On affine Hamiltonians and Lagrangians of higher order and their subspaces

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Abstract

Some new Legendre transformations, which give intrinsic dualities between Lagrangians and affine Hamiltonians of higher order are considered. Some canonical ways to induce vectorial and affine Hamiltonians on a submanifold are given, too.

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Key words: affine bundle, higher order spaces, Lagrangian, affine Hamiltonian, vectorial Hamiltonian, submanifold.

Higher order Finsler and Lagrange spaces were studied by R. Miron using the bundles of accelerations (see for example [15, 9]). A dual theory of higher order Hamilton spaces was recently studied also by R. Miron in [11, 12]. Some Legendre transformations that relates a Lagrangian and a Hamiltonian of the same order, are considered by the same author in [13]; we give here a synthetic description of his construction in subsection 3.1, where a Hamiltonian of R. Miron is called here a vectorial Hamiltonian. These Legendre transformations use essentially an affine section and they are not intrinsic associated with the Lagrangian or the Hamiltonian. In order to remove this inconvenient, we define in subsection 3.2 an affine Hamiltonian, which can be related by Legendre transformations with a Lagrangian of the same order. These Legendre transformations are intrinsic associated with the Lagrangian or the Hamiltonian and give a bijective correspondence between the regular Lagrangians and Hamiltonians of the same order.

The theory of Finsler and Lagrange submanifolds is studied in many papers (for example [1, 2, 3, 17, 18, 19, 20, 21, 23]). A theory of Hamilton submanifolds of order one was also studied by R. Miron in [16, 10] and the case of higher order is considered in [13]. In these approaches one define an induced Hamiltonian on a submanifold, which is not intrinsic (the induction procedure is not uniquely defined by the Hamiltonian and the submanifold). A canonical way to induce a Hamiltonian (of order one) on a submanifold is given in [24, 25], solving a problem of R. Miron [16, 10] concerning the possibility to induce a Hamiltonian on a submanifold *in an intrinsic way*; the

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idea is extended in [26], where (vectorial) Hamiltonians of higher order are induced on submanifolds.

In the first section of the paper we briefly discuss the Legendre transformations between vectorial and affine Hamiltonians on one way and Lagrangians on the other way, defined on open sets of affine spaces. In the second section we briefly revise some known constructions and results related to the geometry of the k-acceleration bundle and its dual [13]. In the second section we study the Legendre transformations associated to Lagrangians and vectorial Hamiltonians of higher order (subsection 3.1) and to Lagrangians and vectorial Hamiltonians of higher order (subsection 3.2). The Legendre transformation studied in subsection 3.1 use essentially an affine section and it is not intrinsic associated with the Lagrangian and the vectorial Hamiltonian. It is used in [13]. The Legendre transformations studied in subsection 3.2 is intrinsic associated with the Lagrangian and the affine Hamiltonian.

Some canonical inductions of Hamiltonians on submanifolds are studied in the third section, extending an idea used by C.-M. Marle in [6] for a classical Hamiltonian. A canonical way to induce a vectorial Hamiltonian of higher order on a submanifold is given first in subsection 4.1, using an adapted section and following [26]. Then a canonical way to induce an affine Hamiltonian of higher order on a submanifold is considered in subsection 4.2.

The aim of the paper is to give only some basic ideas that will be further investigated in some forthcoming papers.

1 Vectorial and affine Hamiltonians and Lagrangians

Let $\mathcal{A}$ be a real affine space, modelled on a real vector space V. The *vectorial dual* of $\mathcal{A}$ is $\mathcal{A}^\dagger = Aff(\mathcal{A}, I\!R)$, where Aff denotes affine morphisms. An *affine frame* on $\mathcal{A}$ is a couple $\mathcal{R} = (o, \mathcal{B})$, where $o \in \mathcal{A}$ is a point and $\mathcal{B} = \{\bar{e}_i\}_{i=\overline{1,n}} \subset V$ is a base. If $z \in \mathcal{A}$ is an arbitrary point, then its *affine coordinates* (or simply *coordinates*) corresponding to this frame, are the (vectorial) coordinates $(z^i)_{i=\overline{1,n}}$ of the vector $\overline{oz}$, i.e. $\overline{oz} = z^i \bar{e}_i$; denote $Z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}$. Let $\mathcal{R}' = (o', \mathcal{B}')$, $\mathcal{B}' = (e_{i'})_{i'=\overline{1,n}}$ be an other affine frame, where $\bar{e}_{i'} = a^i_{i'} \bar{e}_i$ and $\overline{oo'} = a^i \bar{e}_i$, and denote $A = (a^i_{i'})$ and $a = \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix}$. Then

$$\begin{pmatrix} Z \\ 1 \end{pmatrix} = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z' \\ 1 \end{pmatrix}.$$

