$:=eigenvals $(\Phi(t))$ (we write the eigenvalues vector of $\Phi(t)$)
$U(t) := \mathrm{Re}(\lambda(t))$ (we assign to $U(t)$ the real part of the eigenvalues vector)
$U(t) \longrightarrow$ (it is written the vector $U(t)$)
$V(t) := \mathrm{Im}(\lambda(t))$ (we assign to $V(t)$ the imaginary part of the eigenvalues vector)
$V(t) \longrightarrow$ (it is written the vector $V(t)$)
MENIU: INSERT $\longrightarrow$ GRAPH $\longrightarrow$ $x - y$ PLOT

5 Examples

Example 5.1. Let $\Phi(t) = \begin{bmatrix} 1 & \mathrm{e}^{-2\mathrm{i}t} & 0 \\ \mathrm{e}^{\mathrm{i}t} & 1 & \mathrm{e}^{-2\mathrm{i}t} \\ 0 & \mathrm{e}^{\mathrm{i}t} & 1 \end{bmatrix}$

With the algorithm in the previous section we obtain the eigenvalues vector :

$$\lambda(t) = \begin{bmatrix} 1 \\ \frac{1}{2\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} + 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) \\ \frac{1}{2\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} - 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) \end{bmatrix}$$

We obtain the vectors

$$\mathrm{Re}(\lambda(t)) = \begin{bmatrix} 1 \\ \frac{1}{4\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} + 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) + \frac{1}{4} \cdot \overline{\frac{1}{\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} + 2\sqrt{2\mathrm{e}^{\mathrm{i}t}})} \\ \frac{1}{4\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} - 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) + \frac{1}{4} \cdot \overline{\frac{1}{\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} - 2\sqrt{2\mathrm{e}^{\mathrm{i}t}})} \end{bmatrix}$$

and

$$\mathrm{Im}(\lambda(t)) = \begin{bmatrix} 0 \\ -\frac{\mathrm{i}}{2} \cdot \left[\frac{1}{2\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} + 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) - \frac{1}{2} \cdot \overline{\frac{1}{\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} + 2\sqrt{2\mathrm{e}^{\mathrm{i}t}})}\right] \\ -\frac{\mathrm{i}}{2} \cdot \left[\frac{1}{2\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} - 2\sqrt{2\mathrm{e}^{\mathrm{i}t}}) - \frac{1}{2} \cdot \overline{\frac{1}{\mathrm{e}^{\mathrm{i}t}} \cdot (2\mathrm{e}^{\mathrm{i}t} - 2\sqrt{2\mathrm{e}^{\mathrm{i}t}})}\right] \end{bmatrix}$$

Example 5.2. Let $\Phi(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} & 0 \\ \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} \\ 0 & \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} \end{bmatrix}$

The eigenvalues vector is $\lambda(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}t} \\ (\sqrt{2}+1) \cdot \mathrm{e}^{\mathrm{i}t} \\ (1-\sqrt{2}) \cdot \mathrm{e}^{\mathrm{i}t} \end{bmatrix}$

Then $\mathrm{Re}(\lambda(t)) = \begin{bmatrix} \frac{1}{2} \cdot \mathrm{e}^{\mathrm{i}t} + \frac{1}{2} \cdot \mathrm{e}^{-\mathrm{i}t} \\ \frac{1}{2} \cdot (\sqrt{2}+1) \cdot \mathrm{e}^{\mathrm{i}t} + \frac{1}{2} \cdot (\sqrt{2}+1) \cdot \mathrm{e}^{-\mathrm{i}t} \\ \frac{1}{2} \cdot (1-\sqrt{2}) \cdot \mathrm{e}^{\mathrm{i}t} + \frac{1}{2} \cdot (1-\sqrt{2}) \cdot \mathrm{e}^{-\mathrm{i}t} \end{bmatrix}$

and $\mathrm{Im}(\lambda(t)) = \begin{bmatrix} -\frac{\mathrm{i}}{2} \cdot (\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}) \\ -\frac{\mathrm{i}}{2} \cdot \left[(\sqrt{2}+1) \cdot \mathrm{e}^{\mathrm{i}t} - (\sqrt{2}+1) \cdot \mathrm{e}^{-\mathrm{i}t}\right] \\ -\frac{\mathrm{i}}{2} \cdot \left[(1-\sqrt{2}) \cdot \mathrm{e}^{\mathrm{i}t} - (1-\sqrt{2}) \cdot \mathrm{e}^{-\mathrm{i}t}\right] \end{bmatrix}$

