Symplectic classification of elliptic Monge-Ampère operators

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Abstract. We construct an e-structure (absolute parallelism) for classical Monge-Ampère operators of elliptic type. This allows us to solve the problem of local symplectic equivalence for Monge-Ampère operators. As an example we consider non-linear Laplace operator and construct its functional invariants.

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Let M be a smooth 2-dimensional manifold. Let ω be a normed effective differential 2-form (i.e. Pfaffian $\mathrm{Pf}\,(\omega) = 1$) on the cotangent bundles T^*M of M. Let A be the operator that corresponds to ω [6]. Then $A^2 = -1$ and the complexification of the tangent space $T_a(T^*M)$ at a point $a \in T^*M$ splits into the direct sum of two skew-orthogonal complex symplectic planes:

$$T_a\left(T^*M\right)^{\mathbb{C}} = V_+\left(a\right) \oplus V_-\left(a\right),$$

where

$$V_\pm\left(a\right) = \left\{X \in T_a\left(T^*M\right)^{\mathbb{C}} \mid A_a^{\mathbb{C}} X = \pm\iota X\right\}.$$

The ith derivatives of the distributions $V_\pm$ we denote by $V_\pm^{(i)}$. Let us assume that $V_\pm^{(i)}$ are distributions also for $i = 1, 2$. We get the following decomposition of the de Rham complex [2]:

$$\Omega^s\left(T^*M\right)^{\mathbb{C}} = \underset{p+q=s}{\oplus} \Omega^{p,q}\left(T^*M\right),$$

$$d = d_{1,0} \oplus d_{0,1} \oplus d_{2,-1} \oplus d_{-1,2}.$$

where $\Omega^{p,q}\left(T^*M\right) = \Omega^p(V_+) \otimes \Omega^q(V_-)$, and

$$d_{i,j} : \Omega^{p,q}(T^*M) \to \Omega^{p+i,q+k}(T^*M).$$

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Remark that $d_{-1,2}$ and $d_{2,-1}$ are the tensor invariants of the Monge-Ampère equation E_ω [2].

The formula

$$W_\omega \rfloor \Omega^2 = 2d\omega$$

uniquely determines the real vector field W_ω. Let μ_ω be the real differential 1-form

$$\mu_\omega \stackrel{\text{def}}{=} W_\omega \rfloor \Omega.$$

Using decomposition (1), we get: $W = W_+ + W_-$, where $W_+ \in D(V_+)$ and $W_- \in D(V_-)$ are complex vector fields. Since the distributions V_+ and V_- are skew-orthogonal, we see that the differential 1-forms $W_+ \rfloor \Omega^{\mathbb{C}}$ and $W_- \rfloor \Omega^{\mathbb{C}}$ belong to $\Omega^{1,0}(T^*M)$ and $\Omega^{0,1}(T^*M)$ correspondingly. Denote them by

$$\mu_+ \stackrel{\text{def}}{=} W_+ \rfloor \Omega^{\mathbb{C}}, \qquad \text{and} \qquad \mu_- \stackrel{\text{def}}{=} W_- \rfloor \Omega.$$

The 3-dimensional distributions $\ker \mu_+$ and $\ker \mu_-$ define 2-dimensional distribution $\ker \mu_+ \cap \ker \mu_-$. Note that $\mu_+(W_+) = \mu_+(W_-) = \mu_-(W_+) = \mu_-(W_-) = 0$, therefore the distribution $\ker \mu_+ \cap \ker \mu_-$ is generated by the vector fields W_+ and W_-: $\ker \mu_+ \cap \ker \mu_- = \mathcal{F}\langle W_+, W_- \rangle$.

Let Q be their commutator: $Q \stackrel{\text{def}}{=} [W_+, W_-]$. Then using decomposition (1) again, we get two vector fields $Q_+ \in D(V_+)^{\mathbb{C}}$ and $Q_- \in D(V_-)^{\mathbb{C}}$ such that $Q = Q_+ + Q_-$.

Suppose now that the 3-dimensional distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable.

Since this distribution $\mathcal{F}\langle W_+, W_- \rangle$ is completely integrable also, one can define two functional invariants g_+ and g_- of the form ω by the following formula:

$$[W_+, W_-] = g_+ W_+ + g_- W_-.$$

Since the distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable, we see that $\mu_+ \wedge d\mu_+ = \mu_- \wedge d\mu_- = 0$. Then $\mu_+ \wedge (W_+ \rfloor d\mu_+) = W_+ \rfloor (\mu_+ \wedge d\mu_+) = 0$, i.e. the 1-forms μ_+ and $W_+ \rfloor d\mu_+$ are linear dependent. Therefore

$$W_+ \rfloor d\mu_+ = g_0 \mu_+$$

for some complex-valued function g_0.

