

A different line of evolution of Geometry on manifolds endowed with pseudo-Riemannian metrics of Lorentzian signature

Jose G. Vargas and Douglas G. Torr

Abstract

In Cartan's generalization of the Erlangen program, the concepts of group and geometric equality continue to play, as in Klein, the guiding role in geometry, but now within the structure of principal fiber bundles. We develop a line of geometry that emphasizes those Klein-Cartan concepts. We are led to a complex of two structures, of respective Finsler and Kaluza-Klein types, both generated by the differential invariants ω^μ, ω_0^i . These structures in turn give rise in principle to their product structures by $O(n-1)$, represented by their left invariant forms ω_j^i .

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1 Introduction

Generalized Euclidean geometry (inhomogeneous spaces) was born with Riemann, whose "program" was based on the concept of distance. Most Finsler geometers envisage their own work from that perspective. On the other hand, F. Klein's generalization of Euclidean geometry [15] retains homogeneity but replaces the Euclidean group with some other group (affine, projective, etc). With his general theory of connections, Cartan unified the apparently incompatible programs of Riemann [16], [17] and Klein: "...*Riemannian geometry is a simple generalization of Euclidean geometry, but whereas Klein keeps from it above all the notion of geometric equality, Riemann keeps only the notion of distance. Pushed to their last consequences, . . . these viewpoints are radically divergent. The notion of distance disappears from the most general Klein geometries, and the notion of equal figures disappears from the most general Riemannian geometries...*" [9].

Consider the following characterization by Dieudonné (as reported in the introduction of a book by Gardner): *Finally, it is fitting to mention the most unexpected*

extension of Klein's ideas in differential geometry... E. Cartan was able to show...; but it is necessary to replace the group with a more complex object, called the "principal fiber space"; one can roughly represent it as a family of isomorphic groups, parametrized by the different points under consideration; the action of each of these groups attaches objects of an "infinitesimal nature" (tangent vectors, tensors...) at the same point... [13]. Notice the reference to Klein's program, not Riemann's. Notice also that, in addition to groups and geometric equality, fiber bundles are essential in Klein-Cartan geometry. In spite of all this, distance has attracted more attention than these concepts, specially among Finsler practitioners. In this paper, we develop a line of geometry which emphasizes geometric equality, (complexes of) groups and fiber bundles. It is not relevant for global differential geometry, where one is interested in connection-independent results, or in control theory, optimization and, in general, where the metric is the main concept. Rather than competing with the main line of geometry, we are dealing here with a very particular one that implements to the greatest possible extent Klein's ideas, as reinterpreted and extended by Cartan.

In developing this new structure, one touches upon issues that are intimated in Cartan's papers of the early 1920's, but which have not been addressed otherwise. Among these issues are his "Kaluza-Klein" (KK) space [4], his related bivector-valued (rather than vector-valued) energy-momentum 3-forms [4]-[5], and his implicit suggestion that one might extend the theory of connections beyond a theory of moving frames [1] (see section 1 of this very long reference).

The contents of the paper is organized as follows. In section 2, we briefly review Klein's ideas from the modern perspective that emerged from Cartan's revolution in geometry. In section 3, we reproduce the Cartan moving frame approach to the equations of structure of affine and Euclidean spaces, and also his other approach where those equations are obtained from movements of frames while points are left unchanged. In section 4, we consider the Clifton formulation in terms of bundles of the affine-Finsler geometry that underlies Cartan's work on his Finsler connection, to be found only in one of the present authors papers [19]. In section 5, we explain that teleparallel Finsler structures can be considered as product structures in ways that non-teleparallel structures cannot. This motivates consideration of a different space generated by the differential invariants that define the richer factor in the product structure.

