Mond-Weir type dualities in multiobjective nonsmooth programming

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Abstract. In this paper are developed the generalized Mond-Weir duality and the Preda duality be weak, direct and converse duality theorems for a multiobjective nonsmooth program.

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1 Introduction and preliminaries

Let X be a locally convex space and A be a nonempty open set in X. Consider as well the nonsmooth vector functions $f = (f_1, \dots, f_p)' : A \to \mathbf{R}^p$, $g = (g_1, \dots, g_m)' : A \to \mathbf{R}^m$ and $h = (h_1, \dots, h_q)' : A \to \mathbf{R}^q$, where $p, m, q \in \mathbf{N}^*$. We consider the following multiobjective mathematical program:

$$\text{(MP)} \quad \begin{cases} \text{Minimize (Pareto)} \quad (f_1(x), \dots, f_p(x)) \\ \text{subject to} \quad g(x) \leqq 0,\ h(x) = 0,\ x \in A. \end{cases}$$

The domain of this program is the set

$$D = \{x \in A | g(x) \leqq 0,\ h(x) = 0\}.$$

For the multiobjective program (MP), in this paper are developed two dualities of Mond-Weir type, namely the generalized Mond-Weir duality and the Preda by means of weak, direct and converse duality theorems. The mathematical instrument used in this study is the Clarke-extended subdifferential for real functions defined on X.

In this section we present the Clarke-extended subdifferential ([5], [6]) and some Pareto extremum conditions for the vector program (MP).

1. First, we present the Clarke-extended subdifferential ([5], [6]) and some of its properties. Let X^* be the dual space of X and let $F : A \to \mathbf{R}$ be a real function.

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Definition 1.1. The *Clarke directional derivative* of F at the point $x \in A$ in the direction $v \in X$, denoted $F^0(x;v)$, is defined by

$$F^0(x;v) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{F(x' + \lambda v) - F(x')}{\lambda}.$$

This derivative was introduced by Clarke ([1]) in 1973 for Lipschitz functions. In 1989 we defined the Clarke-extended subdifferential of F at x.

Definition 1.2 (Mititelu [5], [6]). The set

$$\partial F(x) = \{\xi \in \mathbf{R}^n | F^0(x;v) \geq \langle \xi, v \rangle, \quad \forall v \in X\},$$

where $\langle \xi, v \rangle = \xi(v)$, is said to be the *subdifferential* (or the *generalized gradient*) of F at $x \in A$. If $\partial F(x) \neq \varnothing$ then F is called *subdifferentiable* at x. The elements of $\partial F(x)$ are called *subgradients* of F at x.

The vector function $f = (f_1, \dots, f_m) : A \to \mathbf{R}^p$ is subdifferentiable when all its components $f_1, \dots, f_p$ are subdifferentiable functions.

We quote from ([6]) some properties of the subdifferential ∂, that will be used in this paper

(P1) *If F is continuously differentiable function at x, then $\partial F(x) = \{\nabla F(x)\}$.*

(P2) *If the direction function $F^0(x;\cdot)$ is finite, then the subdifferential $\partial F(x)$ is a nonempty, convex and compact set. Moreover,*

$$F^0(x;v) = \max\{\xi' v | \xi \in \partial F(x)\}, \quad \forall x \in A.$$

(P3) *If F is subdifferentiable at x, then λF $(\lambda \in \mathbf{R})$ is subdifferentiable at x and*

$$\partial(\lambda F)(x) = \lambda \cdot \partial F(x).$$

(P4) *If the function $f_1, \dots f_p$ are subdifferentiable at $x \in A$, then the function $\sum_{i=1}^{p} f_i$ is subdifferentiable at x and*

$$\partial \left(\sum_{i=1}^{p} f_i \right)(x) \subseteq \sum_{i=1}^{p} \partial f_i(x).$$

2. Also, we use Clarke's tangent cone and the normal cone in Clarke's sense at a point x of a nonempty set C of X.

Clarke's tangent cone to C at $x \in C$ is defined by the set (one of its equivalent forms [1]):

$$T_C(x) = \{v \in X^{\bullet} | \forall t_k \downarrow 0,\ \forall (x^k) \subset C : x^k \to x,\ \exists v^k \to v \text{ such that } x + tv \in C\}.$$

The normal cone to C at $x \in C$ is defined by the set ([1]):

$$N_C(x) = \{\nu \in \mathbf{R}^n | \nu' v \leq 0,\ \forall v \in T_C(x)\}.$$

The cones $T_C(x)$ and $N_C(x)$ are nonempty, closed and convex sets ([1]).

