Variational study of an elliptic boundary problem

Olga Martin

Abstract. Using the Lax-Milgram theorem and the techniques of the abstract functional analysis, we prove the existence and uniqueness of the solution of a boundary value problem for a non-homogeneous Helmholtz equation.

Key words: abstract functional analysis, Lax-Milgram theorem, Helmholtz equation, Sobolev space, Hilbert space.

Let us consider the Helmholtz’s equation rewritten in the form

\( \frac{\partial^2 \varphi(x, y)}{\partial x^2} + \frac{\partial^2 \varphi(x, y)}{\partial y^2} + k^2 \varphi(x, y) = -f(x, y) \) (1)

with an homogeneous boundary value problem

\( \varphi|_{\Gamma} = 0, \) (2)

where the boundary \( \Gamma \) of \( D \) is a smooth contour, \( f \) is a given function in \( \forall (x, y) \in D \) and \( k \) is a constant.

In order to get the solution \( \varphi \) of the problem (1)-(2), we define the following spaces:

(a) the Hilbert space \( H = L^2(D) \) (the quadratically integrable functions) with the scalar product defined by the formula

\( (u, v) = \int\int_D u(x, y)v(x, y)dxdy \) (3)

(b) the Sobolev space defined by

\( W^{1,2}(D) = \{ \tilde{u} \in L^2(D) | \exists g_1, g_2 \in L^2(D) \text{ such that} \} \)

\( \int\int_D \tilde{u} \frac{\partial v}{\partial x} = -\int\int_D g_1v \quad \text{and} \quad \int\int_D \tilde{u} \frac{\partial v}{\partial y} = -\int\int_D g_2v, \forall v \in C_c(D) \} \) (4)

where \( C_c(D) \) is the space of the continuous functions with compact support.

Now, we define in the space \( W^{1,2}(D) \) the scalar product

\[
(5) \quad (u, v) = (u, v)_{L_2} + \sum_{i=1}^{2} \left( \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \right)_{L_2} = \int_D \left( w + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy.
\]

Here, the \( W^{1,2}(D) \) is the prolongation of the space \( C^1(D) \) with the limit points of the Cauchy sequences from this space, hence a Banach space, \( [1] \). The space \( W^{1,2}(D) \) becomes a Hilbert space \( H^1(D) \) with the scalar product (5).

If the functions \( \tilde{u} \) satisfy the boundary condition (2), then \( H^1(D) \) becomes the Hilbert space \( H^1_0(D) \).

**Lemma 1.** Every classical solution of the problem (1)-(2) is a weak solution.

**Proof.** A classical solution of the problem is a function \( \varphi \in C^2(\bar{D}) \), which verifies (1)-(2).

A weak solution of the same problem is a function \( \varphi \in H^1_0(D) \) under the following condition:

\[
(6) \quad \int_D \nabla \varphi \cdot \nabla v - k^2 \int_D \varphi v = \int_D f v, \quad \forall v \in H^1_0(D)
\]

where

\[
\nabla \varphi \cdot \nabla v = \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial v}{\partial y} \quad \text{and} \quad v|_\Gamma = 0.
\]

Let us consider \( \varphi \in H^1(D) \cap C(\bar{D}) \) and \( \varphi = 0 \) on the boundary \( \Gamma \). Then \( \varphi \in H^1_0(D) \) and we shall prove that (6) is verified for every \( v \in H^1_0(D) \).

For this, we multiply (1) by \( v \in C^1(D) \) and integrate over the domain \( D \). We have

\[
(7) \quad \int_D \left( \frac{\partial^2 \varphi}{\partial x^2} v + \frac{\partial^2 \varphi}{\partial y^2} v \right) + k^2 \int_D \varphi v = - \int_D f v.
\]

Applying the Green’s formula to the first term we get

\[
(8) \quad \int_D v \Delta \varphi = \int_\Gamma \left( - \frac{\partial \varphi}{\partial y} v dx + \frac{\partial \varphi}{\partial x} v dy \right) - \int_D \nabla v \cdot \nabla \varphi \Rightarrow \int_D v \Delta \varphi = - \int_D \nabla v \nabla \varphi.
\]

Since \( C^1(D) \) is dense in \( W^{1,2}(D) \), we obtain from (7) and (8) for every \( v \in H^1_0(D); v|_\Gamma = 0 \), the following equality

\[
(9) \quad - \int_D \nabla \varphi \nabla v + k^2 \int_D \varphi v = - \int_D f v
\]

and lemma is proved. \( \square \)
Definition. A bilinear form \( a(u, \varphi) : H \times H \to \mathbb{R} \) is called:

1. continuous, if there exists a constant \( K_1 \) such that

\[
|a(\varphi, v)| \leq K_1|\varphi||v|, \quad \forall \varphi, v \in H; \tag{10}
\]

2. coercive, if there exists a constant \( \gamma > 0 \) such that

\[
a(\varphi, \varphi) \geq \gamma|\varphi|^2, \quad \forall \varphi \in H(D) \tag{11}
\]

Lemma 2. The bilinear form

\[
a(\varphi, v) = \iint_D \nabla \varphi \nabla v - k^2 \iint_D \varphi v \tag{12}
\]

is continuous in \( H^1(D) \times H^1(D) \).

