Palais-Smale condition for multi-time actions that produce Poisson-gradient PDEs

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Abstract. We study the Palais-Smale $(PS)_c$ -condition in the case of the multi-times actions φ that produces Poisson-gradient PDEs, where c is minimum value of φ on a Hilbert space. In Section 1 we will present the well known bound between the $(PS)_c$ - condition and the existance of actional functional's extremas and we establish some conditions in which a function has a minimum in a reflexiv Banach space (Theorem 2). In Section 2 we will prove the existance of the multiple periodical extremals of an action that produce Poisson-gradient systems (Theorems 3 and 4) and we will show that the $(PS)_c$ -condition is satisfied, for the same action (Theorem 5).

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1 $(PS)_c$-condition and minimum value for action in Banach spaces

Let $\varphi : X \to R$ be a differentiable function, where X is a Banach space. For $c \in R$, we say that φ satisfies the $(PS)_c$-condition if the existence of sequence (u_k) in X such that $\varphi(u_k) \to c, \varphi'(u_k) \to 0,\ k \to \infty$, implies that c is a critical value of φ.

We denote by $W_T^{1,2}$ the Sobolev space of the functions $u \in L^2[T_0, R^n]$, having weak derivatives $\dfrac{\partial u}{\partial t} \in L^2[T_0, R^n]$, where $T_0 = [0, T^1] \times ... \times [0, T^p] \subset R^p$ and $T = (T^1, ..., T^p)$. The weak derivatives are defined using the space C_T^∞ of all indefinitely differentiable multiple T-periodic function from R^p into R^n.

We consider the Hilbert space H_T^1 associated to the space $W_T^{1,2}$. The euclidean structure on H_T^1 is given by the scalar product

$$\langle u, v\rangle = \int_{T_0} \left(\delta_{ij} u^i(t)\, v^j(t) + \delta_{ij}\delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha}(t)\, \frac{\partial v^j}{\partial t^\beta}(t)\right) dt^1 \wedge ... \wedge dt^p$$

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and the associated Euclidean norm. These are induced by the scalar product (Riemannian metric)

$$G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\alpha\beta}\delta_{ij} \end{pmatrix}$$

on R^{n+np} (see the jet space $J^1(T_0, R^n)$).

We denote by $\widetilde{H_T^1} = \left\{ u \in H_T^1 \mid \int_{T_0} u(t)\, dt^1 \wedge ... \wedge dt^p = 0 \right\}$, the Hilbert space of all functions from H_T^1 which have mean zero.

Proposition 1. [2] *Let $\varphi : X \to R$ be a function bounded from below and continuously differentiable on a Banach space X. Then, for each minimizing sequence (u_k) of φ, there it exists a minimizing sequence (v_k) of φ such that*

$$\varphi(v_k) \leq \varphi(u_k), \|u_k - v_k\| \to 0, |\varphi'(u_k)| \to 0,\ k \to \infty.$$

Theorem 2. Let *$\varphi : X \to R$ be a continuous, convex and bounded from below function. If X is a Banach reflexive space and φ has a minimizing bounded sequence, then φ has a minimum value in X.*

2 Critical point actions $(PS)_c$-condition for actions that produces Poisson-gradient systems

We will prove the existence of a critical point and of the $(PS)\,c$ -condition for the action $\varphi(u) = \int_{T_0} \left[\frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] dt^1 \wedge ... \wedge dt^p$, on H_T^1.

Theorem 3. *We consider $F : T_0 \times R^n \to R$ a function which satisfies the conditions:*

1. $F(t, x)$ is measurable in t for any $x \in R^n$ and with continuous derivatives in x for any $t \in T_0$,

2. It exists $a \in C^1(R^+, R^+)$ with the derivative a' bounded from above and $b \in C(T_0, R^+)$ such that

$$|F(t, x)| \leq a(|x|)\, b(t), |\nabla_x F(t, x)| \leq a(|x|)\, b(t),$$

for any $x \in R^n$ and any $t \in T_0$,

3. $|\nabla_x F(t, x)| < |h(t)|$ for

$$h \in L^2(T_0, R^n); t \in T_0, x \in R^n,$$

4. $\displaystyle\int_{T_0} F(t, x)\, dt^1 \wedge ... \wedge dt^p \to -\infty$ *when* $|x| \to \infty$.