Example 5.3. Let $\Phi(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{-\mathrm{i}t} & 0 \\ \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{-\mathrm{i}t} \\ 0 & \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} \end{bmatrix}$

The eigenvalues vector is $\lambda(t) = \begin{bmatrix} \frac{1}{\mathrm{e}^{-\mathrm{i}t}} \\ \frac{1+\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} \\ \frac{1-\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} \end{bmatrix}$

Then $\mathrm{Re}(\lambda(t)) = \begin{bmatrix} \frac{1}{2\cdot\mathrm{e}^{-\mathrm{i}t}} + \frac{1}{2\cdot\mathrm{e}^{\mathrm{i}t}} \\ \frac{1}{2} \cdot \frac{1+\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} + \frac{1}{2} \cdot \overline{\frac{1+\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}}} \\ \frac{1}{2} \cdot \frac{1-\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} + \frac{1}{2} \cdot \overline{\frac{1-\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}}} \end{bmatrix}$

and $\mathrm{Im}(\lambda(t)) = \begin{bmatrix} -\frac{\mathrm{i}}{2} \cdot \left(\frac{1}{\mathrm{e}^{-\mathrm{i}t}} - \frac{1}{\mathrm{e}^{\mathrm{i}t}}\right) \\ -\frac{\mathrm{i}}{2} \cdot \left(\frac{1+\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} - \overline{\frac{1+\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}}}\right) \\ -\frac{\mathrm{i}}{2} \cdot \left(\frac{1-\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}} - \overline{\frac{1-\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}t}}{\mathrm{e}^{-\mathrm{i}t}}}\right) \end{bmatrix}$

Example 5.4. Let $\Phi(t) = \begin{bmatrix} e^{it} & e^{-it^2} & 0 \\ e^{it} & e^{it} & e^{-it^2} \\ 0 & e^{it} & e^{it} \end{bmatrix}$

The eigenvalues vector is $\lambda(t) = \begin{bmatrix} e^{it} \\ e^{it} + \sqrt{2 \cdot e^{it-it^2}} \\ e^{it} - \sqrt{2 \cdot e^{it-it^2}} \end{bmatrix}$

Then

$$\mathrm{Re}(\lambda(t)) = \begin{bmatrix} \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot e^{-it} \\ \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot \sqrt{2e^{it-it^2}} + \frac{1}{2} \cdot \overline{e^{it} + \sqrt{2 \cdot e^{it-it^2}}} \\ \frac{1}{2} \cdot e^{it} - \frac{1}{2} \cdot \sqrt{2 \cdot e^{it-it^2}} + \frac{1}{2} \cdot \overline{e^{it} - \sqrt{2 \cdot e^{it-it^2}}} \end{bmatrix}$$

and $\mathrm{Im}(\lambda(t)) = \begin{bmatrix} -\frac{i}{2} \cdot (e^{it} + e^{-it}) \\ -\frac{i}{2} \cdot (e^{it} + \sqrt{2 \cdot e^{it-it^2}} - \overline{e^{it} + \sqrt{2 \cdot e^{it-it^2}}}) \\ -\frac{i}{2} \cdot (e^{it} - \sqrt{2 \cdot e^{it-it^2}} - \overline{e^{it} - \sqrt{2 \cdot e^{it-it^2}}}) \end{bmatrix}$

Example 5.5. Let $\Phi(t) = \begin{bmatrix} e^{2it} & e^{-2it} & 0 \\ e^{it} & e^{2it} & e^{-2it} \\ 0 & e^{it} & e^{2it} \end{bmatrix}$

The eigenvalues vector is $\lambda(t) = \begin{bmatrix} e^{2it} \\ \frac{1}{2e^{it}} \cdot [2e^{3it} + 2 \cdot \sqrt{2e^{it}}] \\ \frac{1}{2e^{it}} \cdot [2e^{3it} - 2 \cdot \sqrt{2e^{it}}] \end{bmatrix}$

