# A moment problem with values positive definite matrices

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**Abstract.** In this article, we study the following generalization of a classical complex moment problem: Given a Hermitian multisequence of k-dimensional matrices with complex entries, when does exist a nonnegative k-dimensional matrix of positive Borelian measures such that every term of the given sequence admits a moment representation with respect to the matrix of measures.

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## 1 Introduction and preliminaries

In this note, following the ideas of K.Schmudgen in [6], we reformulate and solve a k-complex moment sequence having as values complex Hermitian matrices. Obviously, in case $k = 1$ the problem reduces to the classical 1 dimensional complex moment problem. The k-complex moment problem solved is: given a Hermitian multisequence

$$S_{(m,n)} = (a_{i,j}(m,n))_{1\leq i,j\leq k} \forall (m,n) \in Z_+^2$$

of $(k,k)$ matrices with complex entries $a_{i,j}(m,n)$ when does exist a nonnegative $(k,k)$ matrix

$$\Lambda = (\lambda_{i,j})_{1\leq i,j\leq k}$$

of positive Borel measures $\lambda_{i,j}$ on the unit polydisc $D_1 \subset C$ such that $a_{i,j}(m,n) = \int_{D_1} z^m \overline{z^n} d\lambda_{i,j}(z)$ for every $1 \leq i,j \leq k$ and any $(m,n) \in Z_+^2$.
*Notation* Let $k \in N^*, D_1 = \{z \in \mathbb{C}, |z| \leq 1\}$; a matrix $\Lambda = (\lambda_{i,j})_{1\leq i,j\leq k}$ of positive Borel measures on $D_1$ is nonnegative definite on $D_1$ if

$$\sum_{1\leq i,j\leq k} \lambda_{i,j}(B) t_i t_j \geq 0$$

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for any $B \in Bor(D_1)$ and any $t = (t_1, ..., t_k) \in R^k$. We denote with $M_k^*(D_1)$ the set of positive definite matrices of positive measures on $D_1$, having complex moments of all orders. Let $\{S_{(m,n)}\}_{(m,n)\in Z_+^2}$ be a complex Hermitian multisequence of k-dimensional complex matrices that is $S_{m,n} = \overline{S_{n,m}}$ for any $(m, n) \in Z_+^2$.

**Definition 1.** *The Hermitian multisequence of matrices* $\{S_{(m,n)}\}_{m,n} \in Z_+^2$ *is called a k- complex moment sequence on* $D_1$ *if there exists a matrix*

$$\Lambda = (\lambda_{i,j})_{1\leq i,j\leq k} \in M_k^*(D_1)$$

*such that*

$$a_{i,j}(m, n) = \int_{D_1} z^m \overline{z}^n d\lambda_{i,j}(z)$$

*for all* $(m, n) \in Z_+^2$ *and all* $1 \leq i, j \leq k$.

These equalities can also be written as

$$S_{m,n} = \int_{D_1} z^m \overline{z}^n d\Lambda(z).$$

Let $\mathbb{P}_n(\mathbb{C}) = \{P(z, \overline{z}) = \sum_{(m,n)\in H} a_{mn} z^m \overline{z}^n, a_{mn} \in \mathbb{C}\}$ the $\mathbb{C}$-vector space of polynomials in $z, \overline{z}$ variable with complex coefficients. As in the theory of the classical moment problems, it is useful to replace the Hermitian multisequence $\{S(m, n)\}_{m,n}$ of $(k, k)$ matrices by $k \times k$ $\mathbb{C}$-linear mappings

$$\mathbb{S}_{ij} : \mathbb{P}_n(\mathbb{C}) \to \mathbb{C}, \mathbb{S}_{ij}(P(z, \overline{z})) = \sum_{(m,n)\in H finite} a_{mn} a_{ij}(m, n)$$

when $P(z, \overline{z}) = \sum_{(m,n)\in H finite} a_{mn} z^m \overline{z}^n$ for any $1 \leq i, j \leq k$.

The $\mathbb{S}_{ij} - \mathbb{C}$ linear mapping is called positive on $D_1$ iff for any $P \in \mathbb{P}_n$ with $P(z, \overline{z}) \geq 0$ and any $z \in D_1$ we have $\mathbb{S}_{ij}(P) \geq 0$.