2 Retrospective view of Klein's Program

Retrospectively, it is clear that Klein was off the mark in some important respects. For one, he believed that he was defining geometries in general, when he was only defining what nowadays are considered as elementary geometries. But he was not correct even at that. An elementary geometry is defined as a pair (G, G_0) of a group G and a normal subgroup G_0 , not just a group [11], [18]. Thus, for example, G is constituted in Euclidean geometry by the translations, rotations and their products, and G_0 is constituted by just the rotations. In addition to the elementary geometries, there are their respective generalizations, to which Dieudonné's comments of the previous section apply. The isomorphic subgroups that he mentions are copies of the subgroup G_0 . The translations are still at work, infinitesimally, i.e. as differential transformations to be integrated along lines. Needless to say that the group G gives

its name to the elementary geometry and the connections in the spaces that generalize those elementary geometries.

Finally, Klein considered Riemannian geometry, nowadays viewed as generalized geometry, as being a geometry of infinite Lie group. Commenting on this view, Cartan stated: “...the point of view from which we have just envisaged Riemannian geometry does not make evident and we can even say that it hides what, in the intuitive sense of the term, is geometric in Riemannian geometry”.

The remark just made is of great importance vis-à-vis attempts at unification of the interactions, since such attempts do not resort to the natural option of using advanced classical differential geometry. This is due to the falsely perceived gap between the finite Lie groups at the root of gauge theories and the Lie groups that are incorrectly thought to underlie classical differential geometry. The latter geometry is erroneously viewed as pertaining to a group with an infinite number of parameters. But, as Cartan explained, the groups of relevance in the generalization of the elementary geometry remain the same. With regards specifically to Riemannian geometry, he said that “it is a non-holonomic Euclidean space” [6]. More generally, he stated that “a *general space* with an Euclidean connection may be viewed as made of an infinite amount of infinitesimally small pieces of Euclidean space” [7], though in a non-integrable, i.e. non holonomic way.

3 Moving frames versus moving points and frames

We proceed to make the point that Cartan’s theory of connections, and its modern versions for that matter, is a theory of moving frames, not a theory of moving points and moving frames (we shall later propose a line of geometry which amounts to a theory of moving frames interacting with moving particles). In order to make this point most clearly, we first show the derivation of the equations of structure of affine, Euclidean and Minkowski spaces using the method of the moving frame. It will be followed by an alternative derivation of the same equations that explicitly shows that the movement of points has been left out. The first derivation is, however, far more relevant for formal developments. In particular, it will help to understand in the next section what is the elementary geometry that underlies differentiable manifolds endowed with Finsler connections.

Consider the action of the affine group $G(aff, n)$ on the $(n + n^2)$ -differentiable manifold of vector bases constituted as a frame bundle. Affine space $AFF(n)$ can be assimilated to a vector space if we take one of its points to represent the zero. A pair $(\mathbf{P}, \mathbf{e}_\mu)$ consisting of an arbitrary point and an arbitrary basis at that point can be written as

$$(3.1) \quad \mathbf{P} = \mathbf{Q} + A^\mu \mathbf{a}_\mu, \quad \mathbf{e}_\mu = A_\mu^\nu \mathbf{a}_\nu,$$

where $\det A_\mu^\nu \neq 0$, where $(\mathbf{Q}, \mathbf{a}_\mu)$ is a fixed pair and where (A^μ, A_λ^ν) constitute coordinates in the frame bundle. The second of equations 3.1 represent the action of the linear group, which we here write as $G_0(aff, n)$ to emphasize its relation to the first element $G(aff, n)$ in the pair that constitutes affine elementary geometry.

Let $\left\{ \begin{array}{c} \mathbf{P} \\ \mathbf{e}_\mu \end{array} \right\}$ and $\left\{ \begin{array}{c} \mathbf{Q} \\ \mathbf{a}_\mu \end{array} \right\}$ constitute column matrices with $1 + n$ rows each. Equations 3.1 define the group element g in

$$(3.2) \quad \left\{ \begin{array}{c} \mathbf{P} \\ \mathbf{e}_\mu \end{array} \right\} = g \left\{ \begin{array}{c} \mathbf{Q} \\ \mathbf{a}_\mu \end{array} \right\}$$

