The pseudo-Riemannian isometries associated to Sasaki lift

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Abstract

In this paper, using a pseudo-riemannian metric on the base M, of tangent bundle $$(TM, \pi, M)$$ and the pseudo-riemannian Sasaki lift, the results from propositions on the paragraphs 2 and 3 are obtained.

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1 Introduction

Let consider the tangent bundle $$(\xi = (TM, \pi, M))$$, where $$M = (M, [A]; \mathbb{R}^n)$$ is a $$C^\infty$$-differential manifold, paracompact, connex. Moreover, in the general theory knowing of the tangent bundle, an important role has the case that a riemannian metric $$g$$, on $$M$$, is "lifted" to the Sasaki metric $$G$$, on $$TM$$. For this case are working: Kentaro-Yano, Sasaki, Ianus Stere, R. Miron... A generalization for the cotangent bundle is given by P. Stavre.

Now, let be a pseudo-riemannian structure $$(M, g)$$ and $$V = (M, g, \nabla)$$ the pseudo-riemannian space corresponding in these conditions, the riemannian metrics $$\{g\}$$ exists on $$M$$, globally, but generally, the pseudo-riemannian metrics, $$\{g\}$$ do not exists, globally. An example is given by Steenrod for $$n = 2$$, $$V$$-compact. In this case, only the torus and the Klein bottle admit the pseudo-riemannian metrics. For instance, the existence of a vectorial field non-null everywhere is a condition for to exists a pseudo-riemannian metrics, Lorentz, which are necessarily in the generalized theory of relativity. Too Steenrod show that this is equivalent with vanishing of a topological invariant (the characteristic Euler-Poincaré).

We will extend the Sasaki lift in the case of pseudo-riemannian structure $$(M, g)$$. The study is difficult because now we have: vectors with null length; curves (differentiable or differentiable on parts) with null length (or minimal); the curves with null length can be or cannot be geodesics; the distance between two points can be zero:
we will note: if \( g(X, X) < 0 \); \( \varepsilon_x = 0 \) if \( g(X, X) = 0 \); \( \varepsilon_x = 1 \) if \( g(X, X) > 0 \).

We will note: \( C_{-}^{(0)} = \{ X_x \in T_x M \mid \varepsilon_x = -1 \} \) (the set of spatial vectors); \( C_{-}^{(0)} = \{ X_x \in T_x M \mid \varepsilon_x = 0 \} \) (the set of isotropic vectors);

\( C_{+}^{(0)} = \{ X_x \in T_x M \mid \varepsilon_x = 1 \} \) (the set of temporal vectors) and we will define the length of vector \( X \), by the real number, \( |X| \geq 0 \), where \( |X|^2 = \varepsilon_x g(X, X) \).

Evidently, \( X_x \in C_{-}^{(0)} \iff |X| = 0 \). \( \varepsilon_x \), in \( x \), is unitary if \( g(X, X) = \varepsilon_x \). The sets \( C_{-}^{(+)} \), \( C_{+}^{(0)} \) are open and \( C_{-}^{(0)} \) is close. We can write the partitions:

\[
C_{-}^{(-)} = C_{-}^{(1(-)} \cup C_{x}^{2(-)}; C_{x}^{1(-)} \cap C_{x}^{2(-)} = \emptyset; C_{x}^{1(-)} = C_{x}^{1(0)} \cup C_{x}^{2(0)}; C_{x}^{1(0)} \cap C_{x}^{2(0)} = \emptyset; C_{x}^{1(+)} = C_{x}^{1(+)} \cup C_{x}^{2(+)}; C_{x}^{2(+)} \cap C_{x}^{2(+)} = \emptyset, \text{where} \ C_{x}^{1(-)}, C_{x}^{2(-)} \text{are the conexes open sets and analogously,} \\
C_{x}^{1(+)}, C_{x}^{2(+)} \text{. If} \ X_x \in C_{x}^{1(+)} \text{then} -X_x \in C_{x}^{2(+)} \ldots C_{x}^{1(0)}, C_{x}^{2(0)} \text{will be called the} \\
isotropic hipercones, \text{with the vertex in} \ x, \text{they are opposite. We consider} \ C : x \to C_x. \\

