Symplectic classification of elliptic Monge-Ampère operators

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Abstract. We construct an e-structure (absolute parallelism) for classical Monge-Ampère operators of elliptic type. This allows us to solve the problem of local symplectic equivalence for Monge-Ampère operators. As an example we consider non-linear Laplace operator and construct its functional invariants.


Key words: Monge-Ampère operators, symplectic invariants, de Rham complex.

Let \( M \) be a smooth 2-dimensional manifold. Let \( \omega \) be a normed effective differential 2-form (i.e. Pfaffian \( \text{Pf} (\omega) = 1 \)) on the cotangent bundles \( T^*M \) of \( M \). Let \( A \) be the operator that corresponds to \( \omega \) [6]. Then \( A^2 = -1 \) and the complexification of the tangent space \( T_a(T^*M) \) at a point \( a \in T^*M \) splits into the direct sum of two skew-orthogonal complex symplectic planes:

\[
T_a(T^*M)^\mathbb{C} = V_+(a) \oplus V_-(a),
\]

where

\[
V_\pm(a) = \left\{ X \in T_a(T^*M)^\mathbb{C} \mid A_a^\mathbb{C}X = \pm \iota X \right\}.
\]

The \( i \)th derivatives of the distributions \( V_\pm \) we denote by \( V^{(i)}_\pm \). Let us assume that \( V^{(i)}_\pm \) are distributions also for \( i = 1, 2 \). We get the following decomposition of the de Rham complex [2]:

\[
\Omega^s(T^*M)^\mathbb{C} = \bigoplus_{p+q=s} \Omega^{p,q}(T^*M),
\]

\[
d = d_{1,0} \oplus d_{0,1} \oplus d_{2,-1} \oplus d_{-1,2}.
\]

where \( \Omega^{p,q}(T^*M) = \Omega^p(V_+) \otimes \Omega^q(V_-) \), and

\[
d_{i,j} : \Omega^{p,q}(T^*M) \rightarrow \Omega^{p+i,q+j}(T^*M).
\]

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Remark that $d_{-1,2}$ and $d_{2,-1}$ are the tensor invariants of the Monge-Ampère equation $E_{\omega}$ [2].

The formula

$$W_\omega| \Omega^2 = 2 d\omega$$

uniquely determines the real vector field $W_\omega$. Let $\mu_\omega$ be the real differential 1-form

$$\mu_\omega \overset{\text{def}}{=} W_\omega| \Omega.$$

Using decomposition (1), we get: $W = W_+ + W_-$, where $W_+ \in D(V_+)$ and $W_- \in D(V_-)$ are complex vector fields. Since the distributions $V_+$ and $V_-$ are skew-orthogonal, we see that the differential 1-forms $W_+| \Omega^C$ and $W_-| \Omega^C$ belong to $\Omega^{1,0}(T^*M)$ and $\Omega^{0,1}(T^*M)$ correspondingly. Denote them by

$$\mu_+ \overset{\text{def}}{=} W_+| \Omega^C, \quad \text{and} \quad \mu_- \overset{\text{def}}{=} W_-| \Omega^C.$$

The 3-dimensional distributions $\ker \mu_+$ and $\ker \mu_-$ define 2-dimensional distribution $\ker \mu_+ \cap \ker \mu_-$. Note that $\mu_+(W_+) = \mu_-(W_-) = 0$, therefore the distribution $\ker \mu_+ \cap \ker \mu_-$ is generated by the vector fields $W_+$ and $W_-$. Since the distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable, we see that the differential 1-forms $W_+| \Omega^C$ and $W_-| \Omega^C$ belong to $\Omega^{1,0}(T^*M)$ and $\Omega^{0,1}(T^*M)$ correspondingly. Denote them by

$$\mu_+ \overset{\text{def}}{=} W_+| \Omega^C, \quad \text{and} \quad \mu_- \overset{\text{def}}{=} W_-| \Omega^C.$$

Let $Q$ be their commutator: $Q \overset{\text{def}}{=} [W_+, W_-]$. Then using decomposition (1) again, we get two vector fields $Q_+ \in D(V_+)^C$ and $Q_- \in D(V_-)^C$ such that $Q = Q_+ + Q_-$. Suppose now that the 3-dimensional distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable.

Since this distribution $F(W_+, W_-)$ is completely integrable also, one can define two functional invariants $g_+$ and $g_-$ of the form $\omega$ by the following formula:

$$[W_+, W_-] = g_+ W_+ + g_- W_-.$$

Since the distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable, we see that $\mu_+ \land d\mu_+ = \mu_- \land d\mu_- = 0$. Then $\mu_+ \land (W_+| \land d\mu_+ = W_-| \land (\mu_+ \land d\mu_+) = 0$, i.e. the 1-forms $\mu_+$ and $W_+| \land d\mu_+$ are linear dependent. Therefore

$$W_+| \land d\mu_+ = g_0 \mu_+$$

for some complex-valued function $g_0$.