3. According to ([13]), for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we shall use the following notations:

$$\begin{aligned}
x = y \quad &\Leftrightarrow \quad x_i = y_i,\ i = \overline{1,n},\\
x > y \quad &\Leftrightarrow \quad x_i > y_i,\ i = \overline{1,n},\\
x \geqq y \quad &\Leftrightarrow \quad x_i \geq y_i,\ i = \overline{1,n},\\
x \geq y \quad &\Leftrightarrow \quad x_i \geqq y_i,\ x \neq y.
\end{aligned}$$

Definition 1.3 (Geoffrion [2]). A point $x^0 \in D$ is said to be an *efficient solution* (Pareto minimum) for (MP) if there exists no other feasible point $x \in D$ such that $f(x) \leqq f(x^0)$ and $f(x) \neq f(x^0)$ (or equivalently, $f(x) \leq f(x^0)$).

Lemma 1.1 (Kanniapan [3]). *A point $x^0 \in D$ is an efficient solution to* (MP) *if and only if x^0 solves the scalar program*

$$(P_k) \qquad \begin{cases} \text{Minimize} \quad f_k(x) \\ \text{subject to} \quad f_s(x) \leq f_s(x^0),\ \forall s \neq k,\ g(x) \leqq 0,\ h(x) = 0, x \in A, \end{cases}$$

for each $k = \overline{1,p}$.

Definition 1.4 (Geoffrion [2]). A feasible point $x^0 \in D$ is said to be a *properly efficient solution* in (MP) if it is efficient solution in (MP) and there exists a scalar $S > 0$ such that, for each i, we have

$$\frac{f_i(x) - f_i(x^0)}{f_j(x^0) - f_j(x)} \leq S$$

for some j such that $f_j(x) < f_j(x^0)$, whenever $x \in D$ and $f_i(x) > f_i(x^0)$.

Geoffrion considered the following scalar parametric program

$$(P_t) \qquad \begin{cases} \text{Minimize} \quad t' f(x) \\ \text{subject to:} \quad g(x) \leqq 0,\ h(x) = 0, x \in A \\ \quad t > 0,\ t'e = 1,\ e = (1, \dots, 1)' \in \mathbf{R}^p \end{cases}$$

and he established the following:

Lemma 1.2 (Geoffrion [2]). *Let $t > 0$ be fixed with $t'e = 1$. If x^0 is an optimal solution of (P_t), then x^0 is a properly efficient solution of* (MP).

4. Kuhn-Tucker efficiency conditions for (MP). Let $x^0 \in D$. We define the index sets $I^0 = \{i | g_i(x) = 0\}$ and $J^0 = \{1, \dots, m\} \setminus I^0$. Consider the following constraint qualification for D at x^0:

$$R(x^0) \qquad \begin{cases} \exists v \in X : g^0_{I^0}(x^0; v) \leq 0,\ h^0(x^0; v) = 0, \\ \exists \varepsilon > 0 : g_{J^0}(x^0 + \varepsilon v) \leqq 0,\ h(x^0 + \varepsilon v) = 0. \end{cases}$$

Mititelu ([11]) established the following necessary efficiency conditions of the Kuhn-Tucker type for (MP) at x^0:

Theorem 1.1 (Necessary efficiency conditions). *Let x^0 be a local efficient solution of* (MP), *where the functions f, g and h are subdifferentiable, and $h^0(x^0; \cdot)$ is finite* *on X. Also, we suppose that* (MP) *satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $y^0 = (y_1^0, \dots, y_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 are satisfied*:

$$\text{(KT1)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{p} t_k^0 \partial f_k(x^0) + \sum_{i=1}^{m} u_i^0 \partial g_i(x^0) + \sum_{j=1}^{q} v_j^0 \partial h_j(x^0) + N_A(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \ u^0 \geqq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \ t^{0'} e = 1. \end{cases}$$