Consider now an affine frame $\mathcal{R} = (o, \mathcal{B})$ of $\mathcal{A}$ and the affine maps $\tilde{e}^0 : \mathcal{A} \to I\!R$, $\tilde{e}^0(z) = 1$ and $\tilde{e}^i : \mathcal{A} \to I\!R$, $\tilde{e}^i(z) = z^i$, $(\forall) i = \overline{1,n}$. It is easy to see that $\mathcal{R}^\dagger = \{\tilde{e}^0, \tilde{e}^1, \ldots, \tilde{e}^n\} \subset \mathcal{A}^\dagger$ is a base. Let us consider the linear maps $j : I\!R \to \mathcal{A}^\dagger$ and $\pi : \mathcal{A}^\dagger \to V^*$ defined on the bases $j : I\!R \to \mathcal{A}^\dagger$, $j(1) = \tilde{e}^0$, and $\pi : \mathcal{A}^\dagger \to V^*$, $\pi(\tilde{e}^0) = 0$, $\pi(\tilde{e}^i) = \bar{e}^i$, $i = \overline{1,n}$, where $\mathcal{B}^* = \{\bar{e}^i\}_{i=\overline{1,n}} \subset V^*$ is the dual base of $\mathcal{B}$. It is easy to see that the definitions of j and π do not depend on the frame $\mathcal{R}$ and there is a short exact sequence of vector spaces which has the form.

$$0 \to I\!R \xrightarrow{j} \mathcal{A}^\dagger \xrightarrow{\pi} V^* \to 0.$$

Let $\mathcal{R} = (o, \mathcal{B})$ and $\mathcal{R}' = (o', \mathcal{B}')$ be two affine frames as considered previously. We have $z^i = a^i_{i'} z^{i'} + a^i$, thus $\tilde{e}^i = a^i_{i'} \tilde{e}^{i'} + a^i \tilde{e}^0$. Considering the bases $\mathcal{R}^\dagger$, $(\mathcal{R}')^\dagger \subset \mathcal{A}^\dagger$, then $\xi \in \mathcal{A}^\dagger$ has the forms $\xi = \omega \tilde{e}^0 + \Omega_i \tilde{e}^i = \omega' \tilde{e}^0 + \Omega_{i'} \tilde{e}^{i'}$ and the following formulas hold:

$$\begin{aligned} \Omega_{i'} &= a^i_{i'} \Omega_i, \\ \omega' &= \omega + \Omega_i a^i, \end{aligned}$$

or, in a matrix form:

$$\begin{pmatrix} \Omega' & \omega' \end{pmatrix} = \begin{pmatrix} \Omega & \omega \end{pmatrix} \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix},$$

where Ω and Ω' are row matrices of $\dim V$.

Let us consider now Lagrangians and Hamiltonians on the vector space V.

A *Lagrangian* (a *Hamiltonian*) on the real vector space V is a differentiable map $L : V \backslash V_0 \to I\!R$ (respectively $H : V^* \backslash W_0 \to I\!R$), where $V_0 \subset V$ (respectively $W_0 \subset V^*$) is a closed subset (for example an affine subspace). If the hessian of L (respectively H) is non-degenerated in every point, then one say that L (respectively H) is *regular*. In particular, if the hessian of L (respectively H) is strict positively defined, then L (respectively H) is regular. Lagrangians and Hamiltonians are related by Legendre transformations (i.e. their differentials) using the relation $L(z^i) + H(\Omega_i) = z^i \Omega_i$, provided that L or H is regular.

Let us consider now Lagrangians and Hamiltonians on an affine space $\mathcal{A}$.

