The pseudo-Riemannian isometries associated to Sasaki lift

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Abstract

In this paper, using a pseudo-Riemannian metric on the base M, of tangent bundle (TM, π, M) and the pseudo-Riemannian Sasaki lift, the results from propositions on the paragraphs 2 and 3 are obtained.

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1 Introduction

Let consider the tangent bundle $\xi = (TM, \pi, M)$, where $M_n = (M, [A]; \Re^n)$ is a C^∞-differential manifold, paracompact, connex. Moreover, in the general theory knowing of the tangent bundle, an important role has the case that a Riemannian metric g, on M, is "lifted" to the Sasaki metric G, on TM. For this case are working: Kentaro-Yano, Sasaki, Ianus Stere, R. Miron, etc. A generalization for the cotangent bundle is given by P. Stavre.

Now, let be a pseudo-Riemannian structure (M, g) and $\underset{(n)}{V} = (M, g, \nabla)$ the pseudo-Riemannian space corresponding in these conditions, the Riemannian metrics $\{g\}$ exists on M, globally, but generally, the pseudo-Riemannian metrics, $\{g\}$ do not globally exist. An example is given by Steenrod for $n = 2$, $\underset{(2)}{V}$-compact. In this case, only the torus and the Klein bottle admit pseudo-Riemannian metrics. For instance, the existence of an everywhere non-zero vector field is a condition for the existence of pseudo-Riemannian Lorentz metrics, which are essential in the generalized relativity theory. As well, Steenrod showed that this is equivalent with the vanishing of a topological invariant (the Euler-Poincaré characteristic).

We will extend the Sasaki lift in the case of pseudo-Riemannian structure (M, g). The study is difficult because now we have: vectors with zero length; curves (differentiable or differentiable on parts) with zero length (or minimal); the curves with zero length can be or cannot be geodesics; the distance between two points can be zero:

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$x, y \in M, x \neq y, d(x, y) = 0$. So, the metrization of the topological space (M, τ) is not possible like in the Riemannian case and so, the Hopf-Rinow theory does not work.

We will define in $x \in M$ the pseudo-Riemannian indicator, as follows: $\varepsilon_x = -1$ if $g(X, X) < 0$; $\varepsilon_x = 0$ if $g(X, X) = 0$; $\varepsilon_x = 1$ if $g(X, X) > 0$.
We will note: $C_x^{(-)} = \{X_x \in T_xM \backslash \varepsilon_X = -1\}$ (the set of spatial vectors); $C_x^{(0)} = \{X_x \in T_xM \backslash \varepsilon_X = 0\}$ (the set of isotropic vectors);
$C_x^{(+)} = \{X_x \in T_xM \backslash \varepsilon_X = 1\}$ (the set of temporal vectors) and we will define the length of vector X, by the real number, $|X| \geq 0$, where $|X|^2 = \varepsilon_x g(X, X)$.

Evidently, $X_x \in C_x^{(0)} \Leftrightarrow |X| = 0$. The field X, at x, is unitary if $g(X, X) = \varepsilon_x$. The sets $C_x^{(-)}$, $C_x^{(+)}$ are open and $C_x^{(0)}$ is closed. We can write the partitions: $C_x^{(-)} = C_x^{1(-)} \cup C_x^{2(-)}$; $C_x^{1(-)} \cap C_x^{2(-)} = \emptyset$; $C_x^{(0)} = C_x^{1(0)} \cup C_x^{2(0)}$; $C_x^{1(0)} \cap C_x^{2(0)} = \emptyset$; $C_x^{(+)} = C_x^{1(+)} \cup C_x^{2(+)}$; $C_x^{1(+)} \cap C_x^{2(+)} = \emptyset$, where $C_x^{1(-)}$, $C_x^{2(-)}$ are the conexes open sets and analogously, $C_x^{1(+)}$, $C_x^{2(+)}$. If $X_x \in C_x^{1(+)}$ then $-X_x \in C_x^{2(+)}$... $C_x^{1(0)}$, $C_x^{2(0)}$ will be called the isotropic hipercones, with the vertex at x, they are opposite. We consider $C : x \to C_x$.

