Of Finsler fiber bundles and the evolution of the Calculus

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Abstract

In order to develop the Kaluza-Klein (KK) space to which we were led in an accompanying paper, we present the basics of Clifford algebra and of Kähler's generalization of the calculus of differential forms, which is based on that algebra. This generalized calculus contains a "Kähler-Dirac (KD) equation", which appears to supersede the Dirac and Laplace equations and plays a role comparable in significance to that of the equations of structure in differential geometry. We incorporate a KD equation into the structure of our KK space to specify the torsion, thus dressing this type of geometry in a physical attire.

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1 Introduction

In an accompanying paper [7], we have developed a *sui generis* line of evolution of generalized geometry consisting in implementing geometric equality to the largest possible non-trivial extent. This is, needless to say, teleparallelism (TP). Since we are interested in the Lorentzian signature, we have considered Finsler bundles. Indeed, as we showed in the same paper, the Lorentzian signature seems particularly well suited to these bundles. We pointed out that a Finslerian TP structure can be considered as a product structure of the sphere bundle for n dimensions with the rotation group in $n-1$ dimensions. The product structure was there defined as going beyond the topological product of spaces and having to do with how the differential invariants defining the structure of the topological product relate to the differential invariants of the factors.

We pointed out that it was possible to construct in principle another product structure with the differential invariants that define Finslerian TP on (pseudo)-Riemannian metrics of Lorentzian signature. The horizontal invariants will now be used in a different way. The resulting alternative structure may then be multiplied in principle with the second factor in the topological product of TP Finsler connections on those

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metrics, namely the rotation group in $n-1$ dimensions. We pointed out that, in developing this alternative structure, one needs to resort at a very early stage to Clifford algebras, and specially a Clifford algebra of differential forms. At the same time, we shall still need at least the exterior calculus of differential forms, or Cartan calculus, which is the language of differential invariants. The Kähler calculus [3], which incorporates not only Clifford products of differential forms but also exterior differentiation, includes in addition Clifford or Kähler differentiation, which is to exterior differentiation what the Clifford product is to the exterior product.

Kähler differentiation has two pieces, exterior and interior. The interior piece supersedes the codifferential, which involves the Hodge dual and resembles an attachment to the exterior calculus, rather than an operation at par with the exterior differential in its role within the calculus. Those pieces might be referred to as exterior and interior *covariant* differentiations when applied to tensor-valued differential forms, but the term "covariant" is unnecessary in this calculus since it is implicit in the fact that the differentiation operator is operating on a form which is not scalar-valued. Kähler and interior differentiations are not differentiations in the modern sense of the term since they do not satisfy the Leibniz rule. However, it will later become clear why it still makes sense to refer to these operations as differentiations, as Kähler does.

Coming on top of the great achievements of the exterior calculus, Kähler's is unsurpassed by any other. It was proposed to deal not only with general relativity but also with quantum physics. As we shall show, it has an impact on how we can view the equations of structure of the space that we developed in the previous section. It also extends the theory of harmonic functions. The concept of harmonic in the ring of functions is now replaced with the concept of strict harmonic in the ring of differential forms.

Before we provide the main highlights of this calculus, we show the naturalness of Clifford algebra. We are not dealing here with just another algebra, but rather one which corrects our view of vector products and then puts them together with dot products. The magnificent synthesis which thus results has seminal effects on the calculus, analysis and physics, as shown by Kähler [3]. But it is a synthesis which has failed to get the attention that it deserves, and whose implications have barely started to surface. The point of this paper is to show how this calculus can be enhanced further by drawing inspiration from geometry, and perfecting in turn some of it.

The contents of the paper is organized as follows. In section 2, we show how Clifford algebra incorporates into a unity the two products of vectors, symmetric and antisymmetric, for any dimension and not just $n=3$. In section 3, we summarize the Kähler calculus of scalar-valued differential forms (See Ref. [6] for general valuedness, a more controversial case). In section 4, we show how the evolution of geometry along the lines of the accompanying paper [7] has an impact (on) and is impacted upon by the Kähler calculus

2 The Case for a Calculus Based on Clifford Algebra

Different calculi are underlined by different algebras: exterior, tensor, etc. The Kähler calculus is underlined by Clifford algebra, whose basic product is the Clifford product [4]. Let us see the common Euclidean and pseudo-Euclidean rules of this algebra.