## 2 The existence of a solution

A solution of the k-dimensional complex moment problem is given by the following:

**Proposition 1.** *Let* $S_{(m,n)} = ((a_{ij}(m, n))_{1\leq i,j\leq k}$ *for any* $(m, n) \in Z_+^2$ *a Hermitian multisequence of* $(k, k)$ *matrices.The following statements are equivalent:*

*(i)* $\{S_{(m,n)}\}_{(m,n)\in Z_+^2}$ *is a k-complex moment sequence.*

*(ii) The* $\mathbf{C}$*-linear mappings* $\mathbf{S}_{ij}\ _{1\leq i,j\leq k}$ *are all positive on* $D_1$ *and*

$$\sum_{1\leq i,j\leq k} \mathbf{S}_{ij}(P(z, \overline{z})) t_i t_j \geq 0$$

*for any* $t_i \in \mathbb{R}$ *and any positive polynomial on* $D_1$*,* $P \in \mathbb{P}_n(\mathbb{C})$*.*

*Proof* i)$\Rightarrow$ ii) Assume that the Hermitian multisequence $S_{mn} \in M(k, \mathbb{C})$ is a k complex moment sequence on $D_1$. There exists a positive definite matrix

$$\Lambda = (\lambda_{ij})_{1 \le i,j \le k} \in M_k^\star(D_1)$$

such that

$$a_{ij}(m,n) = \int_{D_1} z^m \overline{z}^n d\lambda_{ij}(z)$$

for any $1 \le i, j \le k$ and any $(m,n) \in \mathbb{Z}_+^2$

Let $P(z, \overline{z}) \in \mathbf{P}_n(\mathbb{C})$, $P(z, \overline{z}) = \sum_{(m,n) \in Hfin} a_{mn} z^m \overline{z}^n$ with $P(z, \overline{z}) \ge 0$ for any $z \in D_1$. In this case,

$$\mathbf{S}_{ij}(P(z, \overline{z})) = \sum_{(m,n) \in Hfin} a_{mn} z^m \overline{z}^n d\lambda_{ij}(z)$$

$$= \int_{D_1} P(z, \overline{z}) d\lambda_{ij} \ge 0$$

for any $1 \le i, j \le n$; that means that all $\{\mathbf{S}_{ij}\}_{1 \le i,j \le k}$ are positive on $D_1$. Because of the positivity condition of the matrix

$$\Lambda = (\lambda_{ij})_{1 \le i,j \le k} \in M_k^\star(D_1)$$

we also have:

$$\begin{aligned} \textstyle\sum_{1 \le i,j \le k} \mathbf{S}_{ij}(P(z, \overline{z})) t_i t_j = \; & \textstyle\sum_{1 \le i,j \le k} \int_{D_1} P(z, \overline{z}) d\lambda_{ij}(z) t_i t_j = \\ & = \textstyle\int_{D_1} P(z, \overline{z}) \sum_{1 \le i,j \le k} d\lambda_{ij}(z) t_i t_j \ge 0, \end{aligned}$$

for any $t_i, t_j \in \mathbb{R}$ and any $P \in \mathbf{P}_n(\mathbb{C})$ with $P(z, \overline{z}) \ge 0$ on $D_1$. With this, the statement (ii) is fulfilled.

Conversely, let be $\mathbf{S}_{ij} : \mathbf{P}_n(\mathbb{C}) \to \mathbb{C}$ defined by $\mathbf{S}_{ij}(z^m \overline{z}^n) = a_{ij}(m,n)$ positive definite on $D_1$ for any $1 \le i, j \le k$. Let $\mathbf{P}$ denote the $\mathbb{C}$ vector subspace of $\mathbf{P}_n(\mathbb{C})$, $\mathbf{P} = \{P(z) = \sum_{n \in Hfin} a_n z^n, a_n \in \mathbb{C}\}$ of all analytic polynomials with complex coefficients.Using $\mathbf{S}_{ij}$ we define on $\mathbf{P}$ an inner product by:

$$< P, Q >_{\mathbf{S}_{ij}} = \sum_{m,n \in Hfin} a_{ij}(m,n) b_m \overline{c}_n$$