This function is an invariant of $\omega$. Note also that

$$W_-| \land d\mu_- = -g_0 \mu_-.$$

Since $\mu_+ \in \Omega^{1,0}$, we have:

$$d\mu_+ = d_{1,0} \mu_+ + d_{0,1} \mu_+ + d_{-1,2} \mu_+.$$

By reason of dimension, $\mu_+ \land d_{1,0} \mu_+ = 0$. Then

$$\mu_+ \land d\mu_+ = \mu_+ \land d_{0,1} \mu_+ + \mu_+ \land d_{-1,2} \mu_+.$$

Since $\mu_+ \land d\mu_+ = 0$, $\mu_+ \land d_{0,1} \mu_+ \in \Omega^{2,1}$ and $\mu_+ \land d_{-1,2} \mu_+ \in \Omega^{1,2}$, we see that
Symplectic classification

\[ \mu_+ \wedge d_{0,1} \mu_+ = 0 \]
and \( \mu_+ \wedge d_{-1,2} \mu_+ = 0 \). Since \( d_{-1,2} \mu_+ \in \Omega^{0,2} \) and \( \mu_+ \in \Omega^{1,0} \), the last equality realized if and only if \( d_{-1,2} \mu_+ = 0 \).

Then
\[ d\mu_+ = d_{1,0} \mu_+ + d_{0,1} \mu_+. \]

In the similar way we get
\[ d\mu_- = d_{1,0} \mu_- + d_{0,1} \mu_. \]

From (2) it follows that
\[ d_{0,1} \mu_- = \mu_+ \wedge \gamma_-, \]
for some uniquely determined differential 1-form \( \gamma_- \in \Omega^{0,1} \). In the similar way we get a uniquely determined differential 1-form \( \gamma_+ \in \Omega^{1,0} \) such that:
\[ d_{1,0} \mu_- = \mu_- \wedge \gamma_+. \]

We denote by \( X_+ \) and \( X_- \) by the dual vector fields:
\[ X_{\pm} \Omega = \gamma_{\pm}. \]

**Lemma 1.** \( \gamma_-(W_-) = g_+ \) and \( \gamma_+(W_+) = -g_- \).

**Proof.** Since formulas (3) and (4), and the fact that
\[ W_-]d_{1,0} \mu_+ = W_+]d_{0,1} \mu_- = 0, \]
we get
\[ W_-]d\mu_+ = W_-]d_{0,1} \mu_+ = W_-] (\mu_+ \wedge \gamma_-) = -\gamma_-(W_-) \mu_+ \]
and
\[ W_+]d\mu_- = W_+]d_{1,0} \mu_- = W_+] (\mu_- \wedge \gamma_+) = -\gamma_+(W_+) \mu_-. \]

Using the formula \( \iota_{[X,Y]} = [L_X, \iota_Y] \), we get:
\[ [W_+, W_-]\Omega = [L_{W_+}, \iota_{W_-}] (\Omega) = \]
\[ = L_{W_+} (W_-] \Omega) - W_-] L_{W_+} (\Omega) = W_+] d\mu_- - W_-] d\mu_+ \]
\[ = -\gamma_+(W_+) \mu_- + \gamma_-(W_-) \mu_+ . \]

On the other hand
\[ [W_+, W_-]\Omega = g_+ \mu_+ + g_- \mu_- . \]

Therefore, \( g_+ = \gamma_-(W_-) \) and \( g_- = -\gamma_+(W_+) \). \( \square \)
Note that the complex vector fields $W_+$ and $W_-$ ($Q_+$ and $Q_-$) are complex conjugate, i.e., $W_+ = \overline{W_-}$ and $Q_+ = \overline{Q_-}$. Note also that $W$ and $V \overset{\text{def}}{=} AW$ are linear independent at each point real vector fields.

Define a real vector field $X$: if $\text{Re} \ Q_+ \neq 0$ we put $X \overset{\text{def}}{=} \text{Re} \ Q_+$ and $X \overset{\text{def}}{=} \text{Im} \ Q_+$ otherwise. Moreover, put:

$$Z \overset{\text{def}}{=} AX, \ \eta \overset{\text{def}}{=} V \lceil \Omega, \ \xi \overset{\text{def}}{=} X \lceil \Omega, \ \tau \overset{\text{def}}{=} Z \lceil \Omega.$$ 

The table below indicates of values of the 1-forms $\xi, \tau, \mu, \eta$ on the vector fields $X, Z, W, V$:

<table>
<thead>
<tr>
<th></th>
<th>$X$</th>
<th>$Z$</th>
<th>$W$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>0</td>
<td>0</td>
<td>$w$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0</td>
<td>0</td>
<td>$\nu$</td>
<td>$-w$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$-w$</td>
<td>$-\nu$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$-\nu$</td>
<td>$w$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\nu \overset{\text{def}}{=} \Omega(X, V), w \overset{\text{def}}{=} \Omega(X, W)$.