Proof. Let us consider \( K_1 = \max(1; k^2) \). Using the Cauchy-Schwarz inequality we get

\[
|a(\varphi, v)| = \left| \iint_D \nabla \varphi \nabla v - k^2 \iint_D \varphi v \right| \leq \iint_D |\nabla \varphi \nabla v - k^2 \varphi v| \leq K_1 \left( \iint_D [\varphi^2 + (\nabla \varphi)^2] \right)^{1/2} \left( \iint_D [v^2 + (\nabla v)^2] \right)^{1/2} = K_1 ||\varphi||_{H^1(D)} ||v||_{H^1(D)}.
\]

It follows from (10) that the bilinear form (12) is continuous in \( H^1(D) \times H^1(D) \).

Lemma 3. The bilinear form

\[
a(\varphi, v) = \iint_D \nabla \varphi \cdot \nabla v - k^2 \iint_D \varphi v \tag{13}
\]

is coercive in \( H^1_0(D) \).

Proof. In accordance with (13) we have

\[
a(\varphi, \varphi) = \iint_D \left( \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 \right) - k^2 \iint_D \varphi^2 \tag{14}
\]

where \( \varphi \in H^1_0(D) \) (vanish on \( \Gamma \)).

We enclose the domain \( D \) in a rectangle \( D_1 \) (with sides \( \alpha \) and \( \beta \)), whose two sides are the coordinate axes. Equating this function to zero, we extend it over the entire rectangle \( D_1 \). If \( (x_1, y_1) \) is an arbitrary point, we get

\[
\varphi(x_1, y_1) = \int_0^{x_1} \frac{\partial \varphi(x, y_1)}{\partial x} dx + \varphi(0, y_1) = \int_0^{x_1} \frac{\partial \varphi(x, y_1)}{\partial x} dx.
\]

Using the Cauchy-Schwarz inequality we get

\[
\varphi^2(x_1, y_1) = \left( \int_0^{x_1} 1 \cdot \frac{\partial \varphi(x, y_1)}{\partial x} dx \right)^2 \leq \int_0^{x_1} 1^2 dx \int_0^{x_1} \left( \frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx = x_1 \int_0^{x_1} \left( \frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx \leq \alpha \int_0^\alpha \left( \frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx = \alpha \cdot F(y_1).
\]
Integrating over the entire rectangle $D_1$ we obtain
\[ \iint_{D_1} \phi^2(x_1, y_1) dx_1 dy_1 \leq \int_0^\alpha \alpha dx_1 \int_0^\beta F(y_1) dy_1 \leq \alpha^2 \iint_{D_1} \left( \frac{\partial \phi(x, y)}{\partial x} \right)^2 dx dy. \]

Analogously,
\[ \iint_{D_1} \phi^2(x_1, y_1) dx_1 dy_1 \leq \beta^2 \int_{\Gamma_1} \left( \frac{\partial \phi(x, y)}{\partial y} \right)^2 dx dy. \]

Therefore
\[ \iint_{D} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] dx dy \geq \frac{\alpha^2 + \beta^2}{\alpha^2 \beta^2} \int_{D} \phi^2(x, y) dx dy \geq \frac{2}{\alpha \beta} \int_{D} \phi^2(x, y) dx dy \]

Thus, we obtained the Friedrichs inequality for our problem. If $A = \max(\alpha, \beta)$, we find from (14) and (15) the following inequality
\[ a(\phi, \phi) = \frac{2}{A^2} \int_{D} (|\nabla \phi|^2 - k^2 \phi^2) \geq 2 \alpha \beta \int_{D} \phi^2 \quad \forall \phi \in H^1_0(D) \quad (16) \]

In view of the definition, the bilinear form is coercive if $k \leq 0 \left( \sqrt{\frac{2}{A}} \right)$. \hfill \Box

**Theorem (Lax-Milgram, [1]).** Let $a(\phi, v)$ be a bilinear form, $a : H^1_0 \times H^1_0 \rightarrow \mathbb{R}$, which is continuous and coercive. Then, for every $f \in L^2(D)$ exists an unique $\phi \in H^1_0(D)$ such that
\[ a(\phi, v) = \int_{D} f v, \quad \forall v \in H^1_0(D). \quad (17) \]

Moreover, if $a(\phi, v)$ is symmetric we find $\phi \in H^1_0$ by
\[ \frac{1}{2} a(\phi, \phi) - (f, \phi) = \min_{v \in H^1_0} \left\{ \frac{1}{2} \int_{D} (|\nabla v|^2 - k^2 v^2) - \int_{D} fv \right\}. \quad (18) \]

**Theorem 1.** If $f \in L^2(D)$, then the weak solution of the problem
\[ \Delta \phi(x, y) + k^2 \phi(x, y) = -f(x, y) \quad (19) \]

exists and is unique.