A *Lagrangian* on the real affine space $\mathcal{A}$ is a differentiable map $L : \mathcal{A} \backslash \mathcal{A}_0 \to I\!R$, where $\mathcal{A}_0 \subset \mathcal{A}$ is a closed subset (for example an affine subspace). If the hessian of L is non-degenerated, then we say that L is *regular*. In particular, if the hessian of L is strict positively defined, then L is regular. The Legendre transformation $\mathcal{L} : \mathcal{A} \backslash \mathcal{A}_0 \to V^*$ can relate a Lagrangian on $\mathcal{A}$ to a Hamiltonian on V^* considering a point $z_0 \in \mathcal{A} \backslash \mathcal{A}_0$ using the relation $L(z^i) + H(\Omega_i) = (z^i - z^i_0)\Omega_i$, provided that L or H is regular. The consideration of z_0 gives a H (called a *vectorial Hamiltonian*), but it is not the only one, as we see below.

An *affine Hamiltonian* on the real affine space $\mathcal{A}$ is a differentiable map $h : V^* \backslash W_0 \to \mathcal{A}^\dagger$, such that $\pi \circ h = 1_{V^* \backslash W_0}$, where $W_0 \subset V$ is a closed subset (for example an affine subspace). Using an affine frame $(o, \mathcal{B})$, then h has the form $h(\Omega_i) = (\Omega_i, H_0(\Omega_i))$. If an other affine frame $(o', \mathcal{B}')$ is considered, then $H'_0(\Omega_{i'}) = H_0(\Omega_i) + \Omega_i a^i$. It follows that $\dfrac{\partial^2 H'_0}{\partial \Omega^{i'} \partial \Omega^{j'}} = a^i_{i'} a^j_{j'} \dfrac{\partial^2 H_0}{\partial \Omega^i \partial \Omega^j}$, thus the local functions H'_0 and H_0 have the same hessian, which depend only on h. We call the hessian of H'_0 and H_0 as the *hessian* of h and we say that h is *regular* if its hessian is non-degenerate.

Let $h : V^* \backslash W_0 \to \mathcal{A}^\dagger$ be an affine Hamiltonian and consider a point $z_0 \in \mathcal{A}$. The fact that $H_0(\Omega_i) - \Omega_i z^i_0 = H'_0(\Omega_{i'}) - \Omega_i \cdot (z^i_0 + a^i) = H'_0(\Omega_{i'}) - \Omega_i a^i_{i'} z^{i'}_0 = H'_0(\Omega_{i'}) - \Omega_{i'} z^{i'}_0$

implies that $H(\Omega_i) = H_0(\Omega_i) - \Omega_i z_0^i$ defines a vectorial Hamiltonian which is regular iff h is regular. Conversely, if $H : V^*\backslash W_0 \to I\!R$ is a vectorial Hamiltonian and $z_0 \in \mathcal{A}$ is a point, then denoting $H_0(\Omega_i) = H(\Omega_i) + \Omega_i z_0^i$, the map $h : V^*\backslash W_0 \to \mathcal{A}^\dagger$ given by $h(\Omega_i) = (\Omega_i, H_0(\Omega_i))$ defines an affine Hamiltonian. Thus the vectorial and affine Hamiltonians are related by the following result.

Proposition 1.1. *If $z_0 \in \mathcal{A}$ is a given point and $W_0 \subset V^*$ is a closed subset, then there is a bijective correspondence between affine Hamiltonians and vectorial Hamiltonians on $V^*\backslash W_0$.*

Notice that the correspondence defined above depends on the given point $z_0 \in \mathcal{A}$.

A given point $z_0 \in \mathcal{A}$ and the canonical duality $\varphi : V \times V^* \to I\!R$, define the *Liouville map* $C_{z_0} : \mathcal{A} \times V^* \to I\!R$, given by the formula $C_{z_0}(z, \Omega) = \varphi(z - z_0, \Omega)$, where $z - z_0$ denotes the vector $\overline{z_0 z}$.