2 Considerations on \( \xi = (TM, \pi, M) \)

On \( TM \), like a differentiable manifold, \( 2n \)-dimensional, paracompact, exist a pseudo-riemannian metric, who has the local form:

\[
G = g_{ik}(x)dx^i \otimes dx^k + g_{rs}dy^r \otimes dy^s, 
\]

where, \( \delta y = \nabla y \) (the covariant differential, relative to the pseudo-Levi-Civita connection defined by the pseudo-riemannian \( g \), for \( y \in T_u M, x \in M \) \( u = (x^i, y^j) \in \pi^{-1}(U) \), where \( x \in U \) has the local coordinates \( x^i = x^i(x) \), \( \pi^{-1}(U) \) is the geometric zone of a local chart in \( u \), on \( TM \)). So, we have a Whitney decomposition, \( T_u TM = H_u TM \oplus V_u TM, u \in TM \) and the distributions, \( H : u \to H_u TM \) (horizontal) and \( V : u \to V_u TM \) (vertical). The metric \( G \) is the generalized Sasaki metric for the pseudo-riemannian case.

We have globally and locally, for \( G \),

\[
G = hG + vG, 
\]

\[
hG = g_{ik}(x)dx^i \otimes dx^k; 
\]

\[
vG = g_{rs}(dy^r + N^r_k(x, y)dx^k)(dy^s + N^s_i(x, y)dx^i), 
\]

with,

\[
N^r_k(x, y) = y^r \gamma^r_{kl}(x), 
\]

the coefficients of a nonlinear connection in \( \xi \), defined by \( g \) from the Levi-Civita coefficients \( \{ \gamma \} \), of the Levi-Civita connection \( \nabla \). Evidently, \( N \) is 1-homogeneous relative to the vertical fibres, that is relative to \( y \).

\( G \) is even the Sasaki lift for the riemannian \( g \). Seeming is not differences. But really, now, we cant have \( g(X, X) = \varepsilon_x |X|^2, \varepsilon_x \in \{-1, 0, 1\} \). 
Definition 2.1. Let note $V_n = (TM, g, \nabla)$ the pseudo-riemannian space, base, $V_{2n} = (TM, G, D)$ the tangent pseudo-riemannian space. The submanifold of $V_{2n}$ defined by the restriction, $g_x(y, y) = \varepsilon$, will be called the tangent bundle $(\tilde{TM}, \tilde{\pi}, M)$, on $M$, of the pseudo-spheres.

A locally map, in $u \in TM$ will be with the geometric zone $\tilde{\pi}^{-1}(U)$; $u \in \pi^{-1}(U)$; $\pi(u) = x \in U$, $\pi(u) = x \in U$. Result:

Proposition 2.1. The metric $\tilde{G}$, induced by $G$, on $\tilde{TM}$ is $0$-homogeneous, though $G$ is not $0$-homogeneous.

Proposition 2.2. The horizontal distribution $\tilde{H}$, on $\tilde{TM}$, is the restriction of $H$ in the points $u \in \tilde{\pi}^{-1}(U)$.

From 2.2 - 2.5 result $(vG) \circ h_t \neq vG$, that is $G$ is not $0$-homogeneous on $\tilde{TM} = TM \setminus \{0\}$ ($h_t=$homothety, $t \in \mathbb{R}^+$).

Proposition 2.3. a) On the connected components of sets $C^{(+)}$, $C^{(-)}$, the metric 2.1 is not $0$-homogeneous. b) Its restriction, on $C^{(0)}$ is $0$-homogeneous. c) Its restriction, on $TM$ is $0$-homogeneous.

A generalization of Miron’s results ([4]) is obtained in this way.

For the tangent bundle, having fixed $N$, for each $X \in X(M)$, exists $X^h \in H, X^v \in V$, uniquely. Let be the almost complex structure, natural, $F$, defined by

\[ F(X^h) = -X^v, \quad F(X^v) = X^h (F^2 = -I), \]

which is $G(FX, FY) = G(X, Y), \forall X, Y \in \tilde{TM}$, that is, $(\tilde{TM}, G, F)$ is an almost complex structure, with the 2-form associated, $\theta(X, Y) = G(FX, FY)$. In this case, result:

Proposition 2.4. a) For 2.1, 2.5, 2.6 we have: $d\theta = 0$, that is, the structure is an almost kählerian structure. b) The structure $(\tilde{TM}, G, F)$ is kählerian if and only if $V_n$ is a linear Lorentz space, locally (pseudo-euclidian) that is, if and only if the first fundamental form can be write, locally,

\[ \psi = \sum_{i=1}^{n} \varepsilon_i (dx^i)^2, \quad \varepsilon_i \in \{-1, 1\} \]

The homogenization of $G$ must be making only on the connected components $C^{(+)}$, $C^{(-)}$. Let be $g(y, y) = |y|^2 (\varepsilon \in \{-1, 1\}); \tilde{H} = \sqrt{\varepsilon H}$.

