The continuity of linear physical systems

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Abstract. In a short communication presented at the MENP-3 Colloquium in 2004, we provided several aspects of a rigorous mathematical fundation of physical systems theory, based on the notion of Carleman operator, introduced in its entire generallity by the author, both in the linear and nonlinear cases. Various applications to different specific problems of systems, as time-invariance, causality, passivity and the systems connectivity, were also considered. In the present work, we give some results based on the closed graph theorem, about the continuity of Carleman operators, these being operators generated by some operator-valued functions, named weighted (or admittance) functions. Several applications are given, e.g., the set of hypotheses that assure the coincidence of the two mathematical models for linear physical systems: Carleman operators and linear continuous operators.

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1 Carleman operators

Definitions. Let E, F, G be separated topological vector spaces on the field $\mathbf{K}$ of real or complex numbers, F being composed of functions defined on a set S, with values in G, its topology being finer than the pointwise convergence topology. We denote $L(E,G)$ the vector space of linear continuous operators from E to G. In the papers [4] and [6], the author defined the notion of *Carleman operator* as being the operator $U_\varphi : E \to F$, where φ is an operator-valued function $\varphi : S \to L(E,G)$, named *weighted (admitance) function* of the Carleman operator, such that

$$(U_\varphi(e))(s) = (\varphi(s))(e), \forall s \in S, \forall e \in E. \tag{1.1}$$

Every Carleman operator is linear. We denote by $\mathcal{C}(E,F)$ the vector space of Carleman operators everywhere defined on E with values in F and $\mathcal{FC}(E,F)$ the vector space of the weighted functions that generate the Carleman operators from E

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to F. Consequently, the functions $\varphi \in \mathcal{FC}(E,F)$ are the operator-valued functions $\varphi : S \to L(E,G)$ satisfying the condition

$$(\varphi(.))\,(e) \in F, \forall e \in E. \tag{1.2}$$

Examples of Carleman operators

Scalar case. If $G = \mathbf{K}$, then the elements of F are scalar-valued functions and the weighted function $\varphi : S \to E'$ is a functional-valued function, having its values in the dual E' of E.

Hilbert space case. Consider $G = \mathbf{K}$ and let E be a Hilbert space. According to the Riesz theorem, E is anti-isomorphic to its dual, hence the weighted function φ of a Carleman operator from E to F has its values in E, the operator having the form

$$(U_\varphi(e))\,(s) = \varphi(S) \cdot e, \forall e \in E, \forall s \in S, \tag{1.3}$$

where the product from the right part of the above formula is the scalar product from E.

Initial Carleman case. If $E = F = L^2(E,F)$, the weighted function of a Carleman operator from E to F is a two variables scalar-valued function $\varphi(s,t)$, which satisfies the condition $\varphi(s,\cdot) \in L^2(a,b)$, for almost all $s \in (a,b)$. In this case, the Carleman operators have the integral form

$$(U_\varphi(e(t)))\,(s) = \int_a^b \varphi(s,t)e(t)dt \in L^2(a,b), \quad \forall e(t) \in L^2(a,b), \tag{1.4}$$

for almost all $s \in (a,b)$. This is the initial notion considered by T. Carleman in [2].

Finite dimensional case. Consider $E = \mathbf{K}^n$ and $F = \mathbf{K}^m$, the elements of this last space being considered as scalar-valued functions, defined on the set $\{1,2,...,m\}$. Every linear operator $U : \mathbf{K}^n \to \mathbf{K}^m$, $U(e) = f$, has the matriceal representation $A \cdot e = f$, where $A \in M_{m,n}$ is a matrix with m rows and n columns associated to the operator U, and $e \in M_{n,1} \cong \mathbf{K}^n$, $f \in M_{m,1} \cong \mathbf{K}^m$, are column matrices. Such an operator is Carleman, with the weighted function $\varphi = A$, because $\varphi : S = \{1,...,m\} \to (\mathbf{K}^n)' \cong \mathbf{K}^n \cong M_{1,n}$ and hence $\varphi(s) = L_s \in M_{1,n}, \forall s \in S$, where $L_1,...,L_m$ are the rows of the matrix A.