Then

$$\mathrm{Re}(\lambda(t)) = \begin{bmatrix} \frac{1}{2} \cdot e^{2it} + \frac{1}{2} \cdot e^{-2it} \\ \frac{1}{4e^{it}} \cdot (2e^{3it} + 2 \cdot \sqrt{2e^{it}}) + \frac{1}{4} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{3it} + 2 \cdot \sqrt{2e^{it}})} \\ \frac{1}{4e^{it}} \cdot (2e^{3it} - 2 \cdot \sqrt{2e^{it}}) + \frac{1}{4} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{3it} - 2 \cdot \sqrt{2e^{it}})} \end{bmatrix}$$

and

$$\mathrm{Im}(\lambda(t)) = \begin{bmatrix} -\frac{i}{2} \cdot (e^{2it} - e^{-2it}) \\ -\frac{i}{2} \cdot \left[\frac{1}{2e^{it}} \cdot (2e^{3it} + 2 \cdot \sqrt{2e^{it}}) - \frac{1}{2} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{3it} + 2 \cdot \sqrt{2e^{it}})}\right] \\ -\frac{i}{2} \cdot \left[\frac{1}{2e^{it}} \cdot (2e^{3it} - 2 \cdot \sqrt{2e^{it}}) - \frac{1}{2} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{3it} - 2 \cdot \sqrt{2e^{it}})}\right] \end{bmatrix}$$

Example 5.6. Let $\Phi(t) = \begin{bmatrix} e^{it} & e^{-2it} & 0 \\ e^{it} & e^{it} & e^{-2it} \\ 0 & e^{it} & e^{it} \end{bmatrix}$

The eigenvalues vector is $\lambda(t) = \begin{bmatrix} e^{it} \\ \frac{1}{2e^{it}}[2e^{2it} + 2\sqrt{2 \cdot e^{it}}] \\ \frac{1}{2e^{it}}[2e^{2it} - 2\sqrt{2 \cdot e^{it}}] \end{bmatrix}$

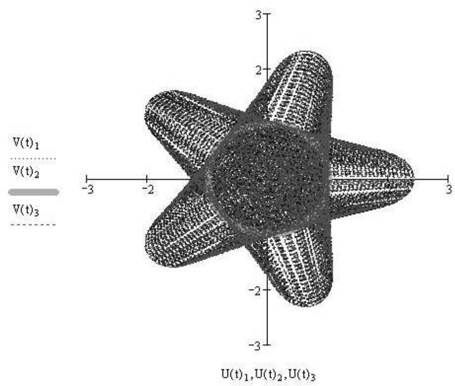
Then

$$\mathrm{Re}(\lambda(t)) = \begin{bmatrix} \frac{1}{2} \cdot e^{it} + \frac{1}{2} \cdot e^{-it} \\ \frac{1}{4e^{it}} \cdot (2e^{2it} + 2 \cdot \sqrt{2e^{it}}) + \frac{1}{4} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{2it} + 2 \cdot \sqrt{2e^{it}})} \\ \frac{1}{4e^{it}} \cdot (2e^{2it} - 2 \cdot \sqrt{2e^{it}}) + \frac{1}{4} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{2it} - 2 \cdot \sqrt{2e^{it}})} \end{bmatrix}$$

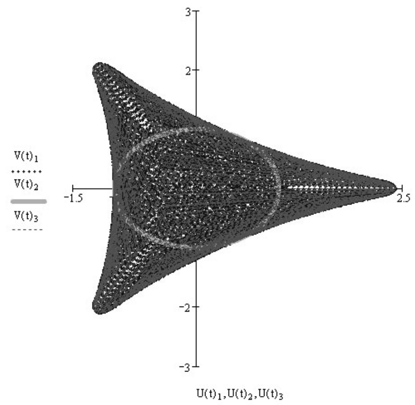
and

$$\mathrm{Im}(\lambda(t)) = \begin{bmatrix} -\frac{i}{2} \cdot (e^{it} - e^{-it}) \\ -\frac{i}{2} \cdot \left[\frac{1}{2e^{2it}} \cdot (2e^{2it} + 2 \cdot \sqrt{2e^{it}}) - \frac{1}{2} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{2it} + 2 \cdot \sqrt{2e^{it}})}\right] \\ -\frac{i}{2} \cdot \left[\frac{1}{2e^{it}} \cdot (2e^{2it} - 2 \cdot \sqrt{2e^{it}}) - \frac{1}{2} \cdot \overline{\frac{1}{e^{it}} \cdot (2e^{2it} - 2 \cdot \sqrt{2e^{it}})}\right] \end{bmatrix}$$

EXAMPLE 5.5.



EXAMPLE 5.6.



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Authors' addresses:

Mircea Olteanu
Department of Mathematics, University Politehnica of Bucharest
e-mail : molteanu@starnets.ro

Tania-Luminita Costache
Department of Mathematics, University Politehnica of Bucharest
e-mail : lumycos@yahoo.com

Mariana Zamfir
Department of Mathematics and Informatics,
Technical University of Civil Engineering Bucharest
e-mail : zacos@k.ro