Note that

$$\Omega^2(W, V, X, Z) = 2(\nu^2 + w^2)$$

and therefore the vector fields $W, V, X, Z$ and the differential 1-forms $\xi, \tau, \mu, \eta$ are linear independent if and only if $\nu^2 + w^2 \neq 0$.

As above we’ll consider two cases:

**Case 1.** $\nu^2 + w^2 \neq 0$.

In this case the vector fields $W, V, X, Z$ form a basis of the module $D(O_a)$ in a some neighborhood $O_a$ of $a$.

Let

$$X_1 \overset{\text{def}}{=} -\frac{1}{\nu^2 + w^2} (\nu V + w W),$$

$$X_2 \overset{\text{def}}{=} X,$$

$$X_3 \overset{\text{def}}{=} \frac{1}{\nu^2 + w^2} (\nu W - w V),$$

$$X_4 \overset{\text{def}}{=} -Z,$$

Then $X_3 = AX_1, X_4 = -AX_2$, and

$$\Omega(X_1, X_2) = \Omega(X_3, X_4) = 1,$$

$$\Omega(X_1, X_3) = \Omega(X_1, X_4) = \Omega(X_2, X_3) = \Omega(X_2, X_4) = 0.$$ 

Let $(\theta_1, \ldots, \theta_4)$ be the dual basis for $(X_1, \ldots, X_2)$. Then we get

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4.$$
Symplectic classification

Calculating values of $\omega$ on the vector fields $X_1, X_2, X_3, X_4$:

$$
\omega(\uparrow, \leftarrow) | \begin{array}{cccc}
X_1 & X_2 & X_3 & X_4 \\
X_1 & 0 & 0 & 0 & 1 \\
X_2 & 0 & 0 & 1 & 0 \\
X_3 & 0 & -1 & 0 & 0 \\
X_4 & -1 & 0 & 0 & 0 \\
\end{array}
$$

we get the following representation of the form $\omega$:

$$\omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$ 

**Theorem 1.** Let $\omega$ be an elliptic normed effective differential 2-form on $T^*M$ and $a \in T^*M$. Suppose that $v^2(a) + w^2(a) \neq 0$. Then in some neighborhood of $a$ there exist an e-structure $X_1, \ldots, X_4$ such that we have the following representation of the forms $\Omega$ and $\omega$:

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4, \quad \omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$ 

**Remark.** Similar e-structures for hyperbolic and elliptic equations was obtained by B. Kruglikov in [1]. He used the Nijenhuis tensor of the operator field $A$.

**Case 2.** The 3-dimensional complex distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable in $O_a$.

Let $X \underset{\text{def}}{=} \text{Re} X_-$ and $Z \underset{\text{def}}{=} AX$. These are real vector fields. As above we put

$$\eta \underset{\text{def}}{=} V \Omega, \quad \xi \underset{\text{def}}{=} X \Omega, \quad \tau \underset{\text{def}}{=} Z \Omega.$$

Then we get the same table (6) of values of the 1-forms on the vector fields, where as above $v \underset{\text{def}}{=} \Omega(P, V)$, $w \underset{\text{def}}{=} \Omega(P, W)$.

We define the new basis $X_1, \ldots, X_4$ of $D(O_a)$ by the same formulas (see Case 1) and get the same canonical representation (7) of the form $\Omega$ and $\omega$.

**Theorem 2.** Let $\omega$ be an elliptic normed effective differential 2-form on $T^*M$ and $a \in T^*M$. Suppose that the 3-dimensional complex distributions $\ker \mu_+$ and $\ker \mu_-$ are completely integrable in $O_a$. Then in some neighborhood of $a$ there exist an e-structure $X_1, \ldots, X_4$ such that we have the following representation of the forms $\Omega$ and $\omega$:

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4, \quad \omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$ 

**Example 1.** As an example we consider the following non-linear Laplace operator

$$\Delta_\omega (v) = (v_q q_1 + v_{p_2} p_2 - f(q, p)) dq_1 \wedge dq_2,$$
which corresponds to the non-linear Laplace equation

$$v_{xx} + v_{yy} = f(x, y, v_x, v_y).$$

For this operator corresponding effective differential 2 form is

$$\omega = -f(q, p) dq_1 \wedge dq_2 + dq_1 \wedge dp_2 + dq_2 \wedge dp_1.$$ 

In the basis $\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}$ the operator $A$ has the following matrix representation:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & f & 0 & -1 \\ -f & 0 & 1 & 0 \end{pmatrix}.$$ 