Suppose the existence of a product of vectors which is associative and distributive with respect to addition, but not necessarily commutative or anticommutative. Any such product, denoted simply through juxtaposition, can always be rewritten as

$$ab \equiv (1/2)(ab + ba) + (1/2)(ab - ba). \tag{2.1}$$

This decomposition applies in particular to the tensor product, but, more interestingly, to the Clifford product, which we shall be defining further down. We introduce symbols to name these two parts of the product as individual products themselves:

$$a \cdot b \equiv (1/2)(ab + ba), \qquad a \wedge b \equiv= (1/2)(ab - ba). \tag{2.2}$$

It is clear that $a \wedge a = 0$ and $a \cdot b = -b \cdot a$.

In order to facilitate the connection with the readers experience, let us use boldface in the illustration that follows of the exterior product. Consider any two non-colinear vectors in a 3-dimensional subspace of an n-dimensional vector space. We shall denote as $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ any basis in this subspace, not necessarily orthonormal. We have

$$\begin{aligned} &(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \wedge (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \\ &= (a_1b_2 - a_2b_1)\mathbf{i} \wedge \mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{j} \wedge \mathbf{k} + (a_3b_1 - a_1b_3)\mathbf{k} \wedge \mathbf{i}, \end{aligned} \tag{2.3}$$

where we have used antisymmetry. In terms of $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{j} \wedge \mathbf{k}$ and $\mathbf{k} \wedge \mathbf{i}$, the components of the expression on the right hand side of 2.3 are the same as for the vector product, but one no longer associates $\mathbf{i} \wedge \mathbf{j}$ with $\mathbf{k}$, $\mathbf{j} \wedge \mathbf{k}$ with $\mathbf{i}$, etc. We shall say that $\mathbf{a} \wedge \mathbf{b}$ represents an oriented plane, a figure of grade 2, determined by a and b, and so does $\lambda(\mathbf{a} \wedge \mathbf{b})$ for any positive scalar. $\lambda(\mathbf{b} \wedge \mathbf{a})$ represents the opposite orientation of the same plane. The contents of the parenthesis on the right hand side of Eq. 2.3 are said to represent the components of the "bivector" $\mathbf{a} \wedge \mathbf{b}$ in terms of the bivectors $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{j} \wedge \mathbf{k}$ and $\mathbf{k} \wedge \mathbf{i}$. We may also say that, up to a multiplicative constant, those quantities are the components of the plane $\mathbf{a} \wedge \mathbf{b}$ relative to the planes $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{j} \wedge \mathbf{k}$ and $\mathbf{k} \wedge \mathbf{i}$. In other words, the product "$\wedge$", called exterior product, of vectors represents oriented planes, not vectors or directions.

We may now associate $\mathbf{a} \cdot \mathbf{b}$ with the standard scalar (i.e. dot) product of Euclidean geometry, a figure of grade zero (i.e. just a number). From the first of the equations 2.2, we have, using $\mathbf{a}_i$ for the vectors of a basis in any number of dimensions,

$$\mathbf{a}_i\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_i - 2g_{ij} = 0. \tag{2.4}$$

where g_{ij} is the usual $\mathbf{a}_{\bar{\imath}} \cdot \mathbf{a}_j$. This equation exemplifies Clifford algebra. Let us show its relation to the general graded tensor algebra.

We start with the exterior algebra, which is simpler. It is defined as the quotient of the graded tensor algebra by the two sided ideal defined by $a \otimes a$, where a represents any tensorial quantity of grade 1 (We return to not bolded symbols, because these symbols may refer in particular to cochains, differential forms, etc). Computing in the exterior algebra is equivalent to computing in the tensor algebra modulo $a \otimes a$. It is important to notice that, if we had

$$...a \otimes b \otimes ... \otimes c \otimes a \otimes d..., \tag{2.5}$$

we still would set this product equal to zero, even if the two factors a are not contiguous. This is so because

$$0 = (a + b) \otimes (a + b) = a \otimes a + a \otimes b + b \otimes a + b \otimes b, \tag{2.6}$$

which shows that $a \otimes a = 0$ for all a implies $a \otimes b = -b \otimes a$ and, therefore, $a \otimes ... \otimes a = 0$ (after performing successive anticommutations). Products so computed (i.e. mod $a \otimes a$) are indicated by the symbol $\wedge$ instead of $\otimes$.

