From curves to extrema, continuity and convexity

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Abstract. Section 1 relates the continuity, extrema and convexity to the unidimensional constraints. Section 2 study the convexity of Monge hypersurfaces along parametrized curves.

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1 Continuity, extrema and convexity by curves

This section develops some results of the authors [1]-[18] regarding the extrema constrained by curves and gives new and interesting results by Theorems 2.6, 2.7, 2.8. To formulate these new results, we start with an open set D in $\mathbf{R}^p$, a function $f : D \to \mathbf{R}$, a point $a \in D$ and a family Γ_a of parametrized curves $\alpha : I \to D$ passing through the point $a = \alpha(t_0)$.

2.1. Definition. We say that a is a extremum point for f constrained by α, if t_0 is an extremum point for $f \circ \alpha$. We say that a is a minimum (maximum) point for f constrained by Γ_a, if a is a minimum (maximum) point for f constrained by each $\alpha \in \Gamma_a$.

2.2. Definition. We say that the point $f(a)$ is a separator for the values of the function f along α, if there exists $\varepsilon > 0$ such that $f(\alpha(t)) \geq f(a)$ $(f(\alpha(t)) \leq f(a))$, $\forall t \in (t_0 - \varepsilon, t_0]$ and $f(\alpha(t)) \leq f(a)$ $(f(\alpha(t)) \geq f(a))$, $\forall t \in [t_0, t_0 + \varepsilon)$.

2.3. Definition. We say that the function f is locally (strict) convex at a along α, if $f(\alpha(tu + (1-t)t_0) \leq (<) tf(\alpha(u)) + (1-t)f(a)$ for u in a neighborhood of t_0 and $t \in (0, 1)$.

For the next theorems we suppose that, the family Γ_a is either the set of all C^1 parametrized curves passing through a and regular at a or the family of all C^2 parametrized curves passing through a, such that either a is a regular point for α, or a is a singular point of the second order for α.

2.4. Theorem ([9],[11]). *Let (x_n) be a sequence of distinct points in $\mathbf{R}^p$ such that $x_n \to a \in \mathbf{R}^p$. Then there exists a parametrized curve $\alpha : \mathbf{R} \to \mathbf{R}^p$ from the family Γ_a, a subsequence (x_{n_k}) of (x_n) and a strictly decreasing sequence (t_k) of real numbers such that*

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1) $\alpha(0) = 0$, *and*

2) $t_k \to 0$ *and* $\alpha(t_k) = x_{n_k}$, $\forall k \in \mathbf{N}$.

Consequently we get

2.5. Theorem. *Let* $f : D \to \mathbf{R}$ *and* $a \in D$. *The point* a *is a minimum (maximum) point for* f *if and only if* a *is a minimum (maximum) point for* f *constrained by* Γ_a.

To formulate a refined result we need

2.6. Theorem. *Let* $f : D \subset \mathbf{R}^p \to \mathbf{R}$ *and* $a \in D$. *Suppose that for each* $\alpha \in \Gamma_a$ *(*$\alpha(t_0) = a$*),* $f \circ \alpha$ *is continuous at* t_0. *Then* f *is continuous at* a.

Proof. Suppose, per absurdum, that f is not continuous at a, i.e., there exists a sequence (x_n) from D with $x_n \to a$ and $f(x_n) \to l$, with $l \neq f(a)$. According to theorem 2.4, there exists a parametrized curve $\alpha \in \Gamma_a$, a subsequence (x_{n_k}) and a sequence of real numbers (t_{n_k}), $t_{n_k} \to 0$ such that $\alpha(t_{n_k}) = x_{n_k}$ and $\alpha(0) = a$. From the continuity of $f \circ \alpha$ at 0, we obtain that $f(x_{n_k}) \to f(a)$, which is a contradiction.

2.7. Theorem. *Let* $f : D \subset R^p \to R$ *and* $a \in D$. *Suppose that* $f \circ \alpha$ *is continuous for each* $\alpha \in \Gamma_a$, *and* a *is a strict extremum point constrained by* α. *Then* a *is a strict extremum point for* f.

Proof. From the previous theorem it follows that f is continuous. Let suppose that a is a strict extremum point for f constrained by each $\alpha \in \Gamma_a$. We prove that a is a strict extremum point for f constrained by Γ_a. In this way, according to theorem 2.5, it follows that a is a point of strict extremum for f.