Proposition 2.5. The $0$-homogeneous metric $\tilde{G}$ is given by:

\[ \tilde{G} = g_{ik} dx^i \otimes dx^k + \frac{r^2}{\varepsilon H} g_{rs} \nabla y^r \otimes \nabla y^s; \quad r = ct > 0 \]

Proposition 2.6. The $0$-homogeneous metric $\tilde{G}$ preserve the signature.
On the connected components $C^{(+)}$, $C^{(-)}$ is defined an almost complex structure, $\tilde{F}$, associated to $\tilde{G}$, locally, from: $\tilde{F}(\delta_k) = -\frac{\partial}{\partial x}^r\delta_k$; $\tilde{F}(\partial_k) = \frac{\partial}{\partial y}\delta_k$, where, $(\delta_k = \frac{\partial}{\partial x}^r - y^s\gamma^r_{ks}\partial_i; \partial_k = \frac{\partial}{\partial y})$ in $u \in \pi^{-1}(U)$, who is given, from $\varepsilon = 1$ (the riemannian case), by R. Miron. Result:

**Proposition 2.7.** We have, a) $\tilde{\theta}(X,Y) = \tilde{G}(FX,Y)$; $d\tilde{\theta} = rd(\frac{1}{\sqrt{r}}) \wedge \theta$

b) $\tilde{G}(FX, FY) = \tilde{G}(X,Y)$, that is $(\tilde{G}, \tilde{F})$ is an almost hermitian structure.

For, $V_n = (M, g, \nabla)$ a riemannian space, $\varepsilon = 1$ and (a) (b) are result from [4].

**Proposition 2.8.** If $N_{\tilde{F}} = 0$, $(N_{\tilde{F}}$ is the Nijenhuis tensor) that is $\tilde{F}$, is complex, then we have, equivalently:

\[
(2.7) \quad r^i_{jkl}(x) = \varepsilon \frac{r}{r^2}(g_{jk}\delta^i_l - g_{jl}\delta^i_k), \varepsilon \in \{-1, 1\},
\]

where $r(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, $X,Y,Z \in \mathcal{X}(M)$ is the curvature tensor of the pseudo-riemannian space, base, $V_n = (M, g, \nabla)$.

**Proposition 2.9.** If $H$ is an integrable distribution, then we have 2.7, that is on the connected components, the curvature is $R = \frac{\varepsilon}{r}$, $\varepsilon \in \{-1, 1\}$, constant but with contrary sign.

**Proposition 2.10.** If $(\tilde{G}, \tilde{F})$ is a hermitian structure, then the first fundamental form $\psi$, of $V_n$, will be bring in to the Riemann form:

\[
\psi = \frac{1}{A} \sum_{i=1}^{n} \varepsilon_i (dx^i)^2 ; \varepsilon_i \in \{-1, 1\}, \text{ where, } A = \left(1 + \frac{\varepsilon}{4r^2} \sum_{i=1}^{n} \varepsilon_i (x^i)^2 \right)^2
\]

In the case $(T^*M)$ these results will be much difficult to obtain. We will study the generalization of these results in the Lagrange theory in to another paper (the pseudo-Lagrange case).

3 The pseudo-riemannian isometries associated

We will start with the observation that, for $n = 2$, $M_2$-compact is not possible to talk about the pseudo-riemannian isometries or the pseudo-riemannian infinitesimal isometries, only if $M_2$ is the torus or the Klein bottle.

In the relativistic mechanic, a vectorial field $\xi \in \mathcal{X}(M)$, with the null divergente, $div\xi = 0$ has an important roll (will be called incompressible). As is known any $\xi$-Killing has this property.

Let be $V = (M, g, \nabla)$ a pseudo-riemannian space and $I(M,g)$ the group of all isometries of $(M,g)$. Like in the riemannian case, $I(M,g)$ is a Lie group who act differential on $(M,g)$, it is a subgroup of the diffeomorphisms group on $M$. Like in the riemannian case, because, $\varphi \in I(M,g)$ preserve $g$, then preserve $\nabla$ (the Levi-Civita connection) and so, preserve the geodesies of $\nabla$, preserve the elementary volume... In the case $g$ riemannian, $\varphi$ preserve also the distances (the property is reciprocally). But, in the pseudo-riemannian case, $g$, we cannot talk about such property. So, look, another difference between the g-riemannian and g-pseudo-riemannian.
Let be $\xi$ and $\omega$, 1-form, dual, given by, $g(\xi, X) = \omega(X); \forall X$.