This function is an invariant of ω. Note also that

$$W_- \rfloor d\mu_- = -g_0 \mu_-.$$

Since $\mu_+ \in \Omega^{1,0}$, we have:

$$d\mu_+ = d_{1,0}\mu_+ + d_{0,1}\mu_+ + d_{-1,2}\mu_+$$

By reason of dimension, $\mu_+ \wedge d_{1,0}\mu_+ = 0$. Then

$$\mu_+ \wedge d\mu_+ = \mu_+ \wedge d_{0,1}\mu_+ + \mu_+ \wedge d_{-1,2}\mu_+.$$

Since $\mu_+ \wedge d\mu_+ = 0$, $\mu_+ \wedge d_{0,1}\mu_+ \in \Omega^{2,1}$ and $\mu_+ \wedge d_{-1,2}\mu_+ \in \Omega^{1,2}$, we see that

$$\mu_+ \wedge d_{0,1}\mu_+ = 0$$

and $\mu_+ \wedge d_{-1,2}\mu_+ = 0$. Since $d_{-1,2}\mu_+ \in \Omega^{0,2}$ and $\mu_+ \in \Omega^{1,0}$, the last equality realized if and only if $d_{-1,2}\mu_+ = 0$.

Then

$$d\mu_+ = d_{1,0}\mu_+ + d_{0,1}\mu_+.$$

In the similar way we get

$$d\mu_- = d_{1,0}\mu_- + d_{0,1}\mu_-.$$

From (2) it follows that

$$d_{0,1}\mu_+ = \mu_+ \wedge \gamma_-,$$

for some uniquely determined differential 1-form $\gamma_- \in \Omega^{0,1}$. In the similar way we get a uniquely determined differential 1-form $\gamma_+ \in \Omega^{1,0}$ such that:

$$d_{1,0}\mu_- = \mu_- \wedge \gamma_+,$$

We denote by X_+ and X_- by the dual vector fields:

$$X_\pm \rfloor \Omega = \gamma_\pm.$$

Lemma 1. $\gamma_-(W_-) = g_+$ and $\gamma_+(W_+) = -g_-$.

Proof. Since formulas (3) and (4), and the fact that

$$W_- \rfloor d_{1,0}\mu_+ = W_+ \rfloor d_{0,1}\mu_- = 0,$$

we get

$$W_- \rfloor d\mu_+ = W_- \rfloor d_{0,1}\mu_+ = W_- \rfloor (\mu_+ \wedge \gamma_-) = -\gamma_-(W_-)\mu_+$$

and

$$W_+ \rfloor d\mu_- = W_+ \rfloor d_{1,0}\mu_- = W_+ \rfloor (\mu_- \wedge \gamma_+) = -\gamma_+(W_+)\mu_-.$$

Using the formula $\iota_{[X,Y]} = [L_X, \iota_Y]$, we get:

$$\begin{aligned}
[W_+, W_-] \rfloor \Omega &= \left[L_{W_+}, \iota_{W_-}\right](\Omega) = \\
&= L_{W_+}(W_- \rfloor \Omega) - W_- \rfloor L_{W_+}(\Omega) \\
&= W_+ \rfloor d\mu_- - W_- \rfloor d\mu_+ \\
&= -\gamma_+(W_+)\mu_- + \gamma_-(W_-)\mu_+.
\end{aligned}$$

On the other hand

$$[W_+, W_-] \rfloor \Omega = g_+\mu_+ + g_-\mu_-.$$

Therefore, $g_+ = \gamma_-(W_-)$ and $g_- = -\gamma_+(W_+)$. □

Note that the complex vector fields W_+ and W_- (Q_+ and Q_-) are complex conjugate, i.e, $W_+ = \overline{W_-}$ and $Q_+ = \overline{Q_-}$. Note also that W and $V \stackrel{\text{def}}{=} AW$ are linear independent at each point real vector fields.

Define a real vector field X: if $\operatorname{Re} Q_+ \neq 0$ we put $X \stackrel{\text{def}}{=} \operatorname{Re} Q_+$ and $X \stackrel{\text{def}}{=} \operatorname{Im} Q_+$ otherwise. Moreover, put:

$$Z \stackrel{\text{def}}{=} AX,\ \eta \stackrel{\text{def}}{=} V \rfloor \Omega,\ \xi \stackrel{\text{def}}{=} X \rfloor \Omega,\ \tau \stackrel{\text{def}}{=} Z \rfloor \Omega.$$

The table below indicates of values of the 1-forms ξ, τ, μ, η on the vector fields X, Z, W, V:

	X	Z	W	V
ξ	0	0	w	v
τ	0	0	v	$-w$
μ	$-w$	$-v$	0	0
η	$-v$	w	0	0

where $v \stackrel{\text{def}}{=} \Omega(X, V)$, $w \stackrel{\text{def}}{=} \Omega(X, W)$.