Proposition 1.2. *Let $L : \mathcal{A}\backslash\mathcal{A}_0 \to I\!R$ be a regular Lagrangian on the real affine space $\mathcal{A}$ and $\mathcal{L} : \mathcal{A}\backslash\mathcal{A}_0 \to V^*\backslash W_0$ be the Legendre transformation. Then for every point $z_0 \in \mathcal{A}$, the map $H : V^*\backslash W_0 \to I\!R$, $H(\Omega) = C_{z_0}(\mathcal{L}^{-1}(\Omega), \Omega) - L(\mathcal{L}^{-1}(\Omega))$ is a Hamiltonian on $V^*\backslash W_0$ and the affine Hamiltonian $h : V^*\backslash W_0 \to \mathcal{A}^\dagger$ corresponding to the point z_0 (according to Proposition 1.1) does not depend on the point z_0, depending only on the Lagrangian L.*

Proof. Using coordinates, the link between L and H is $L(z^i) + H(\Omega_i) = (z^i - z_0^i)\Omega_i$, where $\mathcal{L}^{-1}(\Omega)) = z^i \bar{e}_i$. It is easy to check (classical) that H is a Hamiltonian. The affine Hamiltonian corresponding to the point z_0 according to Proposition 1.1 has the form $(\Omega_i) \xrightarrow{h} (\Omega_i, H_0(\Omega_i))$, where $H_0(\Omega_i) = H(\Omega_i) + z_0^i \Omega_i = z^i \Omega_i - L(z^i)$, thus the conclusion follows. □

A converse correspondence may be performed as follows.

Proposition 1.3. *Let $h : V^*\backslash W_0 \to \mathcal{A}^\dagger$ be a regular affine Hamiltonian on the real affine space $\mathcal{A}$. Consider a point $z_0 \in \mathcal{A}$, the regular vectorial Hamiltonian $H : V^*\backslash W_0 \to I\!R$ corresponding to the point z_0 (according to Proposition 1.1), $\mathcal{H} : V^*\backslash W_0 \to V\backslash W_1$ its Legendre transformation and $\mathcal{A}_0 = z_0 + W_1$. Then*

1. *The map $\mathcal{H}_0 : V^*\backslash W_0 \to \mathcal{A}\backslash\mathcal{A}_0$ given by the formula $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega) + z_0$ is a diffeomorphism (called the* Legendre transformation *of h).*
2. *The real function $L : \mathcal{A}\backslash\mathcal{A}_0 \to I\!R$ given by the formula $L(z) = C_{z_0}(z, \mathcal{H}^{-1}(z - z_0)) - H(\mathcal{H}^{-1}(z - z_0))$ is a regular Lagrangian.*
3. *Both $\mathcal{H}_0$ and L do not depend on the point z_0, depending only on the affine Hamiltonian h.*

Proof. Using coordinates, h has the form $(\Omega_i) \xrightarrow{h} (\Omega_i, H_0(\Omega_i))$ and $H(\Omega_i) = H_0(\Omega_i) - z_0^i \Omega_i$. Thus $\mathcal{H}(\Omega)^i = \dfrac{\partial H}{\partial \Omega_i} = \dfrac{\partial H_0}{\partial \Omega_i} - z_0^i$, then 1. follows, since h is regular. The proof of 2. uses a similar argument as in the Lagrangian case. Using also coordinates, the link between L and H is also $L(z) + H(\Omega) = (z^i - z_0^i)\Omega_i$, where $\Omega = \Omega_i \bar{e}^i = \mathcal{H}^{-1}(z - z_0)$. It is also easy to check (classical) that L is a Lagrangian. If the affine Hamiltonian h has the form $h(\Omega) = (\Omega_i \bar{e}^i, H_0(\Omega_i))$, then H has the form

$H(\Omega) = H_0(\Omega) - z_0^i \Omega_i$, where $\mathcal{H}^{-1}(z - z_0) = \Omega_i \bar{e}^i = \Omega$. Thus $L(z) = (z^i - z_0^i)\Omega_i - H(\Omega) = (z^i - z_0^i)\Omega_i - H_0(\Omega) + z_0^i \Omega_i = z^i \Omega_i + H_0(\Omega)$, thus 2. follows. Using coordinates, from proof of 1. it follows that the affine coordinates of $\mathcal{H}_0(\Omega)$ are $\left(\dfrac{\partial H_0}{\partial \Omega_i}\right)$, thus $\mathcal{H}_0$ depend only on H_0 and implicitly on h. Taking the coordinates (z^i) of $z \in \mathcal{A} \backslash \mathcal{A}_0$ in the form $z^i = \dfrac{\partial H_0}{\partial \Omega_i}$ and denoting, as before, $\mathcal{H}^{-1}(z - z_0) = \Omega_i \bar{e}^i = \Omega$, we have $\mathcal{H}(\Omega) = z - z_0$. Using also 2., we have $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega) + z_0 = z$, thus $\Omega = \mathcal{H}_0^{-1}(z)$. Since $L(z) = z^i \Omega_i + H_0(\Omega)$, the conclusion follows. □