Similarly, a Clifford algebra is defined as the quotient of the graded tensor algebra by the two sided ideal defined by $a \otimes b + b \otimes a - 2Q(a, b).1$, where Q denotes any symmetric, non-singular quadratic form. One then uses the symbol $\vee$, rather than $\otimes$. In terms of a (pseudo)-orthonormal basis pertaining to the Lorentzian signature, the rules for $\vee$ multiplication are the same as for multiplication of the gamma matrices of Dirac's relativistic quantum mechanics [2], [4].

From equations 2.1 and 2.2, we get, now for the Clifford algebra,

$$a \vee b = a \cdot b + a \wedge b. \tag{2.7}$$

Similar considerations, up to issues of sign, apply to the Clifford product of a vector and a multivector (bivector, trivector, etc):

$$a \vee A = a \cdot A + a \wedge A. \tag{2.8}$$

Any element of the Clifford algebra can be written as a sum of a scalar, a vector, a bivector, etc. If A is of homogeneous grade r, $a \vee A$ will have in general a part of grade $r - 1$ and another part of grade $r + 1$. The foregoing few considerations on Clifford algebra are sufficient to understand the basics of the Kähler calculus.

3 The Basics of the Kähler Calculus

We use the symbol $\vee$ for Clifford product of differential forms, rather than the more common juxtaposition used when dealing with the Clifford algebra of tangent vectors, since juxtaposition is often used to designate the exterior product when dealing with differential forms and cochains (integrands). Kähler [3] defined a Clifford algebra of differential forms through the relation

$$\omega^i \vee \omega^j + \omega^j \vee \omega^i - 2\delta^{ij} = 0. \tag{3.1}$$

Here, the ω^i's are the pull-backs of the soldering forms to a section of the bundle $B'(M) \rightarrow M$, where $B'(M)$ is the set of orthonormal bases of tangent vectors to the manifold M. If the signature is Lorentzian, we replace δ^{ij} with η^{ij}, i.e. diagonal $(-1, 1, 1, 1)$ or $(1, -1, -1, -1)$. Actually Kähler used coordinate bases, i.e.

$$dx^i \vee dx^j + dx^j \vee dx^i - 2g^{ij} = 0. \tag{3.2}$$

This equation has to be seen in the context that $dx^i \vee dx^j$ is an element of a basis for an integrand or cochain, normally written as $dx^i dx^j$, where this could be the $dxdy$, or $d\rho d\theta$, etc. He also considered tensor-valued differential forms, i.e. whose coefficients are tensors. For example, he writes a differential r-form of $(1, 1)$ valuedness as u_i^j:

$$u_i^j = u_{ik...l}^j dx^k \wedge ... \wedge dx^l. \tag{3.3}$$

Kähler gave a complicated ansatz for his concept of covariant derivative (it satisfies the Leibniz rule) $d_h u$ such that $dx^h \wedge d_h u$ is the exterior derivative du, and, if the differential form is tensor valued, say $u_{i...}^{j...}$, the expression

$$du_{i...}^{j...} = dx^h \wedge d_h u_{i...}^{j...}. \tag{3.4}$$

gives the usual exterior covariant derivative [3], [6]. His covariant derivative is very cumbersome, appears to have an ad hoc character and is only applicable when the connection of the manifold is the Levi-Civita connection. It is of enormous importance, however, in his calculus since it gives rise to the Kähler or Clifford "derivative" (it does not satisfy the Leibniz rule), which is defined in terms of coordinates bases as

$$\partial u_{i...}^{j...} = dx^h \vee d_h u_{i...}^{j...}. \tag{3.5}$$

We use the symbol ∂ where he uses the symbol δ. We, therefore, have:

$$\partial u_{i...}^{j...} = dx^h \wedge d_h u_{i...}^{j...} + dx^h \cdot d_h u_{i...}^{j...}. \tag{3.6}$$

The definition in terms of contracted Clifford products with the soldering forms ω^h, rather than with the dx^h, is trivial, provided that one treats $d_h u_{i...}^{j...}$ accordingly. It is a matter of a change of basis, which has to be carried out consistently everywhere. The last term in Eq. 3.6 is known as the interior covariant derivative, denoted $\delta u_{i...}^{j...}$, even if it does not satisfy the Leibniz rule. Kähler does not have any dedicated symbol to play the role of our δ. He only defines $\delta u_{i...}^{j...}$ for the Levi-Civita connection (LCC), in which case it becomes equal to the co-derivative. All this, however, is easily extendable to arbitrary connection, in which case the last term in expression 3.6 is no longer the co-derivative in general. The symmetry of the roles of d and δ in the equation

$$\partial u_i^j = du_i^j + \delta u_{i...}^{j...}. \tag{3.7}$$

leads one to still refer to $\delta u_{i...}^{j...}$ as the interior derivative.

