Weak differential operators with periodical test functions

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Abstract

This paper refreshes the theory of weak derivatives and weak differential operators, using periodical test functions, having in mind the needs of our research group. Our point of view has as starting point the paper of J. Mawhin and M. Willem.

Section 1 reviews well-known facts in single-time theory. Section 2 extends the notions to multi-time case and formulates theorems regarding partial weak derivatives with respect to multi-periodical test functions. Section 3 refers to the weak partial differential operators in the context of multi-periodical test functions.

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1 Weak derivative for one variable functions

Let us denote by C_p^k, $k \in N^*$, the set of functions $\varphi : [a,b] \to R$ of class C^k, with the property $\varphi(a) = \varphi(b)$. The index p comes from periodic since these functions extend by periodicity to the set of real numbers R.

Definition 1.1. The function $u : [a,b] \to R$, $u \in L^1$, has *weak derivative*, if there exists $v : [a,b] \to R$, $v \in L^1$ such that

$$\int_a^b u(x)\varphi'(x)dx = -\int_a^b v(x)\varphi(x)dx, \quad \forall \varphi \in C_p^k. \tag{1.1}$$

The function v is called the *weak derivative* of the function u.

Obviosuly, if u has continuous derivative in the usual sense, then $v = u'$.

Proposition 1.1. *If there exists, the weak derivative function v has the property*

$$\int_a^b v(x)dx = 0.$$

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Proof. We take φ as a constant function.

Proposition 1.2. *If there exists, the weak derivative v is unique in L^1.*

Proof. Supposing that for a function u there exists v_1 and v_2 which verify (1.1), we have

$$\int_a^b (v_1(x) - v_2(x))\varphi(x)dx = 0, \quad \forall \varphi \in C_p^k.$$

Then, the fundamental Lemma of Variational Calculus gives $v_1 - v_2 = 0$, a.e.

Proposition 1.3. *If u_1 and u_2 have the same weak derivative, then $u_1 - u_2 = 0$, a.e.*

Proof. From (1.1), it follows

$$\int_a^b (u_1(x) - u_2(x))\varphi'(x)dx = 0, \quad \forall \varphi \in C_p^k,$$

and then, *du Bois Reymond Lemma* gives $u_1(x) - u_2(x) =$const, a.e.

We denote by $\mathcal{F}$ the set of functions $u : [a, b] \to \mathbf{R}$ which have weak derivatives.

Theorem 1.1. *$u \in \mathcal{F}$ and only if it exists $v \in L^1$ such that*

$$\int_a^b v(x)dx = 0 \tag{1.2}$$

and

$$u(x) = c + \int_a^x v(t)dt, \ a.e.; \ a \in \mathbf{R}. \tag{1.3}$$

Proof. Let $w(x) = \displaystyle\int_a^x v(t)dt$. From the condition (1.2) we have $w(a) = w(b) = 0$. Applying the formula of integration by parts (from Lebesgue integral theory) to the functions w and φ, $\varphi \in C_p^k$, and taking into account $w'(x) = v(x)$ a.e., we obtain

$$\int_a^b w(x)\varphi'(x)dx + \int_a^b v(x)\varphi(x)dx = w(b)\varphi(b) - w(a)\varphi(x) = 0.$$

Hence w has weak derivatives v.

If also u has a weak derivative v, then $u - w =$ const. a.e.

Corollary 1.1. *If u has weak derivative, its equivalence class has a representative $\bar{u}$, continuous function (even absolutely continuous), for which $\bar{u}(a) = \bar{u}(b)$.*

2 Weak derivatives for many variables functions

To simplify, let us concern about two variables only; the extension to n variables is obvious.

Let us denote $C_p^k(D)$, $D = [a, b] \times [c, d]$, the set of functions $\varphi : D \to \mathbf{R}$ of class C^k, $k \geq 1$, with the properties $\varphi(a, y) = \varphi(b, y)$, $\forall y \in [c, d]$, and $\varphi(x, c) = \varphi(x, d)$, $\forall x \in$

$[a, b]$. The index p means multi-periodic since these functions extend by periodicity on the whole plane $\mathbf{R}^2$.

Definition 2.1. The function $u : D \to \mathbf{R}$, $u \in L^1$, has *weak partial derivative* with respect to x, if there exists $v : D \to \mathbf{R}$, $v \in L^1$, such that

$$\iint_D u(x,y)\frac{\partial\varphi}{\partial x}(x,y)dxdy = -\iint_D v(x,y)\varphi(x,y)dxdy, \ \forall\varphi \in C_p^k(D). \tag{2.1}$$

The function v is called the *weak partial derivative* of the function u with respect to x.