2 Considerations on $\xi = (TM, \pi, M)$

On the space TM, considered as a paracompact $2n$-dimensional differentiable manifold, there exists a pseudo-Riemannian metric, which has the local form:

$$G = g_{ik}(x) dx^i \otimes dx^k + g_{rs} \delta y^r \otimes \delta y^s, \tag{2.1}$$

where, $\delta y = \nabla y$ (the covariant differential, relative to the pseudo-Levi-Civita connection defined by the pseudo-Riemannian g, for $y \in T_xM$, $x \in M$) ($u = (x^i, y^i) \in \pi^{-1}(U)$, where $x \in U$ has the local coordinates $x^i = x^i(x)$, $\pi^{-1}(U)$ is the geometric zone of a local chart at u, on TM). So, we have a Whitney decomposition, $T_uTM = H_uTM \bigoplus V_uTM$, $u \in TM$ and the distributions, $H : u \to H_uTM$ (horizontal) and $V : u \to V_uTM$ (vertical). The metric G is the generalized Sasaki metric for the pseudo-Riemannian case.

We have globally and locally, for G,

$$G = hG + vG, \tag{2.2}$$

$$hG = g_{ik}(x) dx^i \otimes dx^k, \tag{2.3}$$

$$vG = g_{rs}(dy^r + N_k^r(x, y) dx^k)(dy^s + N_i^s(x, y) dx^i), \tag{2.4}$$

with,

$$N_k^r(x, y) = y^l \gamma_{kl}^r(x), \tag{2.5}$$

the coefficients of a nonlinear connection in ξ, defined by g from the Levi-Civita coefficients $\{\gamma\}$, of the Levi-Civita connection ∇. Evidently, N is 1-homogeneous relative to the vertical fibres, that is relative to y.

G is the Sasaki lift for the Riemannian g. It seems that there are no differences. But, now, we cannot achieve $g(X, X) = \varepsilon_X |X|^2$, $\varepsilon_X \in \{-1, 0, 1\}$.

Definition 2.1. Let *note* $V_n = (TM, g, \nabla)$ *the pseudo-Riemannian space, base,* $V_{2n} = (TM, G, D)$ *the tangent pseudo-Riemannian space. The submanifold of* V_{2n} *defined by the restriction,* $g_x(y, y) = \varepsilon$*, will be called the tangent bundle* $(\overline{T}M, \overline{\pi}, M)$*, on M, of the pseudo-spheres.*

A locally map, in $u\epsilon\overline{T}M$ will be with the geometric zone $\overline{\pi}^{-1}(U)$; $u \in \overline{\pi}^{-1}(U)$; $u \in \pi^{-1}(U)$; $\overline{\pi}(u) = x \in U$, $\pi(u) = x \in U$. Result:

Proposition 2.1. The *metric* $\overline{G}$*, induced by G, on* $\overline{T}M$ *is 0-homogeneous, though G is not 0-homogeneous.*

So, let be $V_{2n-1} = (\overline{T}M, \overline{G}, \overline{D})$, the pseudo-Riemannian subspace (or the Riemannian subspace if $\varepsilon = 1$).

Proposition 2.2. The *horizontal distribution* $\overline{H}$*, on* $\overline{T}M$*, is the restriction of H in the points* $u \in \overline{\pi}^{-1}(U)$.

From 2.2 - 2.5 result $(vG) \circ h_t \neq vG$, that is G is not 0-homogeneous on $\widetilde{T}M = TM \setminus \{0\}$ (h_t=homothety, $t \in \Re^+$).

Proposition 2.3. a*) On the connected components of sets* $C^{(+)}$*,* $C^{(-)}$*, the metric 2.1 is not 0-homogeneous. b) Its restriction, on* $C^{(0)}$ *is 0-homogeneous. c) Its restriction, on* $\overline{T}M$ *is 0-homogeneous.*

A generalization of Miron's results ([4]) is obtained in this way.