**Proof.** It follows from Lemma 2 and Lemma 3 that $a(\phi, v)$ is continuous and coercive in $D$. Applying the theorem Lax-Milgram’s for $f \in L^2(D)$, we get an unique weak solution of our boundary problem. \hfill \Box
Theorem 2. If $\varphi \in H_0^1(D) \subset L_2(D)$ is a weak solution of (19)-(20), then $\varphi \in H^2(D)$.

Proof. Let us consider the function $v$ of the form

$$v(x, y) = \begin{cases} e^{-\varphi}, & \forall (x, y) \in D \\ 0, & \forall (x, y) \notin D \end{cases}$$

where $\varphi \in H_0^1(D) \subset L_2(D)$.

If the boundary $\Gamma$ of $D$ is a smooth contour, it is sufficient that $v \in H^1(D)$ and it is not necessary that $v \in C(\overline{D})$, [1]. We shall show that starting in (4) with the function $v$ and $\tilde{u} = \frac{\partial \varphi}{\partial x} \in L_2(D)$, we obtain

$$\int\int_D \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} = -\int\int_D \left(\frac{\partial \varphi}{\partial x}\right)^2 e^{-\varphi}.$$  (21)

Hence, there exits a function $g_1 = \left(\frac{\partial \varphi}{\partial x}\right)^2 \in L_2(D)$ such that

$$\int\int_D \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} = -\int\int_D g_1 v.$$  (23)

Analogously, there exists

$$g_2 = \left(\frac{\partial \varphi}{\partial y}\right)^2 \in L_2(D)$$

such that

$$\int\int_D \frac{\partial \varphi}{\partial y} \frac{\partial v}{\partial y} = -\int\int_D g_2 v.$$  (24)

According to (4), (21) and (22), it should be observed that $\frac{\partial \varphi}{\partial x} \in H^1(D)$ and $\frac{\partial \varphi}{\partial y} \in H^1(D)$ when $f \in L_2(D)$. Consequently, $f \in H^2(D)$.

Theorem 3. Let the mild solution of (19)-(20) be $\varphi \in C^2(D)$. If $f \in L_2(D)$, then $\varphi$ is a classical solution of the problem.

Proof. Let us consider $\varphi \in C^2(D)$, which verifies (6)

$$\int\int_D \left(\frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial v}{\partial y}\right) - k^2 \int\int_D \varphi v = \int\int_D fv$$

$\forall v \in C^1_0(D)$ and $v|_\Gamma = 0$. Integrating by parts we get

$$\int\int_D \left[\left(-\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2}\right) - k^2 \varphi - f\right] v dx dy = 0.$$  (23)

Since $C^1_0(D)$ is dense in $L_2(D)$, we obtain that

$$\Delta \varphi(x, y) + k^2 \varphi(x, y) = -f(x, y)$$  (25)

$$\varphi|_\Gamma = 0$$

everywhere in $D$. Since $\varphi \in C^2(D)$, the equation (23) is verified in $D$. \qed
Conclusion. Many authors paid attention to the abstract variational formulation of the boundary problem for partial differential equations, [6], [9], [12], [13]. An entertaining and complete survey of the results obtained in this field of functional analysis appears in [1]. Applications of the theory of semi-groups of linear operators to differential equations are presented by Pazy in [13].

An exact solution for the problem (1)-(2) has been presented in [15]. If $D = [0, a] \times [0, b]$, it is of the form

$$
\varphi(x, y) = \int_0^a \int_0^b f(x, y) G(x, y, \xi, \eta) d\xi d\eta
$$

where

$$
G(x, y, \xi, \eta) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(p_n x)(\sin(p_n \xi))}{\beta_n \sinh(\beta_n b)} \cdot H_n(y, \eta), p_n = \frac{\pi n}{a}, \\
\beta_n = \sqrt{p_n^2 - k^2}, \ a \geq b
$$

$$
H_n(y, \eta) = \begin{cases} 
\sinh(\beta_n \eta) \sinh(\beta_n(b-y)), & b \geq y > \eta \geq 0 \\
\sinh(\beta_n y) \sinh(\beta_n(b-\eta)), & b \geq \eta > y \geq 0
\end{cases}, \ n = 1, 2, \ldots
$$

It should be observed that in our case, when we study the solution of a boundary problem using the techniques of the abstract functional analysis, the natural oscillating frequencies $k > 0$, belong to

$$
k \in \left(0, \frac{\sqrt{2}}{A}\right) \subset \left(0, \frac{\pi}{A}\right),
$$

the interval which was obtained from (26). Here $A$ is the greatest side of the rectangle $D_1$.

References

Variational study of an elliptic boundary problem


Author’s address:

Olga Martin
Department of Mathematics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Bucharest, Romania.
e-mail: omartin_ro@k.ro