Proposition 2.1. *If there exists, the weak derivative v has the following properties:*

$$\iint_D v(x,y)\beta(y)dxdy = 0, \quad \beta \in C^k[c,d], \ \beta(c) = \beta(d), \tag{2.2}$$

particularly

$$\iint_D v(x,y)dxdy = 0, \tag{2.3}$$

and

$$\int_a^b v(x,y)dx = 0 \quad a.e. \ on \quad [c,d]. \tag{2.4}$$

Proof. If in (2.1) we put $\varphi(x,y) = \beta(y)$, one obtains (2.2) and for $\varphi = ct$ one obtains (2.3).

Because the function v is integrable on D, the function $v(\cdot, y)$ is integrable on $[a, b]$ for almost all $y \in [c, d]$. Then the relation (2.2) becomes

$$\int_c^d V(y)\beta(y)dy = 0, \quad \forall\beta \in C_p^k,$$

where

$$V(y) = \int_a^b v(x,y)dx.$$

From (8) it follows $V(y) = 0$ a.e. and this is the relation (2.4).

Proposition 2.2. *If there exists, the weak partial derivative v is unique in L^1.*

Proposition 2.3. *If u_1 and u_2 have the same weak partial derivatives with respect to x, then there exists $\beta : [c,d] \to \mathbf{R}$ such that*

$$u_1(x,y) - u_2(x,y) = \beta(y), \ \forall y \text{ and a.e. on } [a,b].$$

The *proofs* of these two results are the same to those in §1.

We denote by $\mathcal{F}$ the set of functions $u : D \to \mathbf{R}$, $u \in L^1$, which have weak partial derivative with respect to x.

Theorem 2.1. *$u \in \mathcal{F}$ if and only if there exists $v : D \to \mathbf{R}$, $v \in L^1$, and $\beta : [c,d] \to \mathbf{R}$ such that*

$$\int_a^b v(x,y)dx = 0 \ a.e.\ y \in [c,d] \tag{2.5}$$

and

$$u(x,y) = \beta(y) + \int_a^x v(t,y)dt \text{ a.e. on } D. \tag{2.6}$$

Proof. We introduce the function

$$w(x,y) = \int_a^x v(t,y)dt.$$

Applying the formula of integration by parts we have, for each $\varphi \in C_p^k(D)$,

$$\iint_D w(x,y)\frac{\partial \varphi}{\partial x}(x,y)dxdy = \int_c^d \left(\int_a^b w(x,y)\frac{\partial \varphi}{\partial x}(x,y)dx\right)dy =$$
$$= \int_c^d (w(b,y)\varphi(b,y) - w(a,y)\varphi(a,y))dy-$$
$$-\int_c^d \left(\int_a^b v(x,y)\varphi(x,y)dx\right)dy = -\iint_D v(x,y)\varphi(x,y)dxdy,$$

the first integral after the second sign "=" being null, because from (2.5)

$$w(b,y) = 0 \text{ a.e. } y \in [c,d].$$

Hence the function w has weak partial derivative with respect to x, namely v. If the function u has the weak derivative v too, then, by Proposition 2.3.,

$$u(x,y) = w(x,y) + \beta(y) \text{ a.e. on } \ D,$$

and this is just the equality (2.6).

Corollary 2.1. *If u has weak partial derivative with respect to x, its equivalence class has a representative $\bar{u}$, absolutely continuous function, for which*

$$\bar{u}(a,y) = \bar{u}(b,y), \quad \forall y \in [c,d].$$

Analogously, there are valid the results with respect to variable y.

So we obtain

Corollary 2.2. *If the functions $u \in L^1$ has weak partial derivatives with respect to x and with respect to y, then its equivalence class has a representative $\bar{u}$, absolutely continuous function which is twice periodic, i.e. $\bar{u}(a,y) = \bar{u}(b,y)$, $\forall y \in [c,d]$ and $\bar{u}(x,c) = \bar{u}(x,d)$, $\forall x \in [a,b]$.*

3 Weak partial differential operators

The notions built up in the previous paragraphs can be extended to several variables, high order derivatives and differential operators.

Let be $T = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and $D_n = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$. Let us denote C_T^∞ the set of the function in the class C^∞ on D_n which are multiple-periodic, i.e. for any $\varphi \in C_T^\infty$ we have

$$\varphi(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) = \varphi(t_1, \ldots, t_{i-1}, a_i, t_{i+1}, \ldots, t_n),$$

$i = \overline{1, n}$ and $t_j \in [0, a_j]$, $j \neq i$.

