Generalized concircular recurrent Weyl spaces

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Abstract. In [2], De and Guha introduced the generalized recurrent Riemannian manifolds and in [4], Maralabhavi and Rathnamma studied the generalized recurrent and concircular recurrent Riemannian manifolds. In this work, the generalized concircular recurrent Weyl spaces are considered and it is proved that a Weyl space which is either generalized concircular recurrent or generalized recurrent is a concircular recurrent Weyl space. In addition, it is shown that every concircular recurrent Weyl space is a conformally recurrent.

Key words: Weyl space, generalized recurrent Weyl space, generalized concircular recurrent Weyl space, conformally recurrent Weyl space.

1 Introduction

An $n$-dimensional differentiable manifold $W_n$ is said to be a Weyl space if it has a conformal metric tensor $g$ and a symmetric connection $\nabla$ satisfying the compatibility condition given by the equation

\[(1.1)\quad \nabla_k g_{ij} - 2T_k g_{ij} = 0 ,\]

where $T_k$ denotes a covariant vector field. Under the renormalization

\[(1.2)\quad \tilde{g} = \lambda^2 g\]

of the metric tensor, $T$ is transformed by the law

\[(1.3)\quad \tilde{T}_k = T_k + \partial_k \ln \lambda\]

where $\lambda$ is a function defined on $W_n$.

An object $A$ defined on $W_n(g, T)$ is called a satellite of weight \{p\} of the tensor $g_{ij}$, if it admits a transformation of the form

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under the renormalization of the metric tensor \( g_{ij} \) ([3],[6]).

The prolonged covariant derivative of a satellite \( A \) is defined by

\[
\hat{\nabla}_k A = \nabla_k A - p T_k A
\]

We note that the prolonged covariant derivative preserves the weight.

Writing (1.1) in local coordinates and expanding it we find that

\[
\partial_k g_{ij} - g_{hj} \Gamma^h_{ik} - g_{ih} \Gamma^h_{jk} - 2 T_k g_{ij} = 0
\]

where \( \Gamma^i_{kl} \) are the connection coefficients of the form

\[
\Gamma^i_{kl} = \left\{ \begin{array}{c} \delta^i_k T_l + \delta^i_l T_k - g^{im} g_{kl} T_m \\ k \end{array} \right. 
\]

Let \( R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \) denote the curvature tensor associated with the connection \( \nabla \).

A non-flat Weyl space is called recurrent, if its curvature tensor \( R \) satisfies

\[
(\hat{\nabla}_U R)(X,Y,Z) = \phi(U) R(X,Y,Z)
\]

where \( \phi \) is a nonzero 1-form of weight zero and \( R(X,Y,Z) \) is the curvature tensor of type (1,3) [1].

In [2], De and Guha have introduced a non-flat Riemannian space whose curvature tensor \( R \) of type (1,3) satisfies the condition

\[
(\nabla_U R)(X,Y,Z) = A(U) R(X,Y,Z) + B(U) (g(Y,Z)X - g(X,Z)Y)
\]

where \( A \) and \( B \) are 1-forms.

A non-flat Weyl space is called generalized recurrent whose curvature tensor \( R \) satisfies the condition

\[
(\hat{\nabla}_U R)(X,Y,Z) = A(U) R(X,Y,Z) + B(U) (g(Y,Z)X - g(X,Z)Y)
\]

where \( A \) and \( B \) are 1-forms of weight 0 and -2, respectively. Note that when \( B = 0 \) a generalized recurrent Weyl space becomes a recurrent Weyl space.

\section{Generalized Concircular Recurrent Weyl Spaces}

The concircular curvature tensor of a Weyl space is given by [7]

\[
z(X,Y,Z) = R(X,Y,Z) - \frac{R}{n(n-1)} [g(Y,Z)X - g(X,Z)Y].
\]

A non-flat Weyl space is called generalized concircular recurrent, if its concircular curvature tensor satisfies the condition
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(2.1) \((\tilde{\nabla}_U Z)(X, Y, Z) = A(U)Z(X, Y, Z) + B(U)(g(Y, Z)X - g(X, Z)Y)\).

**Theorem 2.1** The Weyl space \(W_n(g, T)\) which is either generalized concircular recurrent or generalized recurrent is concircular recurrent.