We have, \(\text{div} \xi = \delta \omega\) (the codifferential of $\omega$).

**Lemma 3.1.** If $\xi_x \in C_x^{(0)}$, then, locally, the equation, $\omega = 0$, admit a set of $(n-1)$-uples \(\{ (\xi, \ldots, \xi) \}_{(a)} \), where $\xi, \ldots, \xi$ are in $x$, linear independent and the directions given by $\xi$, $(a = 1, n-1)$, through $x$, cannot be $g$-conjugate, mutually. We have, locally, $g_{ik} \xi^i \xi^k = 0; g_{ik} \xi^i \xi^k = 0(a = 1, n-1)$, and $\xi$ is a linear combination of $(\xi)_{(a)}$ $(a = 1, n-1)$.

**Lemma 3.2.** Let be a frame, in $x$, \(\{ V \}_{(a)} \) (a = 1, n-1) orthogonal, $g(V_a, V_b) = 0; \forall a \neq b; a, b = 1, n$. Then we have, $V \notin C_x^{(0)} \forall a \neq b; a, b = 1, n$.

That is neither of vectors from the orthogonal frame cannot be with null length.

So, it can be normalized, because $\left| V \right| = \sqrt{\varepsilon_a g(V_a, V_b)} \neq 0$. Let be this $(U_1, \ldots, V_1)$.

We can write:

$g(U_a, U_b) = 0; \forall a \neq b; a, b = 1, n; g(U_a, U_b) = \varepsilon_a; a = 1, n, \varepsilon_a \in \{-1, 1\}$

\[
\text{div} \xi = \sum_{a=1}^{n} \varepsilon_a g(\nabla U_a, \xi) = \delta \omega = \sum_{a=1}^{n} \varepsilon_a (\nabla U_a \omega)(U_a) \omega(\xi)
\]

**Observation.** Sometimes, in the geometry, the codifferential is with (-).

In the case $g$-riemannian, $\varepsilon_a = 1 \ (a=1, n)$ and we will obtain the formula how is know for $\text{div} \xi$, in the orthonormated frame.

A vectorial field, $\xi$ Killing, will be now defined with infinitesimal isometries. Let be $V_{(a)}$ a pseudo-riemannian space and $\xi \in X(M)$. If the 1-parametric group, local, of diffeomorphisms, generate from $\xi$, is given by isometries, local, then $\xi$ will be called vectorial field Killing. The local isometries will be called the infinitesimal isometries or infinitesimal motions.

Equivalently, like for riemannian case we will obtain the Killing equations:

\[
(\nabla_X \omega)(Y) + (\nabla_Y \omega)(X) = 0,
\]

In the definition of $\xi$-Killing we not start from 3.-6 because it is not clear the infinitesimal character of the definition. Using the Lie derivative, we have, like in the riemannian case, $L_\xi g = 0$.

Formal, here is not appear neither differences relative to the riemannian case, just if we take into account the observations make before. Moreover, we must take into
account the case $\xi: x \rightarrow \xi_x \in C^{(0)}_2$ and $\xi$ is Killing. Particulary, when $\nabla \xi = 0$, and its orbits are geodesics (the case of infinitesimal translations).

Only from before we can observe the difficulty of study of pseudo-riemannian isometries, infinitesimal isometries, on $V_{2n} = (TM, G, D)$ where $G$ is the pseudo-riemannian Sasaki lift.

It is normal to not study the infinitesimal isometries $\varphi \in I(TM, G)$ using the general model, so that we study them from correspondence with the infinitesimal isometries, $\varphi \in I(M, g)$. The way of work will be analogously with the riemannian Sasaki case, but, now we will using the adapted basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial \theta^j}\}$. Will appear so the specific results, like in §2, because: $\xi_x \in C^{(-)}_2$ or $\xi_x \in C^{(0)}_2$ or $\xi_x \in C^{(+)}_2$.