when

$$P(z) = \sum_{m \in H_1 fin} b_m z^m, Q(z) = \sum_{n \in H_2 fin} c_n z^n$$

. Because $\mathbf{S}_{ij}(|P(z)|^2) \ge 0$ for any $P \in \mathbf{P}$ this inner product is positive definite.Let $\mathbf{H}_{ij}$ be the separate completion of $\mathbf{P}$ with respect to the mentioned inner product.Let $S_{ij}$ the operator of multiplication by $z$ on $\mathbf{P}$ that is $S_{ij} : \mathbf{P} \to \mathbf{P}, S_{ij} P = zP$.Because

$$\mathbf{S}_{ij}((1 - |z|^2)|P(z)|^2) \ge 0$$

when $z \in D_1$, $S_{ij}$ are all contractions on $\mathbf{P}$. Therefore, since $\mathbf{P}$ is dense in $\mathbf{H}_{ij}$, $S_{ij}$ admits a unique extension to a bounded linear operator on $\mathbf{H}_{ij}$ with the same norm, also denoted by $S_{ij}$. From the positive condition of the $\mathbb{C}$ -linear mappings $\mathbb{S}_{ij}$ on $D_1$, we have:

$$0 \leq \mathbf{S}_{ij}(|\sum_{k=0}^{n} \overline{z}^k P_k(z)|^2) =\| \sum_{k=0}^{n} S_{ij}^{\star k} P_k \|^2=$$

$$= \sum_{p,q} < S_{ij}^{\star p} P_p, S^{\star} q P_q >_{\mathbf{H}_{ij}} = \sum_{p,q} < S^q P_p, S^p P_q >_{\mathbf{H}_{ij}} .$$

These conditions are exactly Ito's necessary and sufficient condition for an operator $S_{ij}$ to be a subnormal one.In this case, for any operator $\mathbb{S}_{ij}$, there exist normals $N_{ij} : \mathbf{K}_{ij} \to K_{ij}$ such that $\mathbf{H}_{ij} \subset \mathbf{K}_{ij}$ and $N_{ij}|_{\mathbf{H}_{ij}} = S_{ij}$ for any $1 \leq i, j \leq k$. Let $E_{ij}$ be the spectral measure associated to the normals $N_{ij}$, $1 \leq i, j \leq k$. Let be also $l_0 = 1$ in $\mathbf{P}$ and the positive Borel measure

$$\lambda_{ij}(B) =< E_{ij}(B) l_0, l_0 >_{\mathbf{S}_{ij}}$$

for any $1 \leq i, j \leq k$. The measures $\lambda_{ij}$ are all supported on $D_1$ because $N_{ij}$ are all contractions. From the properties of the spectral measures, we have

$a_{ij}(m, n) =< S_{ij}^m l_0, S_{ij}^n l_0 >_{\mathbf{S}_{ij}} \int_{D_1} z^m \overline{z}^n d\lambda_{ij}(z) t_i t_j$ for any

$$P(z, \overline{z}) \in \mathbb{P}_n(\mathbb{C}).$$

Because of the uniform approximation of the continuous complex valued functions on $D_1$ with polynomials in $z, \overline{z}$, we have

$$\int_{D_1} \sum_{1 \leq i,j \leq k} |f(z)|^2 d\lambda_{ij}(z) t_i t_j \geq 0$$

for any $t_i \in \mathbb{R}$. From this, it follows that: $\sum_{1 \leq i,j \leq k} \lambda_{ij}(B) t_i t_j \geq 0$ for any $B \in Bor(D_1)$.We have proved with this, that the matrix $\Lambda = (\lambda_{ij})_{1 \leq i,j \leq k}$ of positive Borel measures on $D_1$ is positive definite on $D_1$. $\square$

# References


[1] N.I. Akhizer, *The classical Moment Problem*, Oliver&Boyd, Edinburgh, 1965.

[2] G. Choquet, *Lectures on Analysis*, Vol. 2, Benjamin, New York, 1968.

[3] T. Ito, *On the commuting family of subnormal operators* , J.Fac.Sci. Hokkaido Univ. 14 (1968), 1-5.

[4] L. Lemnete, *A multidimensional moment problem on the unit polydisc*, Rev. Roumaine Math. Pures Appl. 39, 9 (1994), 905-909.

[5] L.Lemnete-Ninulescu, R.Vidican, *On a generalization of a complex moment problem on the unit polydisc*, Scientific Bull.UPB, Series A, 66, 1 (2004), 23-28.

[6] K.Schmudgen, *On a classical moment problem* , J.of Mathematical Analysis and Applications, 125 (1987), 463-470.



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