Suppose, per absurdum, that there exist two parametrized curves $\alpha : I \to D$ and $\beta : J \to D$ in $\Gamma_a\,(a = \alpha(t_0) = \beta(u_0))$ such that a is a point of strict minimum for f constrained by α, and in the same time a is a point of strict maximum for f constrained by β.

Hence we can find two sequences of real numbers (t_n) and (u_n), $t_n \to t_0$ and $u_n \to u_0$, such that $f(\alpha(t_n)) > f(a)$ and $f(\beta(u_n)) < f(a)$. From the continuity of f it follows that on the segment $[\alpha(t_n), \beta(u_n)]$ in $\mathbf{R}^p$ we can find a point x_n with $f(x_n) = f(a)$. Applying theorem 2.4 to the sequence (x_n), we get a parametrized curve γ in Γ_a, a subsequence (x_{n_k}) of (x_n) and a sequence of real numbers (q_k) such that $q_k \to 0$, $\gamma(0) = a$ and $\gamma(q_k) = x_{n_k}$. It follows $f(\gamma(q_k)) = f(a)$, $\forall k \in \mathbf{N}$, i.e., the point a cannot be a strict extremum point for f constrained by γ. This result is a contradiction.

Remark. The previous theorems hold for the case in which $\mathbf{R}^p$ is replaced by a finite dimensional differentiable manifold.

2.8. Theorem. *Let* $f : D \to R$ *and* $a \in D$ *such that*

i) f *is differentiable*

ii) For any $\alpha \in \Gamma_a$ *the point* $f(a)$ *is not a separator point along* α.

iii) For any $\alpha \in \Gamma_a$ *it follows that* f *or* $-f$ *is locally strict convex at* a *along* α. *In these conditions* a *is a point of strict extremum of* f, *and* f *or* $-f$ *is locally convex at* a.

Proof. Let $\alpha \in \Gamma_a$ $(\alpha(t_0) = a)$. Because $f(a)$ is not a separator point along α, it follows that there exists two sequences (u_n) and (v_n) with $u_n < t_0 < v_n$ having the limit t_0 such that $f(\alpha(u_n)) \leq f(a)$, $f(\alpha(v_n)) \geq f(a)$ or $f(\alpha(u_n)) \leq f(a)$, $f(\alpha(v_n)) \leq f(a)$, $n \in \mathbf{N}$. Because $\varphi(t) = f(\alpha(t))$ is differentiable, it follows that $\varphi'(t_0) = 0$. Because φ or $-\varphi$ is locally convex at t_0, it follows that t_0 is a strict extremum point of φ (minimum or maximum). Thus we get that a is a strict extremum point of f constrained by each $\alpha \in \Gamma_a$. In accordance with theorem 2.7 a is a strict extremum point of f. Thus, f is locally strict convex at a along any parametrized curve $\alpha \in \Gamma_a$, or $-f$ is locally strict convex at a along any parametrized curve $\alpha \in \Gamma_a$. It follows that f or $-f$ is locally strict convex at a.

2 Convexity of Monge hypersurfaces

Let $f : D \subset \mathbf{R}^{p-1} \to \mathbf{R}$ be a function of reasonable class, and $\Sigma : x^p = f(x_1, ..., x_{p-1})$ a Monge hypersurface in $\mathbf{R}^p$. Let $a \in D$ and Γ_a the family of parametrized curves specified in the previous section.

3.1. Theorem. *Suppose that for any $\alpha \in \Gamma_a$, the point a is a strict extremum point restricted by α. Then the hyperplane $x^p = f(a)$ is tangent to Σ at $(a, f(a))$, and Σ is locally strict convex at $(a, f(a))$, i.e., around $(a, f(a))$ the Monge hypersurface Σ rests strictly on the same side of the tangent hyperplane.*

This theorem follows from theorem 2.7.

3.2. Theorem. *Suppose that f or $-f$ is locally strict convex at a along any $\alpha \in \Gamma_a$ and $f(a)$ is not a separator point of f along α. In these conditions the hyperplane $x^p = f(a)$ is a tangent hyperplane to Σ at $(a, f(a))$, and Σ is locally strict convex at a.*

This theorem follows directly from theorem 2.8.

In the sequel we suppose that $x^p = f(a)$ is the tangent hyperplane to Σ at $(a, f(a))$. This hyperplane can be identified with $\mathbf{R}^{p-1}$, and the point $(a, f(a))$ with a. Let Γ_a^* be the family of all parametrized curves passing through a and contained in Σ, having the same properties as those for the family Γ_a.

