# Applications of the multiple hybrid Laplace-$Z$ transformation in Control Systems and Queueing Theory

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**Abstract**

The properties of the Multiple Hybrid Laplace-$Z$ transformation are presented. Using them one obtains a direct method of solving differential-difference equations. This method is illustrated by its applications to stochastic processes and to the differential equations of the transition probabilities in queueing theory. The same hybrid transformation is employed to obtain transfer matrices for different classes of hybrid linear control systems.

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**Key words:** Multiple Hybrid Laplace $Z$-Transformation; multiple differential - difference equations, $nD$ linear systems; transfer functions, queueing systems.

## 1 Introduction

A great deal of research is being out today on multidimensional ($nD$) systems, determined by their processing, computer tomography, geophysics etc.

An important subclass of $nD$ systems is represented by the multidimensional systems which are continuous with respect to some variables and descrete with respect to others. Such hybrid systems were studied in [5], [6], [9], [11], [12]. They were used as models in the study of linear repetitive processes [1], [3], [15] or of iterative learning control synthesis [7].

This approach implies the necessity to use a suitable multiple Laplace-type transformation.

In this paper a multiple hybrid Laplace $Z$-transformation is defined and its principal properties are emphasized, by generalizing. The results presented in [12]. These properties are used for solving multiple differential-difference equations, integral equations and integro-differential-difference equations.

The advantages of this method are illustrated by appplying it to some differential recurrence equations which appear in the study of the stochastic processes or in the

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queueing theory can be the same hybrid transformation can be applied to Roesser-type and Fornasini-Marchesini-type models, including descripter and delayed systems, to obtain their transfer matrices.

## 2 Multiple Hybrid Laplace $Z$-Transformation

The following definitions and theorems extend the usual (1D) Laplace and $Z$ transformations and generalize the 2D transformation studied in [12].

**Definition 2.1.** A function $f : R^q \times Z^r \to C$ is said to be an *original function* if $f$ has the following properties:

i) $f(t_1, \ldots, t_q; k_1, \ldots, k_r) = 0$ if $t_i < 0$ or $k_j < 0$ for some $i \in \bar{q}$ or $j \in \bar{r}$,

ii) $f(\cdot, \ldots, \cdot; k_1, \ldots, k_r)$ is piecewise smooth on $\mathbf{R}_+^q$ for any $(k_1, \ldots, k_r) \in \mathbf{Z}_+^r$,

iii) $\exists M_f > 0$, $\sigma_{f_i} \geq 0$, $i \in \bar{q}$, $R_{f_i} > 0$, $j \in \bar{r}$ such that $\forall t_i > 0$, $i \in \bar{q}$, $\forall k_j \in \mathbf{Z}_+$, $j \in \bar{r}$

$$|f(t_1, \ldots, t_q; k_1, \ldots, k_r)| \leq M_f \left( \exp\left( \sum_{i=1}^{q} \sigma_{f_i} t_i \right) \right) \left( \prod_{j=1}^{r} R_{f_j}^{k_j} \right).$$

**Definition 2.2.** For any original function $f$, the function

$$\begin{aligned} F(s_1, \ldots, s_q; z_1, \ldots, z_r) = \int_0^\infty \ldots \int_0^\infty \sum_{k_1=0}^{\infty} \ldots \sum_{k_r=0}^{\infty} f(t_1, \ldots, t_q; k_1, \ldots, k_r) \cdot \\ \cdot e^{-s_1 t_1} \ldots e^{-s_q t_q} z_1^{-k_1} \ldots z_r^{-k_r} dt_1 \ldots dt_r \end{aligned} \tag{2.1}$$

is said to be the *Multiple Hybrid Laplace Z-Transform (MHLZT) of $f$*.

We shall use the notation $F(s; z)$, where $s = (s_1, \ldots, s_q)$ and $z = (z_1, \ldots, z_r)$ and $\mathcal{L}_{q,r}$ for the operator defined by (2.1), hence $F(s; z) = \mathcal{L}_{q,r}[f(t; k)] = \mathcal{L}_{q,r}[f]$.

The following results are proved in [13].