We consider

$$\varphi(u) = \int_{T_0} \left[\frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t)) \right] dt^1 \wedge ... \wedge dt^p.$$

If exists a sequence (u_k) *in* H^1_T, *such that* $\varphi(u_k) \to c, \varphi'(u_k) \to 0$ *when* $k \to \infty$, *then the sequence* (u_k) *is bounded from below on* H^1_T.

Proof. We write $u_k = \overline{u}_k + \widetilde{u}_k$, where $\overline{u}_k = \dfrac{1}{T^1 ... T^p} \displaystyle\int_{T_0} u_k(t)\, dt^1 \wedge ... \wedge dt^p$ and we use the fact that it exists k_0 such that

$$|\langle \varphi'(u_k), v\rangle| \leq \|v\|, \forall k > k_0 \tag{2.1}$$

because $\varphi'(u_k) \to 0$. From the inequality (1) we have

$$|\langle \varphi'(u_k), \widetilde{u}_k\rangle| =$$

$$= \left| \int_{T_0} (\nabla_x F(t, u_k(t)), \widetilde{u}_k(t))\, dt^1 \wedge ... \wedge dt^p + \int_{T_0} \left(\frac{\partial \widetilde{u}_k}{\partial t}(t), \frac{\partial \widetilde{u}_k}{\partial t}(t) \right) dt^1 \wedge ... \wedge dt^p \right|$$

$$\leq \|\widetilde{u}_k\| \tag{2.2}$$

By using [1, Theorem 2] we find

$$\|\widetilde{u}_k\|^2 = \int_{T_0} |\widetilde{u}_k(t)|^2\, dt^1 \wedge ... \wedge dt^p + \int_{T_0} \left| \frac{\partial \widetilde{u}_k}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p$$

$$\leq \left(\frac{\left(\max_i \{T^i\}\right)^2}{4\pi^2} + 1 \right) \int_{T_0} \left| \frac{\partial \widetilde{u}_k}{\partial t} \right|^2 dt^1 \wedge ... \wedge dt^p \tag{2.3}$$

By the inequality $|\nabla F(t,x)| < |h(t)|$ from the hypothesis and Cauchy-Schwartz we obtain

$$-\left(\int_{T_0} (h(t))^2\, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left(\|\widetilde{u}_k\|^2_{L^2} \right)^{\frac{1}{2}}$$

$$= -\left(\int_{T_0} (h(t))^2\, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left(\int_{T_0} |\widetilde{u}_k|^2\, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}$$

$$\leq -\left(\int_{T_0} |\nabla F(t, u_k)|^2\, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left(\int_{T_0} \left|\widetilde{u}_k^2\right| dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}$$

$$\leq -\left| \int_{T_0} (\nabla F(t, u_k(t)), \widetilde{u}_k(t))(t, u_k)\, dt^1 \wedge ... \wedge dt^p \right|. \tag{2.4}$$

Using the inequalities (2), (3) and (4) we have

$$-\left(\int_{T_0} (h(t))^2\, dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \left(\|\widetilde{u}_k\|^2_{L^2} \right)^{\frac{1}{2}} + \left(\frac{\left(\max_i \{T^i\}\right)^2}{4\pi^2} + 1 \right)^{-1} \|\widetilde{u}_k\|^2 \leq \|\widetilde{u}_k\|,$$

so

$$-\|h\|_{L^2}\|\widetilde{u}_k\|+\left(\frac{\left(\max_i\left\{T^i\right\}\right)^2}{4\pi^2}+1\right)^{-1}\|\widetilde{u}_k\|^2\leq\|\widetilde{u}_k\|.$$