3.3. Lemma. *The following statements hold true:*

i) Let α be a parametrized curve passing through a and contained in the tangent hyperplane at a. Then $\alpha \in \Gamma_a$ if and only if $(\alpha, f \circ \alpha) \in \Gamma_a^*$.

ii) The association $\alpha \to (\alpha, f \circ \alpha)$ is a bijective correspondence between Γ_a and Γ_a^*.

iii) a is a strict extremum point of f constrained by α if and only if the parametrized curve $(\alpha, f \circ \alpha)$ rests, around a, on the same side of the tangent hyperplane at a.

Proof. We prove the sentence ii). It is obvious that the association $\alpha \to (\alpha, f \circ \alpha)$ is one-to-one. Let $\beta \in \Gamma_a^*$ ($\beta(0) = \alpha$). It follows that $\beta(t) = (\alpha(t), (f \circ \alpha)(t))$, where α is the projection of β on the tangent hyperplane. Let us prove that $\alpha \in \Gamma_a$. We have

$$\beta'(t) = \left(\alpha'(t), \sum_{i=1}^{p-1} \frac{\partial f}{\partial x^i}(\alpha(t)) \frac{dx^i}{dt}(t) \right)$$

and

$$\beta''(t) = \left(\alpha''(t), \sum_{i=1}^{p-1} \frac{\partial f}{\partial x^i}(\alpha(t)) \frac{d^2 x^i}{dt^2}(t) + \sum_{i,j=1}^{p-1} \frac{\partial^2 f}{\partial x^i \partial x^j}(\alpha(t)) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t) \right).$$

Because $\frac{\partial f}{\partial x^i}(\alpha(0)) = 0$, we obtain $\beta'(0) = (\alpha'(0), 0)$ and

$$\beta''(0) = \left(\alpha''(0), \sum_{i,j=1}^{p-1} \frac{\partial^2 f}{\partial x^i \partial x^j}(\alpha(0)) \frac{dx^i}{dt}(0) \frac{dx^j}{dt}(0) \right).$$

If $\alpha'(0) = 0$ we obtain $\beta'(0) = 0$ and thus $\beta''(0) = (\alpha''(0), 0)$. As $\beta''(0) \neq 0$ we have $\alpha''(0) \neq 0$.

3.4. Definition. We say that the hypersurface Σ is (strictly) convex at a along a parametrized curve $\alpha \in \Gamma_a$ if a is a point of (strict) extremum for f restricted by α. Also, we say that Σ is (strictly) convex at a with respect to the family Γ_a if a is a (strict) extremum point of f constrained by Γ_a.

3.5. Theorem. *The hypersurface Σ is strictly convex at a with respect to the family Γ_a if and only if Σ is strictly convex at a along any parametrized curve α in Γ_a.*

This theorem is a direct consequence of theorem 2.5.

In the sequel let suppose that the hypersurface Σ is defined by the Cartesian implicit equation $F(x_1, ..., x_p) = 0$. Let

$$g(x) = \sum_{i=1}^{p} \frac{\partial F}{\partial x^i}(a)(x^i - a^i)$$

be the linear approximation of F around the point a.

3.6. Definition. We say that the hypersurface Σ is (strictly) convex at a along the parametrized curve $\alpha \in \Gamma_a^*$ if a is a (strict) extremum point for g restricted by α. Also, we say that Σ is (strictly) convex at a with respect to the family Γ_a^* if a is a (strict) extremum point of g restricted by the family Γ_a^*.

Summing the previous results, we obtain:

3.7. Theorem. *The following statements are equivalent:*

a) Σ is strictly convex at a along any parametrized curve in Γ_a;

b) Σ is strictly convex at a along any parametrized curve in Γ_a^*;

c) Σ is strictly convex at a with respect to the family Γ_a;

d) Σ is strictly convex at a with respect to the family Γ_a^*;

e) The point a is a strict extremum point of the function g restricted to Σ.

Reformulating:

3.8. Theorem. *The following statements are equivalent:*

a) Each parametrized curve in the family Γ_a^* rests, around a, strictly on the same side of the tangent hyperplane to Σ at a.

b) All the parametrized curves in the family Γ_a^* rest, around a, strictly on the same side of the tangent hyperplane to Σ at a.

c) The hypersurface Σ rests, around a, on the same side of the tangent hyperplane to Σ at a.

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