Corollary 1.2 (Efficiency necessary conditions). *Let x^0 be a local efficient solution of* (MP), *where the functions f, g and h are subdifferentiable, and $h^0(x^0; \cdot)$ is finite. We also assume that* (MP) *satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions for* (MP) *at x^0 are satisfied*:

$$\text{(KT2)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{p} t_k^0 \partial f_k(x^0) + \sum_{i=1}^{m} u_i^0 \partial g_i(x^0) + \sum_{j=1}^{q} v_j^0 \partial h_j(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \ u^0 \geqq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \ t^{0'} e = 1. \end{cases}$$

Corollary 1.3 (Efficiency necessary conditions). *Let x^0 be a local efficient solution of* (MP), *where the functions f, g and h are subdifferentiable. Also we suppose that* (MP) *satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 for* (MP) *are satisfied*:

$$\text{(KT3)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{p} t_k^0 \partial f_k(x^0) + \sum_{i=1}^{m} u_i^0 \partial g_i(x^0) + \sum_{j=1}^{q} v_j^0 \partial h_j(x^0) + N_A(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \ u^0 \geqq 0, \ v^0 \geqq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \ t^{0'} e = 1. \end{cases}$$

Corollary 1.4 (Efficiency necessary conditions). *Let x^0 be a local efficient solution of* (PV), *where the functions f, g and h are subdifferentiable. Moreover, we suppose that* (MP) *satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 for* (MP) *are satisfied*:

$$\text{(KT4)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{p} t_k \partial f_k(x^0) + \sum_{i=1}^{m} u_i \partial g_i(x^0) + \sum_{j=1}^{q} v_j \partial h_j(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \ u^0 \geqq 0, \ v^0 \geqq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \ t^{0'} e = 1. \end{cases}$$

Remark 1.1. Everywhere, in relations (KT1), (KT2), (KT3) and (KT4) the relation $u^{0'}g(x^0) = 0$ is equivalent with the following relations:

$$u_i^0 g_i(x^0) = 0, \quad i = \overline{1, m}.$$

5. In a recent paper ([11]) Mititelu developed for the multiobjective program (MP) a duality of Wolfe type.

In the following we developed for the program (MP) dualities of Mond-Weir types, namely: the generalized Mond-Weir duality and Preda duality.

2 Generalized Mond-Weir duality for multiobjective program (MP)

In this section is developed a generalized Mond-Weir duality for the multiobjective mathematical program (MP) in the case when its functions are sub-differentiable nonsmooth on A. We consider the sets $M = \{1, \ldots, m\}$ and $Q = \{1, \ldots, q\}$. Let $\{J_0, J_1, \ldots, J_r\}$ be a partition of M, that is

$$J_\alpha \subseteq M,\ J_\alpha \cap J_\beta = \varnothing \quad \text{if} \quad \alpha \neq \beta,\ \bigcup_{\alpha=0}^{r} J_\alpha = M$$

and let $\{K_0, K_1, \ldots, K_r\}$ be a similar partition of Q.

The generalized dual program of Mond-Weir type, associated to the multiobjective nonsmooth program (MP), is the following multiobjective nonsmooth program:

$$\text{(MWD)} \begin{cases} \text{Maximize } L^0(y,u,v) = f(y) + [u'_{J_0} g_{J_0}(y) + v'_{K_0} h_{K_0}]e \\ \text{subject to} : \displaystyle\sum_{k=1}^{p} t_k \partial f_k(y) + \sum_{i=1}^{m} u_i \partial g_i(y) + \sum_{j=1}^{q} v_j \partial h_j(y) + N_A(y) \supset \{0\} \\ \qquad\qquad u'_{J_\alpha} g_{J_\alpha}(y) + v'_{K_\alpha} h_{K_\alpha}(y) \geqq 0,\ \alpha = \overline{1, r} \\ y \in A,\ t \in \mathbf{R}_+^p,\ t'e = 1,\ u \in R^q, \end{cases}$$

where

$$u'_{J_\alpha} g_{J_\alpha}(y) = \sum_{i \in J_\alpha} u_i g_i(y), \quad v'_{K_\alpha} h_{J_\alpha}(y) = \sum_{j \in K_\alpha} v_j h_j(y).$$

We call the function L^0, the generalized Lagrangian associated to program (MP). We also denote by $\Omega = \{(t, y, u, v) | \ldots\}$ the domain of the dual (MWD) and let $\Omega_t = \{(y, u, v) | (t, y, u, v) \in \Omega\}$.