as an $(n+1) \times (n+1)$ matrix where all the elements in the first column except the first one (which is the unity) are zeroes. Thus, in the case of affine space, $G(\text{aff}, n)$ is a (n^2+n) -dimensional hypersurface in the $(n+1)^2$ -dimensional manifold of all $(n+1) \times (n+1)$ matrices. The set of all the $\left\{ \begin{array}{c} \mathbf{P} \\ \mathbf{e}_\mu \end{array} \right\}$ matrices constitutes the bundle of bases. If the associated vector space of the affine space is restricted to be a Euclidean or pseudo-Euclidean space, the corresponding restriction of the bundle of bases is the bundle of (pseudo)orthonormal bases, also called frames. $G(\text{Euc}, n)$ is an $[n + (1/2)n(n+1)]$ -dimensional hypersurface in the same $(n+1)^2$ -dimensional manifold. In every case, one readily gets

$$(3.3) \quad \left\{ \begin{array}{c} d\mathbf{P} \\ d\mathbf{e}_\mu \end{array} \right\} = dg \cdot g^{-1} \left\{ \begin{array}{c} \mathbf{P} \\ \mathbf{e}_\mu \end{array} \right\},$$

and more conveniently as

$$(3.4) \quad d\mathbf{P} = \omega^\mu \mathbf{e}_\mu, \quad d\mathbf{e}_\mu = \omega_\mu^\nu \mathbf{e}_\nu.$$

The bundle is said to be spanned by the 20 independent differential forms $(\omega^\lambda, \omega_\nu^\mu)$ in the affine case. The first column of $dg \cdot g^{-1}$ is now made of all zeroes. The dg hyperplane goes through the zero matrix, which is not contained in the group. $dg \cdot g^{-1}$ is the hyperplane through the zero matrix that is parallel to the hyperplane tangent to the hypersurface at the unit element. It is called the Lie algebra of the group.

In terms of general coordinate systems and their dual bases, the $(\omega^\lambda, \omega_\nu^\mu)$ will take very complicated forms. Any such set of differential forms (the ω^λ and ω_ν^μ having to satisfy certain conditions that we need not specify) are the left-invariant differential forms of affine space (or rather of the affine group) if and only if they satisfy

$$(3.5) \quad d\omega^\nu - \omega^\lambda \wedge \omega_\lambda^\nu = 0, \quad d\omega_\mu^\nu - \omega_\mu^\lambda \wedge \omega_\lambda^\nu = 0.$$

Equations 3.5 constitute the integrability conditions of the differential system 3.4, as can be shown by applying the Frobenius theorem for differential forms to the differential system 3.4 written in the form $d\mathbf{P} - \omega^\mu \mathbf{e}_\mu = 0$, $d\mathbf{e}_\mu - \omega_\mu^\nu \mathbf{e}_\nu = 0$, as required for application of this theorem [2]. Equations 3.5 are referred to as the equations of structure of affine space.

All the foregoing applies mutis-mutandis to the bundles of bases of the Euclidean and Poincaré groups, which are restrictions of the affine bundle by $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$ and $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}$ respectively. The ω_ν^μ 's are no longer independent but satisfy

$$(3.6) \quad \omega_{\nu\mu} + \omega_{\mu\nu} = 0.$$

We now provide an alternative derivation of Eqs. 3.5 by Cartan [1], which emphasizes that we are dealing with fixed points and moving frames. This is not obvious ab initio, given that one computes autoparallel and extremal curves from the invariants forms of the space. Consider a point and a frame. If we leave the point fixed and perform an elementary translation (given by ω_i) and a rotation of the reference frame, the coordinates of the point change so that the following equations are satisfied

$$dx + y\omega_{21} + z\omega_{31} + \omega_1 = 0,$$

$$dy + z\omega_{32} + x\omega_{12} + \omega_2 = 0,$$

$$dz + x\omega_{13} + y\omega_{23} + \omega_3 = 0,$$

with antisymmetric ω_{ij} . In compact form, we have:

$$(3.7) \quad dx^i + x^j \omega_j^i + \omega^i = 0.$$

Exterior differentiating, we obtain

$$(3.8) \quad 0 + dx^j \wedge \omega_j^i + x^j d\omega_j^i + d\omega^i = 0.$$

Substitution of dx from 3.7 in 3.8 yields

$$(-x^k \omega_k^j - \omega^j) \wedge \omega_j^i + x^k d\omega_k^i + d\omega^i = 0,$$

which is further organized as:

$$(3.9) \quad d\omega^i - \omega^j \wedge \omega_j^i + x^k (d\omega_k^i - \omega_k^j \wedge \omega_j^i) = 0.$$