**Proof.** We first note that, in local coordinates the concircular curvature tensor of \(W_n\) is given by [7]

\[ z_{jkh}^i = R_{jkh}^i - \frac{R}{n(n-1)} (g_{jk} \delta_h^i - g_{jh} \delta_k^i) . \]

Suppose now that \(W_n(g, T)\) is a generalized concircular Weyl space. Then, by expressing (2.1) in local coordinates we have

(2.2) \((\tilde{\nabla}_l z_{jkh}) = A_l z_{jkh} + B_l (\delta_h^i g_{jk} - \delta_k^i g_{jh}) . \)

By contracting (2.2) with respect to \(i\) and \(h\), we get

(2.3) \(\tilde{\nabla}_l z_{jki} = \tilde{\nabla}_l z_{jki} = A_l z_{jki} + B_l (n-1) g_{jk} .\)

where

(2.4) \(z_{jki} = R_{jki} - \frac{R}{n} g_{jk} .\)

Transvecting (2.4) by \(g_{jk}\) and using \(g_{jk} z_{jki} = 0\), we get \(B_l = 0\). Thus, from (2.2) we find

(2.5) \(\tilde{\nabla}_l z_{jkh} = A_l z_{jkh} .\)

showing that \(W_n(g, T)\) is a concircularly recurrent.

Next, suppose that \(W_n(g, T)\) is a generalized recurrent Weyl space. By virtue of (1.10), we can write

(2.6) \(\tilde{\nabla}_l R_{jkh} = A_l R_{jkh} + (n-1) B_l g_{jk} .\)

Transvecting (2.6) by \(g_{jk}\) we have

(2.7) \(\tilde{\nabla}_l R = A_l R + (n-1) B_l .\)

On the other hand, taking the prolonged covariant derivative of the concircular curvature tensor of \(W_n\) we have
Substituting (2.7) in (2.8) and using (2.5) we find

\[ \dot{\nabla}_l z^i_{jkh} = A_l z^i_{jkh}, \]

which implies that the generalized recurrent Weyl space \( W_n(g, T) \) is concircularly recurrent.

### 3 Conformally recurrent Weyl spaces

The conformal curvature tensor \( C^i_{jkh} \) of \( W_n \) is given by [5]

\[
C^i_{jkh} = R^i_{jkh} + \frac{2}{n(n-2)} \left( \delta^i_h R_{[jk]} - \delta^i_k R_{[jh]} - g_{jh} g^{im} R_{[mk]} + g_{jk} g^{im} R_{[mh]} - \right.
\]

\[
- (n-2) \delta^i_j R_{[hk]} - \frac{1}{n-2} \left( \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh} g^{im} R_{mk} + g_{jk} g^{im} R_{mh} \right)
\]

\[
+ \frac{R}{(n-1)(n-2)} \left( g_{jk} \delta^i_h - g_{jh} \delta^i_k \right)
\]

where the bracket denotes antisymmetrization.

An \( n \)-dimensional Weyl space is said to be a conformally recurrent Weyl space if its conformal curvature tensor \( C^i_{jkh} \), of weight 0, satisfies the condition

\[
\dot{\nabla}_l C^i_{jkh} = \lambda_l C^i_{jkh}
\]

where \( \lambda_l (\neq T_l) \) is a non-zero covariant vector field of weight 0.

\( C^i_{jkh} \) may be expressed in terms of \( z^i_{jkh} \) and \( z_{jk} = z^i_{jki} \) as follows:

\[
C^i_{jkh} = z^i_{jkh} + \frac{2}{n(n-2)} \left( \delta^i_h z_{[jk]} - \delta^i_k z_{[jh]} - g_{jh} g^{im} z_{[mk]} + g_{jk} g^{im} z_{[mh]} - \right.
\]

\[
- (n-2) \delta^i_j z_{[hk]} - \frac{1}{n-2} \left( \delta^i_h z_{jk} - \delta^i_k z_{jh} - g_{jh} g^{im} z_{mk} + g_{jk} g^{im} z_{mh} \right)
\]

**Theorem 3.1** Every concircular recurrent Weyl space is a conformally recurrent Weyl space.

**Proof.** A concircular recurrent Weyl space is characterized by

\[
\dot{\nabla}_l z_{tjk} = \lambda_l z_{tjk}
\]

where \( \lambda_l (\neq T_l) \) is a non-zero covariant vector field of weight 0.

As an immediate consequence of (3.4), we get
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\begin{equation}
\hat{\nabla}_l z_{jk} = \lambda_l z_{jk}.
\end{equation}

Changing the order \( j \) and \( k \) in (3.5) and substracting the equation so obtained from (3.5), we get

\begin{equation}
\hat{\nabla}_l z_{[jk]} = \lambda_l z_{[jk]}.
\end{equation}

Therefore, by using (3.3),(3.4),(3.5) and (3.6), we get

\[ \hat{\nabla}_l C^i_{jkh} = \lambda_l C^i_{jkh} \]

from which the desired result follows.

References


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