2 Carleman representation of linear continuous operators

Linear continuous operators as Carleman operators. Every linear continuous operator $U : E \to F$ is Carleman, hence

$$L(E,F) \subseteq \mathcal{C}(E,F). \tag{2.1}$$

The weighted function of the linear continuous operator U, considered as Carleman operator, is given by the formula $\varphi(s) = \delta_F^s \circ U, \forall s \in S$ where $\delta_F^s \in L(E,G)$ is the *Dirac operator concentrated at the element* $s \in S$, defined on the space F by the formula $\delta_F^s(f) = f(s)$, $\forall f \in F$.

Carleman operators as closed or continuous operators. In the above mentioned hypotheses, every Carleman operator is closed.

The pair of spaces E and F is said to have *the closed graph property*, if for these spaces takes place the closed graph theorem, hence every linear closed (with closed graph) operator is continuous.

If S, G, E, F are as above and the pair of spaces E and F has the closed graph property, then every Carleman operator from E to F is linear and continuous, hence in these hypotheses, we have

$$L(E,F) = \mathcal{C}(E,F). \tag{2.2}$$

The adjoint $U'_\varphi : F' \to E'$ of the Carleman operator U_φ is weak continuous and is given by formula

$$\left(U'_\varphi(f')\right)(e) = f'\left(U_\varphi(e)\right) = f'\left(\left(\varphi(\cdot)\right)(e)\right), \quad \forall e \in E, \forall f' \in F'. \tag{2.3}$$

In the scalar case, $G = \mathbf{K}$, the weighted function φ of the Carleman operator, is given by the formula $\varphi(s) = U'_\varphi\left(\delta^s_F\right), \forall s \in S$, which justifies another denomination for the weighted function φ, the one of *impulse-response function.*

Linear continuous operators as adjoint of Carleman operators. In addition to the above hypotheses, we suppose E and F to be locally convex spaces, the first tonelate, the second semireflexive and we denote E'_b the dual of E endowed with the strong topology. Then $V : F'_b \to E'_b$ is linear and continuous if and only if it is the adjoint of a Carleman operator, $V = U'_\varphi$. If $G = \mathbf{K}$, then the weighted function is given by the formula $\varphi(s) = V\left(\delta^s_F\right), \forall s \in S$.

3 Particular cases of Carleman representations

Results about the continuity of some particular Carleman operators and hence about Carleman representation of the linear continuous operators, that are special cases of our above results, were given by many authors. In some of these cases, the authors require hypotheses that are not necessary, some of them being dependent of their particular context. Also, in many cases, the initial proofs are different of ours, being not based on the closed graph theorem.

Banach spaces. If E and F are Banach spaces, from the above properties of Carleman operators, there result the known theorems of Izumi and Sunouchi, [11], Taylor, [17] and Jdanov [12].

Hilbert spaces. For E a Hilbert space, $G = \mathbf{K}$ and $F = L^2(S)$, where S is a measure space, are obtained some results given by Weidmann [19] and for $E = F = L^2(a,b)$ by Carleman [2], in connection with his integral operator above mentioned. For $E = \mathbf{K}^n, F = \mathbf{K}^m$ it results the well known property that every linear operator $U : \mathbf{K}^n \to \mathbf{K}^m$ is continuous.

Locally convex spaces. If E and F are locally convex spaces, satisfying several hypotheses, we obtain some results given by V.Ptak [15].

Distributional spaces. In the case $S = \mathbf{R}^n$, $G = \mathbf{K}$, $E = \mathcal{D}\left(\mathbf{R}^m\right)$ and $F = \mathcal{D}\left(\mathbf{R}^n\right)$, from the above considerations it results that a linear operator from $\mathcal{D}\left(\mathbf{R}^m\right)$

to $\mathcal{D}(\mathbf{R}^n)$ is continuous if and only if is Carleman, its weighted function being an indefinite differentiable function $\varphi : \mathbf{R}^n \to \mathcal{D}'(\mathbf{R}^m)$, scalarly with compact support, this meaning that the function $(\varphi(s))(e)$ has compact support in $\mathbf{R}^n$, for every $e \in \mathcal{D}(\mathbf{R}^n)$.

These results was given by the author ([3]), by Pondelicek, ([14]) and by Dolezal, ([10]) and stay at the basis of the linear distributional physical systems theory. The distribution-valued functions that are weighted functions of the Carleman operators between spaces of distributions, was introduced by R.Cristescu ([8]) after the name of *composable famillies of distributions*, they being particular cases of L.Schwartz kernels (distributions of two multi-variables).