**Theorem 2.3.** *(linearity). For any original functions $f$ and $g$ and $\alpha, \beta \in \mathbf{C}$*

$$\mathcal{L}_{q,r}[\alpha f + \beta g] = \alpha \mathcal{L}_{q,r}[f] + \beta \mathcal{L}_{q,r}[g].$$

**Theorem 2.4.** *(first time delay theorem). For any $a = (a_1, \ldots, a_q) \in \mathbf{R}_+^q$ and $b = (b_1, \ldots, b_r) \in \mathbf{Z}_+^r$,*

$$\mathcal{L}_{q,r}[f(t_1 - a_1, \ldots, t_q - a_q; k_1 - b_1, \ldots, k_2 - b_2)] = \left( \exp\left( -\sum_{i=1}^{q} a_i s_i \right) \right) \left( \prod_{j=1}^{r} z_j^{-b_j} \right) F(s; z).$$

**Definition 2.5.** For $\alpha = \{i_1, \ldots, i_l\} \subset \bar{q}$ and $\beta = \{j_1, \ldots, j_h\} \subset \bar{r}$, the $(\alpha, \beta)$-*partial MHLZT* $\mathcal{L}_{q,r}(\alpha, \beta)$ is defined by

$$\begin{aligned} \mathcal{L}_{q,r}(\alpha, \beta)[f(t; k)] = \int_0^\infty \ldots \int_0^\infty \sum_{k_{j_1}=0}^{\infty} \ldots \sum_{k_{j_n}=0}^{\infty} f(t_1, \ldots, t_q; k_1, \ldots, k_r) \cdot \\ \cdot \left( \exp\left( -\textstyle\sum_{i \in \alpha} s_i t_i \right) \right) \left( \textstyle\prod_{j \in \beta} z_j^{+k_j} \right) dt_{i_1} \ldots dt_{i_l}. \end{aligned}$$

Obviously, if $\alpha = \bar{q}$ and $\beta = \bar{r}$, $\mathcal{L}_{q,r}(\bar{q},\bar{r}) = \mathcal{L}_{q,r}$; if $\beta = \emptyset$ then $\mathcal{L}_{q,r}(\alpha,\emptyset) = \mathcal{L}_l =$ the multiple Laplace transformation; if $\alpha = \emptyset$ then $\mathcal{L}_{q,r}(\emptyset,\beta) = \mathcal{Z}_h =$ the multiple $z$ - transformation; if $\alpha = \emptyset$ and $\beta = \emptyset$, $\mathcal{L}_{q,r}(\emptyset,\emptyset)[f] = f$.

We shall use the following notations: if $\alpha = \{i_1,\dots,i_l\} \subset \bar{q}$

$$f(0_\alpha+;k) = f(t_1,\dots,t_{i_1-1},0+,t_{i_1+1},\dots,t_{i_l-1},0+,t_{i_l+1},\dots,t_q;k)$$

and $f(0_i+;k) = f(0_\alpha+;k)$ when $\alpha = \{i\}$; similar significations have $f(t;0_\beta)$ for $\beta = \{j_1,\dots,j_h\} \subset \bar{r}$ and $f(t;0_j)$, $j \in \bar{r}$.

For $\gamma = (\gamma_{i_1},\dots,\gamma_{i_l}) \in (\mathbf{N}^*)^l$ and $\theta = (\theta_{j_1},\dots,\theta_{j_h}) \in (\mathbf{N}^*)^h$, $\dfrac{\partial^\gamma}{\partial t^\gamma}(t,k+\theta)$ denotes

$$\frac{\partial^{\gamma_{i_1}+\ldots+\gamma_{i_l}}}{\partial t_{i_1}^{\gamma_{i_1}}\ldots t_{i_l}^{\gamma_{i_l}}} f\left(t;k+\sum_{j\in\beta}\theta_j e_j\right),$$

$s^\gamma = s_{i_1}^{\gamma_{i_1}}\ldots s_{i_l}^{\gamma_{i_l}}$, $z^\theta = z_{j_1}^{\theta_{j_1}}\ldots z_{j_h}^{\theta_{j_h}}$ and $|\gamma| = l$.