Let be the infinitesimal transformation on base,

$$
(3.-5) \quad \xi_x f = \frac{d}{dt} f(\varphi_t(x))|_{t=0}; x \in M, f \in F(M),
$$

$\{\varphi_t(x)\}$ is an 1-parametric group generate from $\xi$, who, in generally, is local $(t \in (-\varepsilon, \varepsilon) = I, \varepsilon > 0)$. It is global if $\xi$ is complete and reciprocally. If $\xi$ is Killing then, the local diffeomorphisms of group are isometries (local), $\varphi \in I(M, g)$.

Let be the infinitesimal transformation on $TM$,

$$
(3.-4) \quad \bar{\xi}_v \bar{f} = \frac{d}{dt} \bar{f}(\varphi_t(u))|_{t=0}; \pi(u) = x; u \in \pi^{-1}(U); x \in U,
$$

If $\bar{\xi}$ is Killing then the diffeomorphisms of 1-parametric group generate from $\bar{\xi}$, are isometries (local) $\varphi \in I(TM, G)$.

In generally, the orbit of $x$ is an immersion $\gamma_x : I \rightarrow M$ given by $\gamma_x(t) = \varphi_t(x)$ and $\xi(\gamma_x(t)) = \gamma_x(t)$ ($\gamma_x$ is an integral curve of $\xi$, through $x$). Theirs family $\{\gamma_x(t)\}$, $x \in M$ ($x \in U$), $t \in I$, is a congruence of curves on $M$ (if $\xi$ is complete) or locally, through each $x \in U$ pass one curve from family, in contrary case. Analogously for $\bar{\xi}$.

**Problem 1** If $\xi \notin C^{(0)}$, then when are the infinitesimal isometries (local), $\varphi \in I(TM, G)$ a natural prolongation of infinitesimal isometries (local), $\varphi \in I(M, g)$?

**Problem 2** In this prolongation to the infinitesimal translations (infinitesimal motions, who are translations) local, on base, what infinitesimal motions (local) correspond on TM?

**Problem 3** In both cases, induces they on $V_{2n-1}$ infinitesimal motions?

**Problem 4** In this natural prolongation, if the orbits of $\xi$ are geodesics (local) of $V_n$, then the orbits of $\bar{\xi}$ are they the geodesics of $V_{2n}$?

**Problem 5** Because we have the unique decomposition $\bar{\xi} = h\xi + v\xi$ (with respect to the nonlinear connection $N$), then when $h\xi$ and $v\xi$ generate the infinitesimal isometries (local)?

Because $\xi \notin C^{(0)}$ we can write, in $x \in U$, $\varepsilon \xi \in \{-1, 1\}$ and $\xi \in C^{(-)}$ or $\xi \in C^{(+)}$,

$$
(3.-3) \quad |\xi|^2 = \varepsilon \delta_{ik}\xi^i\xi^k,
$$

Let be $\xi$ defined by,

$$
(3.-2) \quad \bar{\xi} = h\xi + v\xi,
$$

where, in $u \in \pi^{-1}(U), \pi(u) = x \in U$,
\( h_{\xi} \overset{\text{def}}{=} \xi^h = \xi^i \frac{\partial}{\partial x^i}; v_{\xi} \overset{\text{def}}{=} (y^s \nabla_s \xi^r) \frac{\partial}{\partial y^r}, \)

if,

\[
\xi = \xi^i(x) \frac{\partial}{\partial x^i},
\]

Let note \( G(u^\alpha, u^\beta) = G_{\alpha\beta}(x, y) \), where \( u^\alpha = (u^i = x^i, u^{n+i} = y^i) \).

**Theorem 3.1.** Let be \( V_n = (TM, G, D), \) with the pseudo-riemannian Sasaki lift \( G \). The necessary and sufficient condition for to exists infinitesimal motions (that is local isometries, infinitesimal) \( \varphi \in I(M, g), \tilde{\varphi} \in I(TM, G), \) who correspond one the other, is to exist a system of local coordinates \( (x, y) \), on \( TM \), such as \( G_{n+i n+k}(x, y) \) is not depend form a local coordinate, \( x^r \) (r, fixed). That is,

\[
\frac{\partial G_{n+i n+k}}{\partial x^r} = 0; \forall i, k = \overline{1, n} (r, \text{fixed}).
\]

**Theorem 3.2.** In the conditions of theorem 1, the equations \( L_{\xi} g = 0 \) admit one solution \( \xi \), who is Killing and for who the diffeomorphisms of 1-parametric group 3.-5, given by \( \xi \), are the infinitesimal motions. Let be \( \xi \). Then \( \xi \not\in \mathcal{C}_c^{(0)}(x \in U) \).