Other cases of representation of linear continuous operators as Carleman operators acting between several spaces from distributions theory, which are also particular cases of our results, were give by Meidan [13] and Dolezal and Sanborn [10].

4 Applications

Using the results about the continuity of the Carleman operators presented above, several old or new theorems can be obtained, in many cases with new and considerably reduced proofs. Some of them will be presented below.

Gelfand-Dunford type theorems. N.Bourbaki's Gelfand-Dunford theorem states that if E is a separated locally convex space with the (GDF) property and if $\varphi : S \to E'_\sigma$ is a scalar essential integrable function on a measure space (S, μ), then the Radon integral $\int \varphi d\mu$ is an element from the dual E'. Here the operator $U_\varphi : E \in L^1(S)$, defined by $U_\varphi(e) = \int \varphi e d\mu, \forall e \in E$ is Carleman hence continuous and $U'_\varphi : \left(L^1(S)\right)' \cong L^\infty(S) \to E'$, which infers $\int \ \varphi d\mu = U'_\varphi(1) \in E'$.

L.Schwartz' Gelfand-Dunford theorem in vectorial distribution theory, states that if E is a locally convex space having the (GDF) property, then every scalar locally summable function $\varphi : \mathbf{R}^n \to E'_\sigma$, defines a vectorial distribution (of function type) with values in E'_σ, namely the linear operator W defined by the formula

$$(W(\alpha))(e) = \int (\varphi(s))(e)\alpha(s)ds, \forall \alpha \in \mathcal{D}(\mathbf{R}^n), \forall e \in E, \tag{4.1}$$

is a continuous operator from $\mathcal{D}(\mathbf{R}^n)$ to E'_σ.

Hellinger-Toeplitz type theorem. The linear operators generated by infinite matrices that act between sequence spaces, being particular cases of Carleman operators, are continuous if the spaces satisfy conditions of the type above mentioned. This theorem has several particular settings: if $E = F = l^2$, this result was given by Hellinger and Toeplitz, if E, F are (FK) spaces by Zeller, [19], if E, F are Banach spaces of sequences by Taylor, [17]. Another extension of the Hellinger-Toeplitz theorem given by Ptak, [15], is also particular case of our results.

Operators commuting with the translations. In case that E and F are both composed of functions with values in G, defined on a semigroup S, having continuous translations and their pair having the closed graph property, then $U : E \to F$ is a linear continuous operator, commuting with the translations, i.e.

$$U(\tau^s e) = \tau^s(U(e)), \forall e \in E, \forall s \in S, \tag{4.2}$$

if and only if it is a convolution operator

$$U(e) = U_{\tau^{\cdot} u}(e) = u \star e, \forall e \in E, \tag{4.3}$$

with a convolutor u, that is an operator $u \in L(E, G)$ having the property $(\tau^{\cdot} u) = u(\tau^{\cdot} e) \in F, \forall e \in E$. If the semigroup has an unity $s_0 \in S$, then $u = \delta_F^{s_0} \circ U = U'(\delta_F^{s_0})$.

Particular cases of the convolution representation of the linear continuous operators that commute with the translations are for example the theorem S of L.Schwartz, for $E = F = \mathcal{D}(\mathbf{R}^n)$ and several results given by B.Brainerd and Edwards, [1].

5 Physical linear systems and Carleman operators

Physical systems. The mathematical model of a *physical system* is an operator $U : E \to F$ and it is said that the system is *governed* by the operator U. The elements e of the definition domain E are called the *inputs* of the system and the elements f of the range F are called the *outputs*. Because the last ones must be measured, the elements f from F are supposed to be functions defined on a set S with values in $\mathbf{R}$. The physical system is named *linear* if E, F, G are vector spaces and the operator U is linear.

Linear physical systems as Carleman operators. We defined in [5] a physical time-varying system as being a Carleman operator. In this case, the properties of the system will be studied on the base of the weighted function φ associated with the Carleman operator, which is usually called *admittance function*. In [8] were presented several properties of the physical systems, both linear and non-linear were presented. Time-invariant systems were particularly considered.

The continuity of the linear physical systems. Because several authors define the linear physical systems as being linear continuous operators is of great interest to compare the notions of Carleman with that of linear continuous operators, problem that was addressed in the present communication.

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