The complex distributions $V_+$ and $V_-$ are

$$V_+ = \left< \frac{\partial}{\partial q_2} + f \frac{\partial}{\partial p_2} + \iota \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_1} + f \frac{\partial}{\partial p_1} - \iota \frac{\partial}{\partial q_2} \right>,$$

$$V_- = \left< \frac{\partial}{\partial q_2} + f \frac{\partial}{\partial p_2} - \iota \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_1} + f \frac{\partial}{\partial p_1} + \iota \frac{\partial}{\partial q_2} \right>.$$ 

The vector field

$$W = -f_{p_2} \frac{\partial}{\partial p_1} + f_{p_1} \frac{\partial}{\partial p_2}$$

falls into two components

$$W_\pm = \frac{1}{2} \left( -f_{p_2} \frac{\partial}{\partial p_1} + f_{p_1} \frac{\partial}{\partial p_2} \mp \iota \left( f_{p_1} \frac{\partial}{\partial p_1} + f_{p_2} \frac{\partial}{\partial p_2} \right) \right).$$

Therefore,

$$\mu_\pm = \frac{1}{2} ( f_{p_2} dq_1 - f_{p_1} dq_2 \pm \iota ( f_{p_1} dq_1 + f_{p_2} dq_2 ) ).$$

We see that $\mu_+ \wedge d\mu_+ = 0$ and $\mu_- \wedge d\mu_- = 0$, therefore the distributions ker $\mu_+$ and ker $\mu_-$ are completely integrable and the equation belongs to the class $H_{2,2}$. The vector field

$$Q = \iota \left( (2 f_{p_1} f_{p_1 p_2} + f_{p_2} (f_{p_2 p_2} - f_{p_1 p_1})) \frac{\partial}{\partial p_1} + (2 f_{p_2} f_{p_1 p_2} + f_{p_1} (f_{p_2 p_2} - f_{p_1 p_1})) \frac{\partial}{\partial p_2} \right)$$

is a linear combination of the vector fields $W_+$ and $W_-$ with coefficients

$$g_+ = \frac{(f_{p_2} - \iota f_{p_1}) (2 f_{p_1 p_2} + \iota (f_{p_1 p_1} - f_{p_2 p_2}))}{2 (f_{p_1} - \iota f_{p_2})},$$

$$g_- = \frac{(f_{p_1} - \iota f_{p_2}) (f_{p_2 p_2} - f_{p_1 p_1} - \iota 2 f_{p_1 p_2})}{2 (f_{p_2} - \iota f_{p_1})}.$$
respectively.

**Example 2.** Let us construct $\epsilon$-structure for the following case of the non-linear Laplace operator:

$$\Delta \omega (v) = (v_{q_1 q_1} + v_{q_2 q_2} - v_q^2) \, dq_1 \wedge dq_2,$$

For this operator

$$W = 2p_1 \frac{\partial}{\partial p_2},$$
$$V = -2p_1 \frac{\partial}{\partial p_1},$$
$$W_{\pm} = p_1 \frac{\partial}{\partial p_2} \mp ip_1 \frac{\partial}{\partial p_1},$$
$$\gamma_{\pm} = \frac{1}{2p_1} \left(-dp_1 \pm i(p_1^2 dq_2 - dp_2)\right),$$
$$v = 1,$$
$$w = 0.$$

Therefore we get the following $\epsilon$-structure:

$$X_1 = 2p_1 \frac{\partial}{\partial p_1},$$
$$X_2 = -\frac{1}{2p_1} \frac{\partial}{\partial q_1},$$
$$X_3 = 2p_1 \frac{\partial}{\partial q_2},$$
$$X_4 = -\frac{1}{2p_1} \frac{\partial}{\partial q_2} - \frac{p_1}{2} \frac{\partial}{\partial p_2}.$$

Indeed, the dual basis is:

$$\theta_1 = \frac{1}{2p_1} dp_1,$$
$$\theta_2 = -p_1 dq_1,$$
$$\theta_3 = \frac{1}{2p_1} dp_2 - \frac{p_1}{2} dq_2,$$
$$\theta_4 = -2p_1 dq_2.$$

and we see that

$$\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \Omega,$$
$$\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3 = -p_1^2 dq_1 \wedge dq_2 + dq_1 \wedge dp_2 + dq_2 \wedge dp_1 = \omega.$$
References


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