The family of all non-empty subsets of $\alpha = \{i_1,\dots,i_l\}$ and $\beta = \{j_1,\dots,j_l\}$ are denoted by $E_\gamma$ and $E_\theta$ respectively. For $\zeta = E_\theta$, $\hat{\zeta} = \{j \in \zeta | \theta_j > 0\}$, $D_{\theta,\zeta} = \prod_{j\in\hat{\zeta}}\{0,1,\dots,\theta_j-1\}$ and if $\hat{\zeta} = \{\zeta_1,\dots,\zeta_p\}$, $\sum\limits_{\theta,\zeta}$ stands for $\sum\limits_{k_{\zeta_1}=0}^{\theta_{\zeta_1}-1}\cdots\sum\limits_{k_{\zeta_p}=0}^{\theta_{\zeta_p}-1}$. If $\varepsilon \in E_\gamma$, and $\varepsilon = \{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m\}$, $\gamma_\varepsilon = (\gamma_{\varepsilon_1},\dots,\gamma_{\varepsilon_m})$ then for $\eta_\varepsilon = (\eta_{\varepsilon_1},\dots,\eta_{\varepsilon_m}) \in \mathbf{N}^m$, $\sum\limits_{\eta_\varepsilon\le\gamma_\varepsilon-1}$ stands for $\sum\limits_{\eta_{\varepsilon_1}=0}^{\gamma_{\varepsilon_1}-1}\cdots\sum\limits_{\eta_{\varepsilon_m}=0}^{\gamma_{\varepsilon_m}-1}$.

**Theorem 2.6.** *(differentiation and second time delay). For any $i \in \bar{q}$, $j \in \bar{r}$, $\gamma = (\gamma_{i_1},\dots,\gamma_{i_l}) \in (\mathbf{N}^*)^l$ and $\theta = (\theta_{j_1},\dots,\theta_{j_h}) \in (\mathbf{N}^*)^h$*

$$\mathcal{L}_{q,r}\left[\frac{\partial f}{\partial t_i}(t;k)\right] = s_i F(s;z) - \mathcal{L}_{q,r}(\tilde{i},\tilde{r})[f(0_i+;k)]$$

$$\mathcal{L}_{q,r}[f(t;k+e_j)] = z_i F(s;z) - \mathcal{L}_{q,r}(\tilde{q},\tilde{j})[f(t;0_j)]$$

$$\mathcal{L}_{q,r}\left[\frac{\partial^\gamma f}{\partial t^\gamma}(t;k+\theta)\right] = s^\gamma z^\theta F(s;z)+$$

$$+z^\theta\sum_{\varepsilon\in E_\gamma}\sum_{\zeta\in E_\theta}(-1)^{|\varepsilon|+|\zeta|}s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}}\sum_{\eta_\varepsilon\le\gamma_\varepsilon-1}s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1}\sum_{\theta,\zeta}\mathcal{L}_{q,r}(\tilde{\varepsilon},\tilde{\zeta})\left[\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon+;0_\zeta)\right]\left(\prod_{j\in\zeta}z_j^{-k_j}\right).$$

**Definition 2.7.** Given two original functions $f$ and $g$, the $(q,r)$-*hybrid convolution* of $f$ and $g$ is the function denoted by $f*g$ which is given by

$$(f*g)(t_1,\dots,t_q;k_1,\dots,k_r) = \int_0^{t_1}\cdots\int_0^{t_q}\sum_{l_1=0}^{k_1}\cdots\sum_{l_r=0}^{k_r}f(u_1,\dots,u_q;l_1,\dots,l_r)\cdot$$

$$\cdot g(t_1-u_1,\dots,t_q-u_q;k_1-l_1,\dots,k_r-l_r)du_1\dots du_q$$

and which equals 0 otherwise.