Equation 3.9 has to be valid for all x . Hence, the equations of structure follow. This emphasizes that these equations are about a geometry where points remain fixed and frames move. We shall later see why the equations of “canonical curves” (autoparallels and extremals) are not counterexamples to that statement.

Of course, one might represent the motion of the point so that it is absorbed into the ω^i . Nothing is thus modified. The interesting fact, retrospectively, is that there is an alternative way where the translation of the point is not equivalent to a translation of the frame. Physically, they are not equivalent. Whereas a translation of a frame is a comparison of two parallel frames, a translation of a point is an actual motion and, in present formalisms, is represented by a boost or by integration of elementary boosts. In any case, it is important to look at descriptions where motions of particles are not assimilated to motions of frames.

4 Canonical signature and elementary Geometry for Finsler bundles

The construction of the Finsler bundle of bases and the definition of affine Finsler connections that follows is due to the differential topologist Y. H. Clifton, who is not very inclined to writing (Some of his papers were presented to journals by the late Prof. Chern). Given the importance of his ideas for the line of evolution of geometry which is of the essence of this paper, we first summarize the main part of a paper on this subject that we wrote largely from notes taken from him and to which we gave the title of “Finslerian Structures: the Cartan-Clifton Method of the Moving Frame” [19]. We take credit for formulating some of the questions and getting Clifton to work on them. The main question was the following. If Riemannian geometry was viewed by Cartan as a particular case of generalized affine geometry (and so is the case also for all the Euclidean, pseudo-Euclidean and Weyl connections), should one

not try to see the Cartan-Finsler connection as a particular case of “affine-Finsler” connection? If affirmative, what is the latter? In the second part of the section, we extract consequences vis-a-vis the elementary geometry that underlies the Finsler bundle. It will become clear that this line of development of Finsler geometry is one which singles out the Lorentzian signature.

We must first address the issue of why should one add the name of Clifton to a method that is known in the literature by the name of Cartan. To start with, Cartan did not create the method. The great user of the method (it escapes us whether he created it or not), was Darboux [12], to whom Cartan gives extensive credit [8]. Cartan brought the use of the moving frame to a higher level than Darboux, and so did Clifton relative to Cartan. To be specific, Clifton contributed as follows:

(a) The concept of the Finsler bundle as a refibration of the standard bundle (this is already implicit in Chern’s 1948 paper [10], though in a far less clear way). This bundle is not only the one where Finsler connections live, but also has implications as to what the elementary or Klein geometry of Finsler geometry is, as we shall later see.

(b) His introduction of both affine connection and equations of structure into just one and the same definition, a definition which, accompanied by a few straightforward theorems, makes the method of the moving frame rigorous in generalized geometry. He did that for affine-Finsler connections, which one easily specializes to the pre-Finsler case [19]. That would constitute a formidable achievement even if it had taken place only for the usual (pre-Finsler) affine connections. Clifton thus proved that one does not need to resort to the modern and cumbersome approach to the equations of structure and clumsy concomitant computations to make the method rigorous. In addition, there is also in the modern approach a complete absence of integrability considerations. This is clearly because vector fields, which dominate that approach, is not the natural language of the theory of integration; differential forms are.