For the tangent bundle, having fixed N, for each $X \in \mathcal{X}(M)$, exists $X^h \in H, X^v \in V$, uniquely. Let be the almost complex structure, natural, F, defined by,

$$F(X^h) = -X^v\,;\; F(X^v) = X^h\; (F^2 = -I), \tag{2.6}$$

which is $G(FX, FY) = G(X, Y)$, $\forall X, Y \in \widetilde{T}\,M$, that is, $(\widetilde{T}M, G, F)$ is an almost complex structure, with the 2-form associated, $\theta(X, Y) = G(FX, Y)$. In this case, result:

Proposition 2.4. a*) For 2.1, 2.5, 2.6 we have:* $d\theta = 0$*, that is, the structure is an almost kählerian structure. b) The structure* $(\widetilde{T}\,M, G, F)$ *is kählerian if and only if* V_n *is a linear Lorentz space, locally (pseudo-euclidian) that is, if and only if the first fundamental form can be write, locally,*

$$\psi = \sum_{i=1}^{n} \varepsilon_i (dx^i)^2\,,\; \varepsilon_i \in \{-1, 1\}$$

The homogenization of G must be making only on the connected components $C^{(+)}$, $C^{(-)}$. Let be $\varepsilon g(y, y) = |y|^2$ ($\varepsilon \in \{-1, 1\}$); $\overline{H} = \sqrt{\varepsilon H}$.

Proposition 2.5. The *0-homogeneous metric G is given by:*

$$\widetilde{G} = g_{ik} dx^i \otimes dx^k + \frac{r^2}{\varepsilon H} g_{rs} \nabla y^r \otimes \nabla y^s; r = ct > 0$$

Proposition 2.6. The *0-homogeneous metric* $\widetilde{G}$ *preserve the signature.*

On the connected components $C^{(+)}$, $C^{(-)}$ is defined an almost complex structure, $\widetilde{F}$, associated to $\widetilde{G}$, locally, from: $\widetilde{F}(\delta_k) = -\frac{\overline{H}}{r}\partial_k$; $\widetilde{F}(\partial_k) = \frac{r}{\overline{H}}\delta_k$, where, $(\delta_k = \frac{\partial}{\partial x^k} - y^s \gamma^i_{ks}\partial_i; \partial_k = \frac{\partial}{\partial y^k})$ in $u \in \pi^{-1}(U)$, who is given, from $\varepsilon = 1$ (the Riemannian case), by R. Miron. Result:

Proposition 2.7. We *have, a)* $\widetilde{\theta}(X,Y) = \widetilde{G}(FX,Y)$; $d\widetilde{\theta} = rd(\frac{1}{\overline{H}}) \wedge \theta$
b) $\widetilde{G}(\widetilde{F}X, \widetilde{F}Y) = \widetilde{G}(X,Y)$, *that is* $(\widetilde{G}, \widetilde{F})$ *is an almost hermitian structure.*

For, $V_n = (M, g, \nabla)$ a Riemannian space, $\varepsilon = 1$ and (a) (b) are result from [4].

Proposition 2.8. If $N_{\widetilde{F}} = 0$, *($N_{\widetilde{F}}$ is the Nijenhuis tensor) that is $\widetilde{F}$, is complex, then we have, equivalently:*

$$r^i_{jkl}(x) = \frac{\varepsilon}{r^2}(g_{jk}\delta^i_l - g_{jl}\delta^i_k),\ \varepsilon \in \{-1, 1\}, \tag{2.7}$$

where $r(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, $X, Y, Z \in \mathcal{X}(M)$ *is the curvature tensor of the pseudo-Riemannian space, base,* $V_n = (M, g, \nabla)$.