**Theorem 2.8.** *(convolution). For any original functions $f$ and $g$*

$$\mathcal{L}_{q,r}[(f*g)(t;k)] = F(s;z)G(s;z).$$

# 3 Multiple differential-difference and integral equations

Let $f$ be an original function; $j \in \bar{r}$, $l \in \mathbf{N}^*$, $\beta = \{j_1, \dots, j_n\} \in \bar{r}$ and $\theta = (\theta_{j_1}, \dots, \theta_{j_h}) \in (\mathbf{N}^*)^h$.

**Definition 3.1.** The *$j$-first difference* (*$(j,1)$-difference*) of $f$ is the function

$$\Delta_j f(t;k) = \begin{cases} 0 \qquad \text{if } t_i < 0 \text{ or } k_l < 0 \text{ for some } i \in \bar{q}, l \in \bar{r} \\ f(t_1, \dots, t_q; k_1, \dots, k_{j-1}, k_j + 1, k_{j+1}, \dots, k_r) - \\ \quad -f(t_1, \dots, t_q; k_1, \dots, k_{j-1}, k_j, k_{j+1}, \dots, k_r) \qquad \text{otherwise.} \end{cases}$$

The $(j,l)$-*difference* of $f$ is defined by induction by

$$\Delta_j^l f(t;k) = \Delta_j(\Delta_j^{l-1} f(t,k))$$

The $(\beta,\theta)$-*difference* of $f$ is defined by induction by

$$\Delta_\beta^\theta f(t;k) = \Delta_{j_n}^{\theta_{j_n}} \dots \Delta_{j_1}^{\theta_{j_1}} f(t;k).$$

Let $\Gamma$ be a subset of $\displaystyle\bigcup_{i=1}^{q} \mathbf{R}_+^i$ and $\Theta$ a subset of $\displaystyle\bigcup_{j=1}^{r} \mathbf{Z}_+^j$. For $\gamma = (\gamma_{i_1}, \dots, \gamma_{i_h}) \in \Gamma$ and $\theta = (\theta_{j_1}, \dots, \theta_{j_l}) \in \Theta$ we denote a coefficient $a_{\gamma_{i_1},\dots,\gamma_{i_n}}; \theta_{j_1}, \dots, \theta_{j_l})$ by $a_{\gamma\theta}$. A *multiple differential-difference equation* has the form

$$\sum_{\gamma\in\Gamma} \sum_{\theta\in\Theta} a_{\gamma\theta} \frac{\partial^\gamma}{\partial t^\gamma} \Delta^\theta x(t;k) = f(t;k) \tag{3.1}$$

where $a_{\gamma\theta} \in \mathbf{R}$, $\forall\gamma \in \Gamma$, $\theta \in \Theta$, $x(t;k)$ is un unknown original function and $f(t;k)$ is a given original function.

We consider the boundary conditions

$$\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon+; 0_\zeta) = g_{\varepsilon,\zeta}(t_{\bar{\varepsilon}}; k_{\bar{\zeta}}), \quad \varepsilon = E_\gamma,\ \zeta \in E_\theta \tag{3.2}$$

where $t_\alpha$ and $k_\beta$ stand for $t_{\alpha_1,\dots,\alpha_l}$ and $k_{\beta_1,\dots,\beta_n}$ if $\alpha = (\alpha_1, \dots, \alpha_l)$ and $\beta = (\beta_1, \dots, \beta_n)$.

By using Definition 3.1, the equation (3.1) can be rewritten as

$$\sum_{\gamma\in\Gamma} \sum_{\theta\in\Theta} b_{\gamma\theta} \frac{\partial^\gamma}{\partial t^\gamma} x(t; k+\theta) = f(t,k) \tag{3.3}$$

By Theorem 2.6 and by applying the MHLZT to the equation (3.3) with boundary conditions (3.2) it is transformed into the algebraic equation

$$B(s;z)X(s;z) + C(s;z) = F(s;z)$$

having the solution $X(s;z) = \dfrac{F(s;z) - C(s;z)}{B(s;z)}$, where $B(s;z)$ and $C(s;z)$ are the polynomials

$$B(s;z) = \sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta} b_{\gamma\theta} s^{\gamma} z^{\theta} = \sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta} b_{\gamma_{i_1},\dots,\gamma_{i_l};\theta_{j_1},\dots,\theta_{j_n}} s_{i_1}^{\gamma_{i_1}} \dots s_{i_l}^{\gamma_{i_l}} z_{j_1}^{\theta_{j_1}} \dots z_{j_n}^{\theta_{j_n}}$$