(c) His reduction of the technique of doing Finsler geometry to a relatively simple variation of the techniques of pre-Finsler geometry. This is to be compared with the complexity of the same problems when other methods are used. An example of this is Cartan’s obtaining of his own Finsler connection, based on a string of postulated properties for the connection [3]. Clifton replaced all those postulates with the solving for the connection with the system of equations constituted by the first equation of structure with zero torsion and the statement of metric compatibility [20]. When the metric is Finslerian, one cannot avoid having one of the terms of the Finslerian torsion be different from zero. With that caveat, the solving for that system is similar to the solving for the Levi-Civita connection in Riemannian geometry. The difficulty lies just in the fact that one is not used to compute with exterior equations, much less in the sphere bundle. But this, rather than a difficulty, is an opportunity to acquire computational virtuosity.

We proceed to summarize contributions (a) and (b), avoiding the few required theorems [19]. Let $S(M)$ and $T(M)$ be the sphere and tangent bundles of a differentiable manifold M , and let $T[S(M)]$ be the tangent bundle of $S(M)$. Let π and $T\pi$ be the projections $\pi : T(M) \rightarrow M$ and $T\pi : T[S(M)] \rightarrow S(M)$. We say that two vectors at $s \in S(M)$ are equivalent if they differ by a vector tangent to the fibers of π (these go to zero under $T\pi$). Let $RT[S(M)]$ be the quotient set of $T[S(M)]$ by this equivalence relation and let $\hat{\pi} : RT[S(M)] \rightarrow T(M)$ be the projection of the induced

isomorphic map of tangent spaces. Many tangent spaces to $S(M)$ go to each tangent space to M , but we need consider only “adapted vectors” of $RT[S(M)]$. The vectors $\hat{\pi}^{-1}\mathbf{e}$ of $RT(s)$ will be called adapted if \mathbf{e} belongs to the equivalence class of s . A basis $(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ in $RT[S(M)]$ will be called adapted if \mathbf{a}_0 is adapted (notice that the choice of \mathbf{a}_0 for this role is arbitrary). Let b be a point of the bundle $B(M)$ of bases of vectors that are in $T(M)$, i.e. a basis $(\mathbf{e}_0, \dots, \mathbf{e}_{n-1})$ of $T(x)$ for some $x \in M$. Since \mathbf{e}_0 is different from zero, it belongs to one of the equivalence classes s that constitute $S(M)$. This defines a map $\tilde{\pi} : B(M) \rightarrow S(M)$. Since the basis $(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ that corresponds to the basis b of $T(\hat{\pi}^{-1})$ is adapted, the bundle $B(M)$ can also be regarded as the bundle of adapted bases over $S(M)$. The fiber of $\tilde{\pi}$ is constituted by the $(n^2 - n + 1)$ -dimensional group G of transformations that leaves the direction of \mathbf{e}_0 fixed. We call Finsler bundle of bases the principal fiber bundle $[B(M), S(M), \tilde{\pi}, G]$. If M is endowed with a Riemannian (or pseudo-Riemannian) metric, we define $B'(M)$ as the restriction of $B(M)$ to the normal or pseudo-orthonormal bases. When the metric is Riemannian, we refer to $[B'(M), S(M), \tilde{\pi}, G]$ as the Finsler bundle of frames.

Definition 4.1. *An affine-Finsler connection is a 1-form $(\omega^\mu, \omega^\lambda_\nu)$ on a $(n^2 + n)$ -dimensional manifold $B(M)$ taking values in the Lie algebra of the affine group and satisfying the conditions:*

- (1) *The $n^2 + n$ real-valued 1-forms are linearly independent.*
- (2) *The forms ω^μ are the soldering forms.*
- (3) *The forms ω^μ vanish on the fibers of $\tilde{\pi} : B(M) \rightarrow S(M)$.*
- (4) *The pullbacks of $\omega^\mu, \omega^\lambda_\nu, \omega^i_j$ into the fibers of $\tilde{\pi} : B(M) \rightarrow S(M)$ are the left invariant forms of the linear (sub)group that leaves the direction of a vector unchanged.*
- (5) *The forms $\Omega^\nu = d\omega^\nu - \omega^\lambda \wedge \omega^\nu_\lambda$, called torsion, and $\Omega^\nu_\mu = d\omega^\nu_\mu - \omega^\lambda_\mu \wedge \omega^\nu_\lambda$, called affine curvature, are quadratic exterior polynomials in the $2n - 1$ forms ω^μ, ω^i_0 :*