The set $\Omega_0 = \displaystyle\bigcup_{\substack{t \geq 0 \\ t'e=1}} \Omega_t$ is the domain of the generalized Lagrangian objective $L^0(y, u, v)$. Therefore, the domain Ω of the restrictions is different from the domain Ω_0 of the objective L^0.

Definition 2.1. A point $(t^0, x^0, u^0, v^0) \in \Omega$ is said to be a t^0-*efficient solution* of (MWD) if (x^0, u^0, v^0) is an efficiency point of maximum type for $L^0(y, u, v)$.

Theorem 2.1 (Weak duality). *We suppose that*:

a) *The domain* D *and* Ω *of the dual programs* (MP) *and* (MWD) *are nonempty.*

b) *For each* $(t, y, u, v) \in \Omega$, *the inequality* $L^0(y, u, v) \geq L^0(x, u, v)$ *is false,* $\forall x \in A$. *Then, for* $\forall x \in D$ *and* $\forall(t, y, u, v) \in \Omega$, *the inequality* $f(x) \leq L^0(y, u, v)$ *is false.*

Proof. According to b), for each $(t, y, u, v) \in \Omega$ and $\forall x \in A$ the inequality

$$L^0(y, u, v) \geq f(x) + [u'_{J_0} g_{J_0}(x) + v'_{K_0} h_{K_0}(x)]e \tag{2.1}$$

is false. Taking into account the true relations:

$$u_i g_i(x) \leq 0,\ i = \overline{1, m},\ \forall x \in D,\ \forall u \geq 0,$$

$$v_j h_j(x) = 0,\ j = \overline{1, q},\ \forall x \in D,\ \forall v \in \mathbf{R}^q,$$

we obtain

$$u'_{J_\alpha} g_{J_\alpha}(x) + v'_{K_\alpha} h_{K_\alpha}(x) \leq 0$$

and then, we infer that $(t, y, u, v) \in \Omega$ and moreover, from (2.1), it results that for each $(t, y, u, v) \in \Omega$ and $\forall x \in D$ the inequality $f(x) \leq L^0(y, u, v)$ is false. But the point (t, y, u, v) being arbitrarily taken in Ω, it results that for $\forall x \in D$ and $\forall(t, y, u, v) \in \Omega$ the relation $f(x) \leq L^0(y, u, v)$ is false.

Theorem 2.2 (Direct duality). *Let* x^0 *be a local efficient solution of the primal* (MP), *where the functions* f, g *and* h *are subdifferentiable. Also, we assume the next hypotheses*:

(d1) *The domain* D *satisfies the constraint qualifications* $R(x^0)$.

(d2) *For every* $(t, y, u, v) \in \Omega$, *the inequality* $L^0(y, u, v) \geq L^0(x, u, v)$ *is false,* $\forall x \in A$.

Then there exist the vectors $t^0 \in \mathbf{R}^p$, $u^0 \in \mathbf{R}^m$ *and* $v^0 \in \mathbf{R}^q$ *such that* (t^0, x^0, u^0, v^0) *is a* t^0*-efficient solution of the dual* (MWD) *and* $f(x^0) = L^0(x^0, u^0, v^0)$.