and

$$C(s,z) = \sum_{\gamma\in\Gamma}\sum_{\theta\in\Theta} b_{\gamma\theta} z^{\theta} \sum_{\varepsilon\in E_\gamma}\sum_{\zeta\in E_\theta} (-1)^{|\varepsilon|+|\zeta|} s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}} \sum_{\eta_\varepsilon\le\gamma_\varepsilon-1} s_{\varepsilon}^{\gamma_\varepsilon-\eta_\varepsilon-1}\cdot$$
$$\cdot \sum_{D_{\theta,\eta}} \mathcal{L}_{q,r}(\tilde{\varepsilon},\tilde{\zeta}) g_{\varepsilon,\eta}(t_{\tilde{\varepsilon}}, k_{\tilde{\zeta}}) (\prod_{j\in\zeta} z_j^{-k_j})$$

**Application 3.2.** A Poisson process $(X_t)_{t\in\mathbf{R}^+}$ is described by the probabilities $P_k(t) = P(X_t = k)$, $k\in\mathbf{N}$, which verify the system of differential equations

$$P_0'(t) = -\lambda P_0(t) \tag{3.4}$$

$$P_{k+1}'(t) = -\lambda P_{k+1}(t) + \lambda P_k(t),\ k = 0, 1, \dots \tag{3.5}$$

with the initial conditions $P_0(0) = 1$ and $P_k(0) = 0$, $k = 1, 2, \dots$ .

By using the notation $P_k(t) = x(t,k)$ the system (3.4), (3.5) becomes a differential-difference system which is transformed (by Theorem 2.6) as above, by applying the MHLZT to the second equation and the usual (1D) Laplace transform $\mathcal{L}$ to the first one, into an algebraic system

$$\begin{aligned}
&s\mathcal{L}[x(t,0)] - x(0,0) = -\lambda\mathcal{L}[x(t,0)]\\
&szX(s,z) - sz\mathcal{L}[x(t,0)] - z\mathcal{Z}[x(0,h)] + zx(0,0) =\\
&= -\lambda z(X(s,z) - \mathcal{L}[x(t,0)]) + \lambda X(s,z).
\end{aligned}$$

Since $x(0,0) = P_0(0) = 1$ and the 1D $\mathcal{Z}$-transform $\mathcal{Z}[x(0,k)]$ equals

$$\sum_{k=0}^{\infty} x(0,k) z^{-k} = \sum_{k=0}^{\infty} P_k(0) z^{-k} = 1,$$

this system has the solution $\mathcal{L}[x(t,0)] = \dfrac{1}{s+\lambda}$ and

$$X(s,z) = \frac{z(s+\lambda)}{sz+\lambda z-\lambda}\mathcal{L}[x(t,0)] = \frac{z}{sz+\lambda z-\lambda}$$

whose original is the usual solution $P_k(t) = x(t,k) = e^{-\lambda t}\dfrac{(\lambda t)^k}{k!}$, $k\in\mathbf{N}$.

**Application 3.3.** In Queueing theory, the transition probabilities $p_{ij}(t) = P(\xi_{\tau+t} = j|\xi_\tau = i)$ of a system $M/M/1$ verify the system of differential equations (see [8])

$$\begin{aligned}
&p_{i0}'(t) = -\lambda p_{i0}(t) + \mu p_{i1}(t)\\
&p_{ij}'(t) = \lambda p_{i,j-1}(t) - (\lambda+\mu) p_{ij}(t) + \mu p_{i,j+1}(t)
\end{aligned}$$

with the initial conditions $p_{ij}(0) = \delta_{ij}$, where $0 < \lambda < \mu$.