$$(4.1) \quad \Omega^\nu = R^\nu_{\lambda\mu} \omega^\lambda \wedge \omega^\mu + S^\nu_{\lambda i} \omega^\lambda \wedge \omega^i_0$$

$$(4.2) \quad \Omega^\nu_\pi = R^\nu_{\pi\lambda\mu} \omega^\lambda \wedge \omega^\mu + S^\nu_{\pi\lambda i} \omega^\lambda \wedge \omega^i_0 + T^\nu_{\pi ij} \omega^i_0 \wedge \omega^j_0,$$

where $R^\nu_{\lambda\mu}, R^\nu_{\pi\lambda\mu}$ and $T^\nu_{\pi ij}$ are antisymmetric in the last 2 subscripts.

Consider a differentiable manifold endowed with a standard Lorentzian pseudo-metric. The adaptation of the preceding definition to the Finsler bundle of frames consists in restricting it to pseudo-orthonormal bases. The first element in the basis may be chosen to be timelike or spacelike. In both cases, the subgroup of the linear group that leaves a direction unchanged is now replaced by the subgroup of the Lorentz group in n -dimensions that leaves a direction unchanged. If the direction is timelike, the subgroup is $O(n - 1)$. It is, however, $O(1, n - 2)$ if that direction is spacelike. But why should the “special element in the frame”, which we arbitrarily chose to be the first one, be other than timelike? The metric already chooses one of the four elements (if the frame does not contain null vectors), namely the timelike element. By choosing the direction that is instrumental in refracting over $S(M)$ the frames tangent to M to be timelike, one conforms the affine structure with the metric structure. Conversely, if one chooses the first element in the frames to be spacelike, one is arbitrarily choosing the $(n - 2)$ -dimensional spatial subspace that the rotations in the group of the fibers act upon. That arbitrariness also exists if the metric is properly Riemannian, i.e. definite. In this case, however, the arbitrariness is not removable, unlike in the Lorentzian case. Because the Lorentzian signature singles out one of the

elements in the bases (if null vectors are not used), it is the canonical signature of metric-Finsler bundles. The adapted vectors are naturally timelike in the Lorentzian metrics, and arbitrary for other metrics. Since any pre-Finslerian connection can be lifted to the Finsler bundle, this result applies in particular to the Finsler bundle of differentiable manifolds endowed with the usual metric-compatible affine connections.

It is clear that an elementary geometry of metric-Finsler connections on pseudo-Riemannian metrics of Lorentzian signature is constituted now by a triple consisting of a group, a subgroup and a subgroup, namely the Poincaré, Lorentz and rotation groups, the latter in $n - 1$ dimensions. In the case $n = 4$, the space to which this elementary geometry refers is the spacetime of special relativity, viewed from this new perspective and not as pseudo-Riemannian. Hence, the theory of Finsler frame bundles has the Lorentz-Minkowski's spacetime as the elementary geometry whose generalization are the differentiable manifolds endowed with Finslerian connections.

5 A different line of evolution of Geometry

Equality is an equivalence relation. Only teleparallel connections endow a manifold with equality of vectors at a distance, since only they have the property of path independent equality of vectors. In fact, path-independent is a misnomer which has to do with historical accident, namely that, mathematically, non-teleparallel connections were known before the teleparallel ones. Hence, maximization of Klein's ideas will include teleparallelism. It will also include Finsler frame bundles, rather than pseudo-Riemannian ones, for the reason stated in the previous section. A concomitant of Finsler bundles is that their connections are then generalizations of an elementary geometry constituted by a group G , a subgroup G' and a subgroup G'' .