For each multiple-index $l = (l_1, l_2, \ldots, l_n)$ denote

$$D^l \varphi = \frac{\partial^{|l|} \varphi}{\partial x_1^{l_1} \partial x_2^{l_2} \ldots \partial x_n^{l_n}},$$

where $|l| = l_1 + l_2 + \ldots + l_n$.

Definition 3.1. The function $u : D_n \to \mathbf{R}$, $u\imath L^1$, has *weak derivative* D^l, if there exists a function $v : D_n \to \mathbf{R}$, $v\imath L^1$, such that

$$\int_{D_n} u(x) D^l \varphi(x) dx = (-1)^{|l|} \int_{D_n} v(x) \varphi(x) dx, \quad \forall \varphi \in C_T^\infty,$$

where $x = (x_1, \ldots, x_n)$ and $dx = dx_1 \ldots dx_n$.

The results of the previous paragraphs hold true, but the theorems about recapturing the function u from its weak derivatives are more and more complicated.

One can introduce the notion of *weak linear differential operator* of any order: Let

$$L(\varphi) = a_0(x)\varphi + \sum_{j=1}^{k} \sum_{|l|=j} a_l(x) D^l \varphi = \sum_{|l| \leq k} a_l(x) D^l \varphi$$

be a *linear differential operator* and

$$L^*(\varphi) = \sum_{|l| \leq k} (-1)^{|l|} D^l (a_l(x)\varphi)$$

its *formal-adjoint operator*. For $u \in L^1(D_n)$ define the *weak differential operator* $\mathcal{L}(u)$ by equality

$$\int_{D_n} u(x) L^*(\varphi)(x) dx = \int_{D_n} \mathcal{L}(u)(x) \varphi(x) dx, \quad \forall \varphi \in C_T^\infty. \tag{3.1}$$

We can easily prove

Proposition 3.1. *If it exists, the weak operator has the following properties:*

1° $\displaystyle\int_{D_n} \mathcal{L}(u)(x) \beta(x) dx = 0, \quad \forall \beta \in Ker\ L^*$;

2° *if $Ker\ L^*$ contains the constant functions (i.e. $a_0(x) = 0$, $\forall x$), then*

$$\int_{D_n} \mathcal{L}(u)(x)dx = 0.$$

The recapturing of the function u from $v = \mathcal{L}(u)$ depends essentially upon the operator L. For instance, if $\mathcal{L} = \Delta_2$, i.e.,

$$\mathcal{L}(\varphi) = \Delta_2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \mathcal{L}^*(\varphi),$$

the equality (3.1) becomes

$$\iint_{D_2} u(x,y)\Delta\varphi(x,y)dxdy = \iint_{D_2} v(x,y)\varphi(x,y)dxdy, \quad \forall\varphi \in C_T^\infty.$$

Then

$$u(x,y) = \frac{1}{2\pi}\iint_{D_2} v(s,t)\ln\sqrt{(x-s)^2+(y-t)^2}dsdt + \beta(x,y),$$

where β is a harmonic functions.

Remark 1. The existence of the function $v = \mathcal{L}(u)$ does not imply the existence of each individual weak derivative which appears in the operator L.

Remark 2. The general form of the integration by parts formula for the Lebesgue integral is the following: if $f, g \in L^1$ and

$$F(x) = \lambda + \int_a^x f(t)dt, \quad G(x) = \mu + \int_a^x g(x)dt, \quad \lambda,\mu \in \mathbf{R},$$

then

$$\int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a).$$

Remark 3. Let us denote by $W_T^{1,2}$ the Sobolev space of the functions $u \in L^2[T_0, R^n]$, which have the weak derivative $\dfrac{\partial u}{\partial t} \in L^2[T_0, R^n]$, $T_0 = [0, T^1] \times ... \times [0, T^p] \subset R^p$. The weak derivatives are defined using the space C_T^∞ of all indefinitely differentiable multiple T-periodic function from R^p into R^n. We consider H_T^1 the Hilbert space associated to $W_T^{1,2}$. The definition of the weak divergence of the weak Jacobian matrix $\dfrac{\partial u}{\partial t}$ (i.e., the definition of the weak Laplacian of the function u) is

$$\int_{T_0} \delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial v^j}{\partial t^\beta}dt^1 \wedge ... \wedge dt^p = -\int_{T_0} \delta^{\alpha\beta}\delta_{ij}\frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta}v^j dt^1 \wedge ... \wedge dt^p,$$

for all $v \in H_T^1$ and hence for all $v \in C_T^\infty$.

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