By denoting $p_{ij}(t) = x_i(t, j)$ and $\mathcal{L}[x_i(t, j)] = X_i(s, z)$ and by applying the MHLZT and Theorem 2.6 as in Application 3.2, the differential-difference system is transformed directly intoa set of algebraic equations having the solutions

$$X_i(s, z) = \frac{z^{-i+1} + \mathcal{L}[x_i(t, 0)]\mu z(1 - z)}{(s + \lambda + \mu)z - \lambda - \mu z^2}$$

whose original has the usual expression of the probabilities $p_{ij}(t)$.

A *multiple continuous-discrete convolution integral equation* has the form

$$Ax(t_1, \ldots, t_q; k_1, \ldots, k_2) + \int_0^{t_1} \ldots \int_0^{t_q} \sum_{l_1=0}^{k_1} \ldots \sum_{l_2=0}^{k_2} x(u_1, \ldots, u_q; l_1, \ldots, l_r) \cdot \tag{3.6}$$
$$\cdot g(t_1 - u_1, \ldots, t_q - u_k; k_1 - l_1, \ldots, k_2 - l_2) du_1 \ldots du_q = f(t_1, \ldots, t_q; k_1, \ldots, k_r)$$

where $A \in \mathbf{R}$, $x(t; k)$ is an unknown original function and $f$ and $g$ are given original functions.

By applying MHLZT, due to Theorem 2.8 (3.6) is transformd into the algebraic equation

$$AX(s; z) + X(s; z)G(s; z) = F(s; z)$$

and the solution of (3.6) is the original of the function

$$X(s; z) = \frac{F(s; z)}{A + G(s; z)}.$$

Similarly, by combining the equations (3.1) and (3.6) one obtains multiple integro-differential-difference equations which can easily by solved following the same approach.

# 4 Transfer matrices of multiple hybrid control systems

The 2D discrete-time Roesser [14] and Fornasini-Marchesini [2] models were extended to 2D or $nD$ continuous-discrete linear systems in [4], [5], [6], [10] and [11]. The MHLZT can be applied to obtain the transfer matrices of different classes of such systems.

Because of lack of space, we limit ourselves to present only the transfer matrices of the descriptor and delayed systems, obtained from the state space representations by using MHLZT and Theorems 2.3, 2.4 and 2.6.

The Roesse type multiple hybrid descriptor and delayed systems has the transfer matrix

$$H(s; z) = \left[ C_0 + C_1 \left( \exp\left( \sum_{i=1}^{q} a_i s_i \right) \right) \left( \prod_{j=1}^{r} z_j^{-b_j} \right) \right] \times$$
$$\times \left[ E \left( \bigoplus_{i=1}^{q} s_i I_{n_{c_i}} \right) \oplus \left( \bigoplus_{j=1}^{r} z_j I_{n_{d_j}} \right) - A_0 - A_1 \left( \exp\left( -\sum_{i=1}^{q} a_i s_i \right) \right) \left( \prod_{j=1}^{r} z_j^{-b_j} \right) \right]^{-1} \times$$
$$\times \left[ B_0 + B_1 \left( \exp\left( -\sum_{i=1}^{q} a_i s_i \right) \right) \left( \prod_{j=1}^{r} z_j^{-b_j} \right) \right] + D_0 + D_1 \left( \exp\left( -\sum_{i=1}^{q} a_i s_i \right) \right) \left( \prod_{j=1}^{r} z_j^{-b_j} \right)$$

and the transfer matrix of the corresponding Fornasini-Marchesini system is

$$H(s;z) = (C_0 + C_1 e^{-as} z^{-b})(E s^{\bar{q}\bar{r}} - \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} A_{0,\tau,\delta} s^{z} z^{\delta} - \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} A_{1,\tau,\delta} s^{\tau} z^{\delta} e^{-as} z^{-b})^{-1} \cdot$$
$$\cdot \left( \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} B_{0,\tau,\delta} s^{\tau} z^{\delta} + \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} B_{1,\tau,\delta} s^{\tau} z^{\delta} e^{-as} z^{-b} \right) + D_0 + D_1 e^{-as} z^{-b},$$

where $e^{-as}$ denotes $\exp\left(-\sum_{i=1}^{q} a_i s_i\right)$ and $z^{-b}$ denotes $\prod_{j=1}^{r} z_j^{-b_j}$.

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