Note that

$$\Omega^2(W, V, X, Z) = 2\left(v^2 + w^2\right)$$

and therefore the vector fields W, V, X, Z (and the differential 1-forms ξ, τ, μ, η) are linear independent if and only if $v^2 + w^2 \neq 0$.

As above we'll consider two cases:

Case 1. $v^2 + w^2 \neq 0$.

In this case the vector fields W, V, X, Z form a basis of the module $D(\mathcal{O}_a)$ in a some neighborhood $\mathcal{O}_a$ of a.

Let

$$\begin{aligned}
X_1 &\stackrel{\text{def}}{=} -\frac{1}{v^2 + w^2}\left(vV + wW\right),\\
X_2 &\stackrel{\text{def}}{=} X,\\
X_3 &\stackrel{\text{def}}{=} \frac{1}{v^2 + w^2}\left(vW - wV\right),\\
X_4 &\stackrel{\text{def}}{=} -Z,
\end{aligned}$$

Then $X_3 = AX_1$, $X_4 = -AX_2$, and

$$\begin{aligned}
&\Omega(X_1, X_2) = \Omega(X_3, X_4) = 1,\\
&\Omega(X_1, X_3) = \Omega(X_1, X_4) = \Omega(X_2, X_3) = \Omega(X_2, X_4) = 0.
\end{aligned}$$

Let $(\theta_1, \dots, \theta_4)$ be the dual basis for $(X_1, \dots, X_2)$. Then we get

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4.$$

Calculating values of ω on the vector fields X_1, X_2, X_3, X_4:

$\omega(\uparrow,\leftarrow)$	X_1	X_2	X_3	X_4
X_1	0	0	0	1
X_2	0	0	1	0
X_3	0	-1	0	0
X_4	-1	0	0	0

we get the following representation of the form ω:

$$\omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$

Theorem 1. *Let ω be an elliptic normed effective differential 2-form on T^*M and $a \in T^*M$. Suppose that $v^2(a) + w^2(a) \neq 0$. Then in a some neighborhood of a there exist an e-structure $X_1, \ldots, X_4$ such that we have the following representation of the forms Ω and ω:*

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4, \qquad \omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$

Remark. *Similar e-structures for hyperbolic and elliptic equations was obtained by B. Kruglikov in [1]. He used the Nijenhuis tensor of the operator field A.*

Case 2. *The 3-dimensional complex distributions* $\ker\mu_+$ *and* $\ker\mu_-$ *are completely integrable in $\mathcal{O}_a$.*

Let $X \overset{\text{def}}{=} \operatorname{Re} X_-$ and $Z \overset{\text{def}}{=} AX$. These are real vector fields. As above we put

$$\eta \overset{\text{def}}{=} V \rfloor \Omega, \ \xi \overset{\text{def}}{=} X \rfloor \Omega, \ \tau \overset{\text{def}}{=} Z \rfloor \Omega.$$

Then we get the same table (6) of values of the 1-forms on the vector fields, where as above $v \overset{\text{def}}{=} \Omega(P,V)$, $w \overset{\text{def}}{=} \Omega(P,W)$.

We define the new basis $X_1, \ldots X_4$ of $D(O_a)$ by the same formulas (see Case 1) and get the same canonical representation (7) of the form Ω and ω.

Theorem 2. *Let ω be an elliptic normed effective differential 2-form on T^*M and $a \in T^*M$. Suppose that the 3-dimensional complex distributions* $\ker\mu_+$ *and* $\ker\mu_-$ *are completely integrable in $\mathcal{O}_a$. Then in a some neighborhood of a there exist an e-structure $X_1, \ldots, X_4$ such that we have the following representation of the forms Ω and ω:*

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4,$$
$$\omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$

Example 1. As an example we consider the following non-linear Laplace operator

$$\Delta_\omega(v) = (v_{q_1 q_1} + v_{q_2 q_2} - f(q,p))\, dq_1 \wedge dq_2,$$

which corresponds to the non-linear Laplace equation

$$v_{xx} + v_{yy} = f(x, y, v_x, v_y).$$

For this operator corresponding effective differential 2 form is

$$\omega = -f(q,p)dq_1 \wedge dq_2 + dq_1 \wedge dp_2 + dq_2 \wedge dp_1.$$

In the basis $\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}$ the operator A has the following matrix representation:

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & f & 0 & -1 \\ -f & 0 & 1 & 0 \end{Vmatrix}.$$

The complex distributions V_+ and V_- are

$$V_+ = \left\langle \frac{\partial}{\partial q_2} + f\frac{\partial}{\partial p_2} + \iota\frac{\partial}{\partial q_1}, \ \frac{\partial}{\partial q_1} + f\frac{\partial}{\partial p_1} - \iota\frac{\partial}{\partial q_2} \right\rangle,$$
$$V_- = \left\langle \frac{\partial}{\partial q_2} + f\frac{\partial}{\partial p_2} - \iota\frac{\partial}{\partial q_1}, \ \frac{\partial}{\partial q_1} + f\frac{\partial}{\partial p_1} + \iota\frac{\partial}{\partial q_2} \right\rangle.$$

The vector field

$$W = -f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}$$

falls into two components

$$W_\pm = \frac{1}{2}\left(-f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2} \mp \iota\left(f_{p_1}\frac{\partial}{\partial p_1} + f_{p_2}\frac{\partial}{\partial p_2}\right)\right).$$

Therefore,

$$\mu_\pm = \frac{1}{2}\left(f_{p_2}dq_1 - f_{p_1}dq_2 \pm \iota\left(f_{p_1}dq_1 + f_{p_2}dq_2\right)\right).$$

We see that $\mu_+ \wedge d\mu_+ = 0$ and $\mu_- \wedge d\mu_- = 0$, therefore the distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable and the equation belongs to the class $H_{2,2}$. The vector field

$$\begin{aligned} Q =\iota \Bigg(& \left(2f_{p_1}f_{p_1p_2} + f_{p_2}\left(f_{p_2p_2} - f_{p_1p_1}\right)\right)\frac{\partial}{\partial p_1} + \\ & \left(-2f_{p_2}f_{p_1p_2} + f_{p_1}\left(f_{p_2p_2} - f_{p_1p_1}\right)\right)\frac{\partial}{\partial p_2}\Bigg) \end{aligned}$$

is a linear combination of the vector fields W_+ and W_- with coefficients

$$g_+ = \frac{\left(f_{p_2} - \iota f_{p_1}\right)\left(2f_{p_1p_2} + \iota\left(f_{p_1p_1} - f_{p_2p_2}\right)\right)}{2\left(f_{p_1} - \iota f_{p_2}\right)},$$
$$g_- = \frac{\left(f_{p_1} - \iota f_{p_2}\right)\left(f_{p_2p_2} - f_{p_1p_1} - \iota 2f_{p_1p_2}\right)}{2\left(f_{p_2} - \iota f_{p_1}\right)}$$

respectively.

Example 2. Let us construct e-structure for the following case of the non-linear Laplace operator:

$$\Delta_\omega(v) = \left(v_{q_1q_1} + v_{q_2q_2} - v_{q_1}^2\right) dq_1 \wedge dq_2,$$

For this operator

$$\begin{aligned} W &= 2p_1 \frac{\partial}{\partial p_2}, \\ V &= -2p_1 \frac{\partial}{\partial p_1}, \\ W_\pm &= p_1 \frac{\partial}{\partial p_2} \mp \iota p_1 \frac{\partial}{\partial p_1}, \\ \gamma_\pm &= \frac{1}{2p_1}\left(-dp_1 \pm \iota\left(p_1^2 dq_2 - dp_2\right)\right), \\ v &= 1, \\ w &= 0. \end{aligned}$$

Therefore we get the following e-structure:

$$\begin{aligned} X_1 &= 2p_1 \frac{\partial}{\partial p_1}, \\ X_2 &= -\frac{1}{2p_1}\frac{\partial}{\partial q_1}, \\ X_3 &= 2p_1 \frac{\partial}{\partial p_2}, \\ X_4 &= -\frac{1}{2p_1}\frac{\partial}{\partial q_2} - \frac{p_1}{2}\frac{\partial}{\partial p_2}. \end{aligned}$$

Indeed, the dual basis is:

$$\begin{aligned} \theta_1 &= \frac{1}{2p_1} dp_1, \\ \theta_2 &= -p_1 dq_1, \\ \theta_3 &= \frac{1}{2p_1} dp_2 - \frac{p_1}{2} dq_2, \\ \theta_4 &= -2p_1 dq_2. \end{aligned}$$

and we see that

$$\begin{aligned} &\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \Omega, \\ &\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3 = -p_1^2 dq_1 \wedge dq_2 + dq_1 \wedge dp_2 + dq_2 \wedge dp_1 = \omega. \end{aligned}$$

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