Because \( \xi \) is Killing, result that \( \xi \) 3.-2, 3.-1 is Killing, for the space \( V_{2n} \) and so \( L_{\xi} G = 0 \), and reciprocally. So, the 1-parametric group, local, generated by \( \xi \) is given by local isometries. We have the reciprocal.

In the local coordinates for 3.1, we have:

\[
|\xi|^2 = \varepsilon \xi_{g_{rr}}; g_{ik} = g_{ik}(x^1, \ldots, x^{r-1}, x^{r+1}, \ldots, x^n), i, k = \overline{1, n}
\]

Evidently, \( \varepsilon \in \{-1, 1\} \), if \( g_{rr} < 0 \) or \( g_{rr} > 0 \). We cannot have \( g_{rr} = 0 \) because \( g \) is nondegenerate. \( |\xi| \neq 0 \), so the infinitesimal motion is not minimal.

The infinitesimal motion is a translation, \( |\xi| = ct \) if and only if \( g_{rr} = ct \neq 0 \). From that result \( G_{n+r n+r} = ct \) for the first fundamental form of \( V_n \), such as, of \( V_{2n} \). The linear element will be: \( ds^2 = \varepsilon g_{ik} dx^i dx^k + \varepsilon_r g_{rr} (dx^r)^2; g_{ii} \neq g_{rr}(i = \overline{1, n}) \) (without summation from r) with \( \frac{\partial g_{ll}}{\partial x^r} = 0; j, l = \overline{1, n} \).

**Theorem 3.3.** If exist a local chart in \( u \), on \( TM (u \in \pi^{-1}(U), \pi(u) = x \in U) \), thus we have 3.1, then exist \( \xi, \tilde{\xi} \) 3.-2 3.-1, thus we have the relations:

\[
D^2_{\xi Y} \tilde{\xi} = R(X, \xi) Y; \forall X, Y \in \mathcal{X}(TM),
\]

\[
L_{\xi} D = 0,
\]

(\( \tilde{\xi} \) is an affine colliniation, infinitesimal of space \( V_{2n} \)) and we have the relations for \( \xi \), analogously.

**Proof.** The conditions given are equivalent with the existence of a vectorial field, Killing \( \xi \) on \( V_n \) and so, with the existence of a vectorial field Killing \( \tilde{\xi} \), on \( V_{2n} \), who has
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the form 3.-2 3.-1. So we have 3.2 3.3 and the homologue relations, \( \nabla^2_{XY} \xi = r(X, \xi)Y; \)
\( \forall X, Y \in \mathcal{X}(M), L_\xi \nabla = 0. \)\( \square \)

If \( \xi \notin C^{(0)}_x \), then like in the Sasaki-riemannian case, we can show that, taking into account the transformations \( (x, y) \to (\tau, \varphi) \), for the relations 3.-5, 3.-3 are obtain the infinitesimal transformations:

\[
(3.4) \quad \tau^i = x^i + \xi^i(x) \delta t,
\]
\[
(3.5) \quad \tau^i = x^i + \xi^i(x) \delta t, \varphi^i = y^i + (y^s \nabla_s \xi^i - N^i \xi^r) \delta t,
\]

Starting from here we can prove that if 3.4 are the isometries (local) \( \varphi \in I(M, g) \), then 3.5 are the isometries (local) \( \tilde{\varphi} \in I(TM, G) \). So, if \( \xi \notin C^{(0)} \) is Killing, then \( \xi \) 3.-2 3.-1 is Killing and reciprocally. Other aspects relative to the problems who appear before will be given in another paper.

**Proposition 3.1.** If \( V_n = (M, g, \nabla) \)-pseudo-riemannian has \( \xi \) parallel, \( \nabla \xi = 0 \), then result \( \tilde{\xi} = \xi^h \) and \( \xi = \xi^h \) is Killing.

Moreover, the orbits of \( \xi \) are the geodesics of space \( V_n \) and the orbits of \( \tilde{\xi} \) are the horizontal lifts of the orbits of \( \xi \), and these are the geodesics of \( V_2n \). Their restrictions to \( V_{2n-1} \) are the geodesics of \( V_{2n-1} \).

The general problem of geodesics on \( V_{2n} \)-pseudo-riemannian, who is tie from geodesics on \( V_n \), is solve. We show that, local, the equations of geodesics, on \( V_{2n} \) can be write only in the terms of \( V_n \), using \( \nabla \). The problem of general translations for \( V_n, V_{2n} \) is follow to solve.

**References**


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