Proposition 2.9. If *H is an integrable distribution, then we have 2.7, that is on the connected components, the curvature is* $R = \frac{\varepsilon}{r^2}$, $\varepsilon \in \{-1, 1\}$, *constant but with contrary sign.*

Proposition 2.10. If $(\widetilde{G}, \widetilde{F})$ *is a hermitian structure, then the first fundamental form ψ, of V_n, will be bring in to the Riemann form:*
$\psi = \frac{1}{A}\sum_{i=1}^{n} \varepsilon_i (dx^i)^2$; $\varepsilon_i \in \{-1, 1\}$, *where,* $A = \left(1 + \frac{\varepsilon}{4r^2}\sum_{i=1}^{n}\varepsilon_i(x^i)^2\right)^2$

In the case (T^*M) these results will be much difficult to obtain. We will study the generalization of these results in the Lagrange theory in to another paper (the pseudo-Lagrange case).

3 The pseudo-Riemannian associated isometries

We will start with the observation that, for $n = 2$, M_2-compact is not possible to talk about the pseudo-Riemannian isometries or the pseudo-Riemannian infinitesimal isometries, only if M_2 is the torus or the Klein bottle.

In the relativistic mechanic, a vector field $\xi \in \mathcal{X}(M)$, with the null divergente, $div\xi = 0$ has an important roll (will be called incompressible). As is known any ξ-Killing has this property.

Let be $\underset{(n)}{V} = (M, g, \nabla)$ a pseudo-Riemannian space and $I(M, g)$ the group of all isometries of (M, g). Like in the Riemannian case, $I(M, g)$ is a Lie group who act differential on (M, g), it is a subgroup of the diffeomorphisms group on M. Like in the Riemannian case, because, $\varphi \in I(M, g)$ preserve g, hence preserve ∇ (the Levi-Civita connection) and also preserve the geodesics of ∇, the elementary volume, etc. When g is Riemannian, φ also preserves the distances (the property is reciprocal). But, in the pseudo-Riemannian case for g, we cannot talk about such property. Tis is another difference between the g-Riemannian and the g-pseudo-Riemannian cases.

Let be ξ and $\underset{(\xi)}{\omega}$, dual 1-form, given by $g(\xi, X) = \underset{(\xi)}{\omega}(X); \forall X$.

We have, $div\xi = \delta \underset{(\xi)}{\omega}$ (the codifferential of $\underset{(\xi)}{\omega}$).

Lemma 3.1. If $\xi_x \in C_x^{(0)}$, *then, locally, the equation,* $\underset{(\xi)_x}{\omega} = 0$, *admit a set of (n-1)-uples* $\{(\underset{(1)}{\xi}, ..., \underset{(n-1)}{\xi})\}$, *where* $\underset{(1)}{\xi}, ..., \underset{(n-1)}{\xi}$ *are in x, linear independent and the directions given by* $\underset{(a)}{\xi}$, $(a = \overline{1, n-1})$, *through x, cannot be g-conjugate, mutually. We have, locally,* $g_{ik}\xi^i\xi^k = 0; g_{ik}\xi^i \underset{(a)}{\xi^k} = 0 (a = \overline{1, n-1})$, *and* ξ *is a linear combination of* $(\underset{(a)}{\xi})$ $(a = \overline{1, n-1})$.

Lemma 3.2. Let *be a frame, in x,*$\{\underset{(a)}{V}\}$ $(a = \overline{1, n-1})$ *orthogonal,* $g(\underset{(a)}{V}, \underset{(b)}{V}) = 0; \forall a \neq b; a, b = \overline{1, n}$. *Then we have,* $\underset{(a)_x}{V} \notin C_x^{(0)}$, $\forall a \neq b$; *a,b*$=\overline{1, n}$.