Proof. Since x^0 is a local efficient solution of (MP) and (MP) satisfies the constaint qualification $R(x^0)$, then (MP) verifies the consitions (KT3) of Corollary 2.1. Using Remark 1.1, from these conditions it results that $(t^0, x^0, u^0, v^0) \in \Omega$ and the relations $u^0_{J_0} g_{J_0} = 0$, $v^0_{K_0} h(x^0) = 0$. Then,

$$f(x^0) = f(x^0) + [u^0_{J_0} g_{J_0}(x^0) + v^0_{K_0} h_{K_0}(x^0)]e = L^0(x^0, u^0, v^0).$$

From (d2), according to Theorem 2.1, it results that $f(x^0) \leq L^0(y, u, v)$ is false. Moreover, it results that $L^0(x^0, u^0, v^0) \leq L^0(y, u, v)$, $\forall(t, y, u, v) \in \Omega$ is false. Then (t^0, x^0, u^0, v^0) is a t^0-efficient point of maximum type for L^0.

Theorem 2.3 (Converse duality). *Let* (t^0, x^0, u^0, v^0) *be a* t^0*-efficient solution of* (MWD), *where* $t^0 > 0$. *We suppose that*:

(c1) *The primal program* (MP) *admits the efficient solution* $\bar{x}$, *where* D *verifies the constraint qualification* $R(\bar{x})$.

(c2) *The function* $L^0(x^0, u^0, v^0)$ *admits at* x^0 *an efficient minimum on* A.

Then $x^0 = \bar{x}$, *where* x^0 *is a properly efficient solution of* (MP) *and* $f(x^0) = L^0(x^0, u^0, v^0)$.

Proof. We suppose that $x^0 \neq \bar{x}$ and we shall a contradiction. Because $\bar{x}$ is an efficient solution of (MP) which satisfies the constraint qualification $R(\bar{x})$, then there exist the vectors $\bar{t} \in \mathbf{R}^p_+$, $\bar{u} \in \mathbf{R}^m$, $\bar{u} \geqq 0$ and $\bar{v} \in \mathbf{R}^q$, $\bar{v} \geqq 0$, such that (MP) satisfies at $\bar{x}$ efficiency Kuhn-Tucker conditions of the form (KT3) (with $\bar{x}$ instead of x^0); consequently, $(\bar{t}, \bar{x}, \bar{u}, \bar{v}) \in \Omega$ and

$$\bar{u}' g(\bar{x}) = 0. \tag{2.2}$$

Also, we have

$$\bar{v}' h(\bar{x}) = 0. \tag{2.3}$$

Obviously, the relation $L^0(x^0, u^0, v^0) \leq L^0(\bar{x}, \bar{u}, \bar{v})$ is false, because (t^0, x^0, u^0, v^0) is a t^0-efficient solution of (MWD). Multiplying this inequality by $t^0 > 0$ it results that the relation $t^{0'} L^0(x^0, u^0, v^0) \leq t^{0'} L^0(\bar{x}, \bar{u}, \bar{v})$ is false. Then the next relation is true:

$$t^{0'} L^0(x^0, u^0, v^0) > t^{0'} L^0(\bar{x}, \bar{u}, \bar{v}). \tag{2.4}$$

For every $x \in D$ the relation $L^0(x, u^0, v^0) \leq L^0(x^0, u^0, v^0)$ is false (see (c2)). Then it results that the relation $t^{0'} L^0(x, u^0, v^0) \leq t^{0'} L^0(x^0, u^0, v^0)$ is false too. Consequently we have

$$t^{0'} L^0(x, u^0, v^0) > t^{0'} L^0(x^0, u^0, v^0), \quad \forall x \in D \tag{2.5}$$

and particularly, for $x = \bar{x}$, we obtain

$$t^{0'} L(\bar{x}, u^0, v^0) > t^{0'} L(x^0, u^0, v^0). \tag{2.6}$$

From the relations (2.4) and (2.6) we obtain

$$t^{0'} L(\bar{x}, u^0, v^0) > t^{0'} L(\bar{x}, \bar{u}, \bar{v}),$$

or equivalently (using (2.2), (2.3)), $u^{0'} g(\bar{x}) > 0$, that is a contradiction. Therefore, $x^0 = \bar{x}$.