When the differential form is scalar-valued, the expression for ∂u is simplified to become

$$\partial u = dx^h \vee u_{,h} - e^h(\omega_h^l \wedge e_l u), \tag{3.8}$$

where $u_{,h}$ is the partial derivative of u with respect to x^h. The operator e_l acts on u to give $e_l u = u'$, with u' uniquely defined by $u = \omega^l \wedge u' + u$" (where neither u' nor u" contains the factor ω^l). It is worth noticing that operating with e^h is equivalent to the left multiplication "$dx^h \cdot ...$" or "$\omega^h \cdot ...$".

Formula 3.8 still is largely unscrutable. In order to make smooth contact with the exterior calculus, one can try to proceed in reverse and infer the covariant derivative from the exterior derivative, $du = dx^h \wedge d_h u$. Although there are infinite solutions for $d_h u$ in this equation for given du, only a few suggest themselves. Among these, only one extracts the information contained in the partial derivatives –or from the connection from which the covariant derivatives are made– without introducing spurious elements [5]. Proceeding in that way, one avoids Kähler's ad hoc ansatz for derivatives.

The relation of the Kähler derivative to the Laplacian takes a simple form when the affine curvature is zero,

$$\partial\partial u = d^k d_k u = \Delta u, \tag{3.9}$$

where d^k results from raising indices in d_k. A differential form u is called strict harmonic if $\partial u = 0$. It is clear that strict harmonicity implies harmonicity, but not the other way around. It goes without saying then that the solving of the equation $\partial u = 0$ is a problem of great significance ab initio.

Of particular interest is Green's formula for differential forms, which requires a couple of definitions. Following Kähler, we define the scalar product of order one of two differential forms as

$$(u, v) \equiv (\zeta u \vee v) \wedge z = (\zeta u \vee v)_0 \; z, \tag{3.10}$$

where $(...)_0$ is the 0-form part of whatever differential form constitutes the contents of the parenthesis, where ζ is the operator that reverses the order of all 1-form factors (in u in this case) and where z is the unit pseudo-scalar in the Clifford algebra of differential forms. Similarly, one defines the scalar product of order one as the $(n-1)$-form

$$(u, v)_1 \equiv e_i(dx^i \vee u, \; v), \tag{3.11}$$

which can be shown to take the alternative form

$$(u, v)_1 = e_i[(\zeta u \vee dx^i \vee v)_0 \; z] = (\zeta u \vee dx^i \vee v)_0 \; e_i z \tag{3.12}$$

Green's formula for differential forms of arbitrary dimensions then reads

$$(u, \partial v) + (v, \partial u) = d(u, v)_1. \tag{3.13}$$

In the Kähler calculus, differential forms are actually cochains (integrands) rather than antisymmetric multilinear functions of vectors (or fields thereof). Thus integrations involving formulas such as this one result from making explicit the evaluation (i.e integration) of the differential forms and application of the generalized Stokes theorem to the integration of the right hand side of Eq. 3.13. All sorts of formulas can in turn be derived from 3.13 (for instance replacing v with ∂v, etc). In order to handle that situation, one would have to give here explicit rules for Kähler and exterior differentiations of Clifford and exterior products. This transcends the scope of this paper.

The central equation of this calculus is the Kähler-Dirac (KD) equation,

$$\partial u = a \vee u, \tag{3.14}$$

of which the equation $\partial u = 0$ for strict harmonicity is but a special case. If one defines a conjugate KD equation,

$$\partial v = -\zeta a \vee v, \tag{3.15}$$

one can use Green's formula, Eq. 3.13, to prove the following conservation law:

$$d(u, v)_1 = 0. \tag{3.16}$$

The relationship between the solutions of a KD equation and its conjugate depends on the form of a. Kähler showed that, if u is a solution of the first equation with electromagnetic coupling using a complex algebra and metric with signature $(-, +, +, +)$,

$\eta\bar{u}$ is a solution of the conjugate equation, i.e. $\partial(\eta\bar{u}) = -\zeta a \vee \eta\bar{u}$ (overbar denotes complex conjugate and η changes the sign of the odd part of the differential forms).