That is neither of vectors from the orthogonal frame cannot be with null length. So, it can be normalized, because $\left|\underset{(a)}{V}\right| = \sqrt{\varepsilon_a g(\underset{(a)}{V}, \underset{(a)}{V})} \neq 0$. Let be this $(\underset{(1)}{U}, ..., \underset{(a)}{V})$. We can write:

$$g(\underset{(a)}{U}, \underset{(b)}{U}) = 0; \forall a \neq b; a, b = \overline{1, n};\ g(\underset{(a)}{U}, \underset{(a)}{U}) = \varepsilon_a; a = \overline{1, n}, \varepsilon_a \in \{-1, 1\}$$

$$div\xi = \sum_{a=1}^{n} \varepsilon_a g(\nabla_{\underset{(a)}{U}} \xi, \underset{(a)}{U}) = \delta \underset{(\xi)}{\omega} = \sum_{a=1}^{n} \varepsilon_a (\nabla_{\underset{(a)}{U}} \underset{(\xi)}{\omega})(\underset{(a)}{U})$$

Remark. Sometimes, in the geometry, the codifferential is with (-).

In the case when g is Riemannian, we have $\varepsilon_a = 1$ (a=$\overline{1, n}$) and we will obtain the formula for $div\xi$ in the orthonormal frame.

We consider a Killing vector field ξ defined with by means of infinitesimal isometries. Let be $V_{(n)}$ a pseudo-Riemannian space and let $\xi \in \mathcal{X}(M)$. If the local 1-parametric group of diffeomorphisms, generated by ξ is locally given by isometries, then ξ is called Killing vector field. The local isometries will be called the infinitesimal isometries or infinitesimal motions.

Equivalently, like in the Riemannian case, we obtain the Killing equations:

$$(\nabla_X \underset{(\xi)}{\omega})(Y) + (\nabla_Y \underset{(\xi)}{\omega})(X) = 0, \tag{3.-6}$$

In the definition of ξ being Killing we start from 3.-6, to clarify the infinitesimal character of the definition. Using the Lie derivative, we have, like in the Riemannian case, $L_\xi g = 0$.

Formally, there are no differences relative to the Riemannian case, while taking into account the previous remarks. Moreover, we must take into account the case

$\xi : x \to \xi_x \in C_x^{(0)}$ and ξ is Killing, particulary, when $\nabla\xi = 0$ and its orbits are geodesics (the case of infinitesimal translations).

From all these we can observe the difficulty of studying pseudo-Riemannian isometries, infinitesimal isometries, on $V_{2n} = (TM, G, D)$, where G is the pseudo-Riemannian Sasaki lift.

It is normal not to study the infinitesimal isometries $\widetilde{\varphi} \in I(TM, G)$ using the general model, but using the correspondence with the infinitesimal isometries, $\varphi \in I(M, g)$. The study will be similar with the Riemannian Sasaki case, but now we shall be using the adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$. This leads to specific results, like in §2, because: $\xi_x \in C_x^{(-)}$ or $\xi_x \in C_x^{(0)}$ or $\xi_x \in C_x^{(+)}$.

Consider the infinitesimal transformation on base,

$$\xi_x f = \frac{d}{dt} f(\varphi_t(x))|_{t=0}; x \in M, f \in \mathcal{F}(M), \tag{3.-5}$$

$\{\varphi_t(x)\}$ is an 1-parametric group generate from ξ, which is generally local ($t \in (-\varepsilon, \varepsilon) = I, \varepsilon > 0$). It is global if ξ is complete and conversely. If ξ is Killing, then the local group of diffeomorphisms consists of local isometries $\varphi \in I(M, g)$.

Let be the infinitesimal transformation on TM,

$$\overline{\xi_u f} = \frac{d}{dt} \overline{f}(\overline{\varphi_t}(u))|_{t=0}; \pi(u) = x; u \in \pi^{-1}(U); x \in U, \tag{3.-4}$$

If $\overline{\xi}$ is Killing then the diffeomorphisms of 1-parametric group generate from $\overline{\xi}$, are isometries (local) $\overline{\varphi} \in I(TM, G)$.