From here, it results that

$$\|\widetilde{u}_k\|\leq C_1, \tag{2.5}$$

for $k>k_0$. Because $\varphi(u_k)\to c$, $\varphi(u_k)$ bounded and we get the result

$$\varphi(u_k)=\int_{T_0}\frac{1}{2}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p+\int_{T_0}F(t,u_k(t))\,dt^1\wedge...\wedge dt^p$$

$$=\int_{T_0}\frac{1}{2}\left|\frac{\partial u_k}{\partial t}(t)\right|^2 dt^1\wedge...\wedge dt^p+\int_{T_0}F(t,\overline{u}_k(t))\,dt^1\wedge...\wedge dt^p$$

$$+\int_{T_0}\left[F(t,u_k(t))-F(t,\overline{u}_k)\right]dt^1\wedge...\wedge dt^p$$

$$=\int_{T_0}\frac{1}{2}\left|\frac{\partial u_k}{\partial t}(t)\right|^2 dt^1\wedge...\wedge dt^p+\int_{T_0}F(t,\overline{u}_k)\,dt^1\wedge...\wedge dt^p$$

$$+\int_{T_0}\int_0^1\nabla F(t,\overline{u}_k+s\widetilde{u}(t))\,dsdt^1\wedge...\wedge dt^p\geq C_2.$$

Using the relation (5) and the propertie 3 from hypothesis, we obtain

$$|\overline{u}_k|\leq C_4, k\in N, \tag{2.6}$$

because we know that $\displaystyle\int_{T_0}F(t,x)\,dt^1\wedge...\wedge dt^p\to-\infty$ when $|x|\to\infty$.

From the relations (5) and (6) (u_k) is bounded in H_T^1.

Theorem 4. *Some hypothesis as in Theorem 3.*

If $\varphi_1(u)=\int_{T_0}F(t,u(t))\,dt^1\wedge...\wedge dt^p$ *is weakly lower semi-continuous, then it exist* $u\in\widetilde{H_T^1}$ *such that* $\varphi(u)=\min\limits_{v\in\widetilde{H_T^1}}\varphi(v)$.

Proof. According to the Theorem 3, the action φ is bounded from below on $\widetilde{H_T^1}$. We will note by $c=\inf\limits_{v\in\widetilde{H_T^1}}\varphi(v)$. The action $\varphi_2(u)=\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial u^j}{\partial t^\beta}dt^1\wedge...\wedge dt^p$ is weakly lower semi-continuous because is convex action on reflexiv Banach space H_T^1. Consequently, $\varphi(u)=\varphi_1(u)+\varphi_2(u)$ is weakly lower semi-continuous. Let (v_k) be a minimizing sequence for φ in $\widetilde{H_T^1}$. According to the Proposition 1 there it exists a minimizing sequence (u_k) such that $\varphi(u_k)\leq\varphi(v_k)$, $\|u_k-v_k\|\to 0$, $|\varphi'(u_k)|\to 0$, $k\to\infty$. From the Theorem 3 the sequence (u_k) is bounded on $\widetilde{H_T^1}$ and by the Theorem 2 it exist a minimum value $\varphi(u)$ on $\widetilde{H_T^1}$.

Theorem 5. *If the action φ verifies the properties from the Theorem 4 and* $c = \inf_{v \in \widetilde{H_T^1}} \varphi(v)$ *then φ satisfies the (PS) c-condition.*

Proof. We consider the sequence u_k in $\widetilde{H_T^1}$ which satisfies the properties $\varphi(u_k) \to c$, $\varphi'(u_k) \to 0$ when $k \to \infty$. This means that u_k is a minimizing sequence for φ. From the Theorem 4 it exists $u \in \widetilde{H_T^1}$ such that $\varphi(u) = c$. According to [7, Theorem 3] φ is continuously differentiable and from here it results that $\varphi'(u) = 0$, meaning that c is a critical value for φ. As consequence, φ satisfies the (PS) c-condition.

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