Of particular interest is the concept of constant differentials, c, defined by

$$d_i c = 0 \qquad for\ all\ i, \tag{3.17}$$

since they generate new solutions of KD equations. If u is a solution of $\partial u = a \vee u$, we also have

$$\partial(u \vee c) = a \vee (u \vee c), \tag{3.18}$$

If the space is (pseudo-)Euclidean, it admits (pseudo-)Cartesian coordinates. Their differentials are constant differentials, and so are their polynomials with constant coefficients. Examples are $dx \vee dy$, $dx \vee dy \vee dz$, etc. Constant differentials constitute a tool to uncover algebraic structure in the set of solutions of KD equations with symmetries (Lie operators are another).

4 Interplay of the Evolutions of Geometry and of the Calculus

In the previous paper [7], we stopped at a point where we required use of Clifford algebra in order to deal with our new geometric structure, a kind of Kaluza-Klein (KK) space, $M^4 \oplus M^1$, where M^4 is spacetime and where M^1 is a 1-dimensional manifold. This KK space, to be described below, is canonically determined by the same differential invariants that define the base space of the Finsler bundle, namely ω^μ and $d\mathbf{u}$. We have written $d\mathbf{u}$ instead of $d\mathbf{e}_0$ or ω_0^i because specifying $d\mathbf{u}$ is equivalent n the Finsler bundle to specifying ω_0^i, but not in the KK space in question.

Relative to pre-Finslerian connections, the Finslerian ones have additional degrees of freedom, and so do the KK ones. It is not clear whether the respective additional degrees of freedom of one case correspond to those of the other. Explicit calculations of autoparallels, contracted second equation of structure (geometric Einstein equations) and Bianchi identities for both structures reproduce familiar equations of the physics for torsions proportional to $\mathbf{u}$ [6]. The additional terms that appear in the KK space offer tantalizing possibilities because of their potential association with radiation terms. In addition, the KK structure does not have the standard problems with the Laplacian that Finsler geometry has, since the Kähler calculus unambiguously defines the Laplacian.

The translation element of our KK space is

$$d\wp = d\mathbf{P} + \mathbf{u}d\tau = \omega^\mu \mathbf{e}_\mu + \mathbf{u}d\tau, \tag{4.1}$$

The unit vector ($\mathbf{u} \cdot \mathbf{u} = -1$) spans the fifth dimension but is not perpendicular to spacetime. $d\tau$ is clearly its dual differential form. The dot products of $\mathbf{u}$ with the elements of bases of the spacetime subspaces, $g_{\alpha\mu} = \mathbf{u} \cdot \mathbf{e}_\mu = u_\mu$, become on curves the spacetime components of the (former) 4-velocity.

The restricting condition

$$d\wp(\vee, \vee)d\wp = d\wp(\wedge, \wedge)d\wp, \tag{4.2}$$

becomes

$$d\wp(\cdot,\cdot)d\wp = d\tau\cdot d\tau - \omega^0\cdot\omega^0 + \omega^1\cdot\omega^1 + \omega^2\cdot\omega^2 + \omega^3\cdot\omega^3 = 0, \tag{4.3}$$

since the terms $d\wp(\wedge,\cdot)d\wp$ and $d\wp(\cdot,\wedge)d\wp$ in the expansion of the left hand side of 4.2 cancel each other out. We have thus caused the usual metric to become part of the Clifford algebra. In the process, the length of curves (whose square is represented by $d\tau\cdot d\tau$) becomes an expression in terms of spacetime differentials.

In such a KK space, one has the opportunity to make geometry and calculus more like each other, as we now show. The output differential form in the equations of structure of geometry are differential invariants (ω^μ, ω^λ_ν) that define the connected manifold, though, in physics, they are obtained through integration of the equations of structure. KD equations, on the other hand, do not involve in principle the differential invariants of a space. But whereas the set of equations of structure and Bianchi identities involves the exterior (covariant) derivative of the torsion but not its exterior (covariant) derivative, KD equations involve both derivatives for their respective output differential forms. One may specify the torsion through its interior and exterior derivatives, rather than through the first equation of structure. The issue then is to find an appropriate KD equation for the torsion. The input differential form must be such that the output differential be vector-valued, as torsions are. The issue of KD equations when the input is not scalar-valued is not a settled issue [6].