We have

$$t^{0'} f(x) \geq t^{0'} f(x) + u^{0'} g(x) + v^{0'} h(x) = t^{0'} L^0(x, u^0, v^0), \tag{2.7}$$

or shortly, $t^{0'} f(x) > t^{0'} L^0(x, u^0, v^0)$. Using now in this inequality relation $(t^0, x, u^0, v^0) \in \Omega$ it result $t^{0'} f(x) > t^{0'} f(x^0)$, $\forall x \in D$. Taking into account Lemma 1.2 we infer that x^0 is a properly efficient solution of (MP).

Remark 2.1. Using other Kuhn-Tucker conditions, given by Theorem 1.1 or Corollaries 1.2 or 1.4, instead of (KT3), result other duality theorems.

Remark 2.2. For the following particular partitions of M and Q:

$$J_0 = M,\ J_\alpha = \varnothing\ ,\ \alpha = \overline{1, r};\ K_0 = Q,\ K_\alpha = \varnothing\ ,\ \alpha = \overline{1, r},\ r = \max\{m, q\}$$

is obtained a Wolfe duality, generated by (MP) and duality which was studied by us in ([11]).

3 The Preda duality for multiobjective program (MP)

Let $\{J_1, \dots, J_r\}$ be a partition of the set M and $\{K_1, \dots, K_r\}$ a partition of Q. The dual program Preda associated to the primal multiobjective program (MP) is the following multiobjective program

$$\text{(MPD)}\begin{cases} \text{Maximize (Pareto) } L(y,u,v) = f(y) + [u'g(y) + v'h(y)]e \\ \text{subject to}: \sum_{k=1}^{p} t_k \partial f_k(y) + \sum_{i=1}^{m} u_i \partial g_i(y) + \sum_{j=1}^{q} v_j \partial h_j(y) + N_A(y) \supset \{0\} \\ \qquad u'_{J_\alpha} g_{J_\alpha} \overset{\leq}{=} 0,\ v'_{K_\alpha} h_{K_\alpha} = 0,\ \alpha = \overline{1,r} \\ \qquad y \in A,\ t \in \mathbf{R}^p_+,\ t'e = 1,\ u \in R. \end{cases}$$

Between the multiobjective programs (MP) and (MPD) we develop a duality by means of the following weak, direct and converse duality theorems:

Theorem 3.1. (Weak duality). *We suppose that*:

a) *The domains D and Ω_p of the dual programs* (MP) *and* (MPD) *are nonempty.*

b) *For each $(t, y, u, v) \in \Omega_p$ the inequality $L(y, u, v) \geq L(x, u, v)$, $\forall x \in A$, is false.*

Then $\forall x \in D$, $\forall (t, y, u, v) \in \Omega_p$, the inequality $f(x) \leq L(y, u, v)$ is false.

Theorem 3.2. (Direct duality). *Let x^0 be a local efficient solution of the primal* (MP), *where the functions f, g and h are subdifferentiable. Also, we suppose that the following hypotheses are satisfied*:

(d1) *The domain D satisfies the constraint qualifications $R(x^0)$.*

(d2) *For every $(t, y, u, v) \in \Omega_p$, the inequality $L(y, u, v) \geq L(x, u, v)$ is false $\forall x \in A$.*

Then there exist the vectors $t^0 \in \mathbf{R}^p$, $u^0 \in \mathbf{R}^m$ and $v^0 \in \mathbf{R}^q$, such that (t^0, y^0, u^0, v^0) is a t^0-efficient solution of the dual (MPD) *and $f(x^0) = L(x^0, u^0, v^0)$.*

Theorem 3.3. (Converse duality). *Let (t^0, x^0, u^0, v^0) be a t^0-efficient solution of* (MPD), *where $t^0 > 0$. We suppose that*

(c1) *The primal program* (MP) *admits the efficient solution $\bar{x}$, where D verifies the constraint qualification $R(\bar{x})$.*

(c2) *The function $L(x, u^0, v^0)$ admits at x^0 an efficient minimum on A.*

Then $x^0 = \bar{x}$, where x^0 is a properly efficient solution of (MP) *and $f(x^0) = L(x^0, u^0, v^0)$.*

The proofs of Theorems 3.1, 3.2 and 3.3 are similar to those of Theorems 2.1, 2.2, 2.3, respectively.

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