Finsler bundle structures enlighten the fact that one can separate (ω^μ, ω_i^0) from the remaining ω_μ^ν . The ω_i^0 are now horizontal. Teleparallelism (TP) adds something more. Consider first the usual TP, i.e. in pre-Finsler frame bundles. When we say that these bundles are product structures, we do not mean only that they are locally topological products of the base space by the fiber space. We mean that, in addition, one can reconstruct the product structure from just the invariants (ω^μ) and from the off-the-shelf invariants ω_μ^ν of a group, the linear group in n dimensions in this case. Ditto for metric compatible pre-Finsler TP, except for the reduction of the dimensionality of the bundle by virtue of $\omega_{\nu\mu} = -\omega_{\mu\nu}$, the group now being $O(n)$. In affine-Finsler TP, we reconstruct $(\omega^\mu, \omega_\mu^\nu)$ from (ω^μ, ω_i^0) with the left invariant forms of the linear group that leaves a direction in $n - 1$ dimensions unchanged. Ditto for metric compatible metric-Finsler TP on (pseudo)-Riemannian metrics, except, again, for the reduction of the dimensionality of the bundle by virtue of $\omega_{\nu\mu} = -\omega_{\mu\nu}$ (the group is now $O(n - 1)$). Of course, the construction of a product structure in the sense indicated will require knowing what the "frames" of the structure are. In the following, we simply construct an alternative main factor in the product, i.e. one generated by the horizontal invariants of a constant section of a TP Finsler connection on a pseudo-Riemannian metric of Lorentzian signature.

Consider a space, $M^4 \oplus M^1$, where M^4 is spacetime and where M^1 is a 1-dimensional space. The translation element of this "Kaluza-Klein" (KK) space will be defined as

$$(5.1) \quad d\varphi = d\mathbf{P} + \mathbf{u}d\tau = \omega^\mu \mathbf{e}_\mu + \mathbf{u}d\tau,$$

where the vector \mathbf{u} spans the fifth dimension or, equivalently, the M^1 space with coordinate τ . The vector \mathbf{u} is a unit vector of square minus one, but not orthogonal to spacetime. We reserve the index 4 for the fifth dimension in terms of orthonormal bases. We have

$$(5.2) \quad d\mathbf{e}_4 = \omega_4^\mu \mathbf{e}_\mu.$$

where the ω_4^μ ($\mu = 0, 1, \dots, 3$) play the role of the ω_0^i in the Finsler bundle [21]. Equivalently,

$$(5.3) \quad d\mathbf{u} = \omega_4^{\prime\mu} \mathbf{e}_\mu + \omega_4^{\prime 4} \mathbf{u}.$$

This space is thus canonically determined by the differential invariants that define the sphere bundle, in the sense that we proceed to explain. The interpretation of \mathbf{u} is that, on curves of $M^4 \oplus M^1$, it becomes the tan least in a weak sense that we proceed to explain. It appears that ω_4^4 represents some extra information which is peculiar to the KK space. However, ω_4^4 is a vector, and its dual coordinate becomes proper time. It is in that sense that we say that the translation element 5.1 is defined by the set of differential invariants (ω^μ, ω_i^0) of appropriate cross sections of the Finsler bundle of a spacetime endowed with a teleparallel Finsler connection. The involvement is partially direct (ω^μ) and partially indirect (through $d\mathbf{u}$). In both scenarios, these two sets of differential invariants represent the metric and $d\mathbf{u}$.

6 Concluding remarks

The inner workings of our KK space require involvement with Clifford algebras and the unfamiliar Kähler calculus [14]. In the KK structure, this calculus allows one to specify the torsion of the space (i.e. half of the equations of structure, the other half specifying the affine curvature) as a Kähler-Dirac (KD) equation. The latter generalizes without gamma matrices the Dirac equation. The reader is referred to the accompanying paper for the foundations of the Kähler calculus [22] (available only in German) and for further development of this line of geometry.

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Authors' addresses:

Jose G. Vargas
 PST Associates, 48 Hamptonwood Way, Columbia, SC 29209, USA
 email: josegvargas@bellsouth.net

Douglas G. Torr
 PST Associates, 215 Ridgeview Rd
 Southern Pines, NC 28387, USA
 email: dougtorr@earthlink.net