Generally, the orbit of x is an immersion $\gamma_x : I \to M$ given by $\gamma_x(t) = \varphi_t(x)$ and $\xi(\gamma_x(t)) = \dot{\gamma_x}(t)$ (γ_x is an integral curve of ξ, through x). Their family $\{\gamma_x(t)\}$, $x \in M$ ($x \in U$), $t \in I$, is a congruence of curves on M (if ξ is complete) or locally, through each $x \in U$ passes one curve of the family, in opposite case. The similar happens for $\overline{\xi}$.

Problem 1 If $\xi \notin C^{(0)}$, then we consider the infinitesimal isometries (local), $\overline{\varphi} \in I(TM, G)$ natural prolongations of infinitesimal isometries (local), $\varphi \in I(M, g)$?

Problem 2 If these prolongations are infinitesimal translations (infinitesimal motions, who are translations) local, on the base, what infinitesimal motions (local) correspond on TM?

Problem 3 In both cases, do they induce induces on V_{2n-1} infinitesimal motions?

Problem 4 In this natural prolongation, if the orbits of ξ are geodesics (local) of V_n, then are the orbits of $\overline{\xi}$ geodesics of V_{2n}?

Problem 5 Considering the unique decomposition $\overline{\xi} = h\overline{\xi} + v\overline{\xi}$ (with respect to the nonlinear connection N), when do $h\overline{\xi}$ and $v\overline{\xi}$ locally generate the infinitesimal isometries?

Because $\xi \notin C^{(0)}$ we can write, in $x \in U$, $\varepsilon_\xi \in \{-1, 1\}$ and $\xi \in C^{(-)}$ or $\xi \in C^{(+)}$,

$$|\ \xi\ |^2 = \varepsilon_\xi g_{ik} \xi^i \xi^k, \tag{3.-3}$$

Let be $\overline{\xi}$ defined by,

$$\overline{\xi} = h\overline{\xi} + v\overline{\xi}, \tag{3.-2}$$

where, in $u \in \pi^{-1}(U)$, $\pi(u) = x \in U$,

$$h\overline{\xi} \overset{\text{def}}{=} \xi^h = \xi^i \frac{\delta}{\delta x^i}; v\overline{\xi} \overset{\text{def}}{=} (y^s \nabla_s \xi^r)\frac{\partial}{\partial y^r}, \tag{3.-1}$$

if,

$$\xi = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{3.0}$$

Let note $G(u^\alpha, u^\beta) = G_{\alpha\beta}(x, y)$, where $u^\alpha = (u^i = x^i, u^{n+i} = y^i)$.

Theorem 3.1. Let *be $V_n = (M, g, \nabla)$-pseudo-Riemannian and $V_{2n} = (TM, G, D)$, with the pseudo-Riemannian Sasaki lift G. The necessary and sufficient condition for to exists infinitesimal motions (that is local isometries, infinitesimal) $\varphi \in I(M, g)$, $\widetilde{\varphi} \in I(TM, G)$, who correspond one the other, is to exist a system of local coordinates (x,y), on TM, such as $G_{n+i\ n+k}(x, y)$ is not depend form a local coordinate, x^r (r, fixed). That is,*

$$\frac{\partial G_{n+in+k}}{\partial x^r} = 0; \forall i, k = \overline{1, n}(r, fixed), \tag{3.1}$$

Theorem 3.2. In *the conditions of theorem 1, the equations $L_\xi g = 0$ admit one solution ξ, who is Killing and for who the diffeomorphisms of 1-parametric group 3.-5, given by ξ, are the infinitesimal motions. Let be ξ 3.0. Then $\xi_x \notin C_x^{(0)}$ $(x \in U)$. Because ξ is Killing, result that $\overline{\xi}$ 3.-2, 3.-1 is Killing, for the space V_{2n} and so $L_{\overline{\xi}}G = 0$, and reciprocally. So, the 1-parametric group, local, generated by $\overline{\xi}$ is given by local isometries. We have the reciprocal.*

In the local coordinates for 3.1, we have:

$$|\xi|^2 = \varepsilon_\xi g_{rr}; g_{ik} = g_{ik}(x^1, \ldots, x^{r-1}, x^{r+1}, \ldots, x^n), i, k = \overline{1, n}$$

Evidently, $\varepsilon_\xi \in \{-1, 1\}$, if $g_{rr} < 0$ or $g_{rr} > 0$. We cannot have $g_{rr} = 0$ because g is nondegenerate. $|\xi| \neq 0$, so the infinitesimal motion is not minimal.