The statement that the affine curvature in our KK space is zero, $(d\omega^{MN} - \omega^{ML}\wedge\omega_L^{\ N})\mathbf{e}_{MN} = 0$, is rewritten as

$$\begin{aligned} &(d\alpha^{MN} - \alpha^{ML}\wedge\alpha_L^{\ N})\mathbf{e}_{MN} + \beta^{ML}\wedge\beta_L^{\ N}\mathbf{e}_{MN} + \\ &+(d\beta^{MN} - \omega^{ML}\wedge\beta_L^{\ N} - \beta^{ML}\wedge\omega_L^{\ N})\mathbf{e}_{MN} = 0, \\ &+(d\beta^{MN} - \omega^{ML}\wedge\beta_L^{\ N} - \beta^{ML}\wedge\omega_L^{\ N})\mathbf{e}_{MN} = 0, \end{aligned} \tag{4.4}$$

where α and β are the Levi-Civita and contorsion bivector-valued differential 1-forms. The first of the three high level terms on the left hand side of Eq. 4.4 is the metric curvature of the 5-dimensional manifold and, in essence, also of spacetime. The last of those three terms is the exterior covariant derivative of the contorsion. Since the torsion is vector-valued, a more specific form of the KD equation would be something along the lines of $\partial\breve{\mathbf{\Omega}} = \mathbf{a}(\vee,\vee)\breve{\mathbf{\Omega}}$, or $\partial\breve{\mathbf{\Omega}} = \mathbf{a}(\vee,\otimes)\breve{\mathbf{\Omega}}$, where the second product in each parenthesis refers to the valuedness factor of Clifford-valued or tensor-valued differential forms. The markings over characters are meant to remind us of the five dimensions. Torsions of the type $\Omega^4\mathbf{u}$ are of particular interest, and so are forms $\mathbf{a}$ of the type $\mathbf{u}\vee d\mathbf{P}$ and $\mathbf{u}\vee d\wp$. Hence the structure equation 4.4 would have to be complemented by a reformulated equation of structure such as

$$\partial\breve{\mathbf{\Omega}} = \mathbf{u}\vee d\wp(\vee,\vee)\breve{\mathbf{\Omega}}. \tag{4.5}$$

The trivector part of the right hand side drops out for $\breve{\mathbf{\Omega}}$ proportional to $\mathbf{u}$. This equation may be viewed not as an equation on the torsion, but as an equation on $d\wp$, since $\breve{\mathbf{\Omega}} = d(d\wp)$. Thus, in the system of two equations of structure 4.4-4.5, the first equation retains its geometric flavor. The second one is of the KD type, i.e. the central equation of the Kähler calculus of differential forms.

5 Concluding Remarks

In a paper titled of *The Tragedy of Grassmann, Dieudonné*speaks of *"...the horrible 'Vector Analysis', which we now see as a perversion of Grassmann's best ideas"*. He goes on to speak of genuine applications of Grassmann's ideas, "which have made *exterior algebra an indispensable tool of modern mathematics...: first of all E. Cartan's calculus of differential forms, which is now the basis of Differential geometry and of the theory of Lie groups;..."* [1]. Kähler has brought that calculus to new heights. It has a tremendous impact on the branch of differential geometry of the previous section, to which we were led in the accompanying paper. That geometry was an outgrowth of emphasizing maximum implementation of some concepts which are prominent in Klein's work (groups), in Poincaré's interpretation of the role of groups (geometric equality) and in Cartan's generalization of his program (principal fiber bundles). Equations 4.4 and 4.5 complement each other in determining together the connection in that branch of differential geometry. And there is no input differential form in this system of equations. All the differential invariants in these equations can be expressed in terms of the fundamental differential invariants, i.e. the one-index and two-indices low case omegas, both in Finsler geometry and in the KK space. Hence, we are dealing with a very sophisticated closed geometric system. One half of it has a distinctly gravitational flavor and, the other one, quantum mechanical.

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