The infinitesimal motion is a translation, $|\xi| = ct$ if and only if $g_{rr} = ct \neq 0$. From that result $G_{n+r\,n+r} = ct$ for the first fundamental form of V_n, such as, of V_{2n}. The linear element will be: $ds^2 = \varepsilon g_{ik} dx^i dx^k + \varepsilon_r g_{rr}(dx^r)^2; g_{ii} \neq g_{rr}(i = \overline{1, n})$ (without summation from r) with $\frac{\partial g_{jl}}{\partial x^r} = 0$; $j, l = \overline{1, n}$.

Theorem 3.3. If *exist a local chart in u, on TM ($u \in \pi^{-1}(U)$, $\pi(u) = x \in U$), thus we have 3.1, then exist ξ, $\overline{\xi}$ 3.-2 3.-1, thus we have the relations:*

$$D^2_{XY}\overline{\xi} = R(X, \xi)Y; \forall X, Y \in \mathcal{X}(TM), \tag{3.2}$$

$$L_{\overline{\xi}}D = 0, \tag{3.3}$$

($\overline{\xi}$ is an affine collineation, infinitesimal of space V_{2n}) and we have the relations for ξ, analogously.

Proof. The conditions are equivalent with the existence of a Killing vector field ξ on V_n and hence, with the existence of a Killing vector field $\overline{\xi}$, on V_{2n}, who has the form 3.-2 3.-1. So we have 3.2, 3.3 and the similar relations, $\nabla^2_{XY}\xi = r(X,\xi)Y$; $\forall X, Y \in \mathcal{X}(M)$, $L_\xi \nabla = 0$. □

If $\xi_x \notin C^{(0)}_x$ then, like in the Sasaki-Riemannian case, we can show that, taking into account the transformations $(x, y) \to (\overline{x}, \overline{y})$, for the relations 3.-5, 3.-3 are obtain the infinitesimal transformations:

$$\overline{x}^i = x^i + \xi^i(x)\delta t, \tag{3.4}$$

$$\overline{x}^i = x^i + \xi^i(x)\delta t, \overline{y}^i = y^i + (y^s \nabla_s \xi^i - N^i_r \xi^r)\delta t, \tag{3.5}$$

Starting from here we can prove that if 3.4 are the isometries (local) $\varphi \in I(M, g)$, then 3.5 are the isometries (local) $\widetilde{\varphi} \in I(TM, G)$. So, if $\xi \notin C^{(0)}$ is Killing, then $\overline{\xi}$ 3.-2 3.-1 is Killing and reciprocally. Other aspects relative to the previously discussed problems will be provided in a forecoming paper.

Proposition 3.1. If $V_n = (M, g, \nabla)$*-pseudo-Riemannian has* ξ *parallel with* $\nabla\xi = 0$, *then* $\overline{\xi} = \xi^h$ *is Killing.*

Moreover, the orbits of ξ are the geodesics of space V_n and the orbits of $\overline{\xi}$ are the horizontal lifts of the orbits of ξ, and these are the geodesics of V_{2n}. Their restrictions to V_{2n-1} are the geodesics of V_{2n-1}.
The general problem of geodesics on V_{2n}-pseudo-Riemannian, tied to the geodesics on V_n, is solved. We show that, the equations of geodesics on V_{2n} can be written locally using ∇, only in terms of V_n. The problem of general translations for V_n, V_{2n} is subject of further research.

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