2 Short review on higher acceleration bundles

Let M be a manifold of dimension m and $\tau M = (TM, \pi, M)$ its tangent bundle. Considering an atlas of M, we denote by (x^i) the coordinates on an arbitrary domain $U \subset M$ and by (x^i, y^j) the coordinates on the domain $\pi^{-1}(U) \subset TM$ $(i, j = \overline{1, m})$. On the intersection of two open domains of coordinates on TM, the coordinates change according the rule

$$x^{i\prime} = x^{i\prime}(x^i), y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i.$$

A surjective submersion $E \xrightarrow{\pi} M$ is usually called a *fibered manifold*. An *affine bundle* $E \xrightarrow{\pi} M$ is a fibered manifold that the change rules of the local coordinates on E have the form

(2.1) $$\bar{x}^i = \bar{x}^i(x^j), \bar{y}^\alpha = g_\beta^\alpha(x^j) y^\beta + v^\alpha(x^j).$$

An *affine section* in the bundle E is a differentiable map $M \xrightarrow{s} E$ such that $\pi \circ s = id_M$ and its local components change according to the rule $\bar{s}^\alpha(\bar{x}^i) = g_\beta^\alpha(x^j) \bar{s}^\beta(x^j) + v^\alpha(x^j)$. The set of affine sections is denoted by $\Gamma(E)$ and it is an affine module over $\mathcal{F}(M)$, i.e. for every $f_1, \ldots, f_p \in \mathcal{F}(M)$ such that $f_1 + \cdots + f_p = 1$ and $s_1, \ldots, s_p \in \Gamma(E)$, then $f_1 s_1 + \cdots + f_p s_p \in \Gamma(E)$, where the affine combination is taken in every point of the base. Using a partition of unity on the base M it can be easily proved that every affine bundle allows an affine section.

A vector bundle $\bar{E} \xrightarrow{\bar{\pi}} M$ can be canonically associated with the affine bundle $E \xrightarrow{\pi} M$. More precisely, using local coordinates, the coordinates change on $\bar{E}$ following the rules $\bar{x}^i = \bar{x}^i(x^j)$, $\bar{z}^\alpha = g_\beta^\alpha(x^j) z^\beta$, when the coordinates on E change according to the formulas (2.1). Every vector bundle is an affine bundle, called a *central affine bundle*. In this case $v^\alpha(x^j) = 0$.

We consider the natural projection $T^k M \to T^{k-1} M$, where the change rule of the local coordinates on the fibers is:

$$k y^{(k)i'} = k \frac{\partial x^{i'}}{\partial x^i} y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'}).$$

Notice that $\Gamma_U^{(k)} = y^{(1)i} \dfrac{\partial}{\partial x^i} + 2 y^{(2)i} \dfrac{\partial}{\partial y^{(1)i}} + \cdots + k y^{(k)i} \dfrac{\partial}{\partial y^{(k-1)i}}$ and on the intersection of two domains corresponding to U and U', we have $\Gamma_{U'}^{(k)} = \Gamma_U^{(k)} - \Gamma_U^{(k)}(y^{(k)i'}) \dfrac{\partial}{\partial y^{(k)i'}}$ (see [9]).

Proposition 2.1. *The fibered manifold $(T^kM, p_k, T^{k-1}M)$ is an affine bundle, for $k \geq 2$.*

Let $\ker \pi_{k*} = V_0T^kM$, be the vertical vector bundle of T^kM (viewed as a fibered manifold over $T^{k-1}M$) and $\Gamma(V_0T^kM)$ be the module of the vertical sections. The local coordinates on the fibers of V_0T^kM have the form (Y^i) and change according to the rules $Y^{i'} = \dfrac{\partial x^{i'}}{\partial x^i}Y^i$. If $S : T^kM \to V_0T^kM$ is a section, then it has the local form $S = S^i \dfrac{\partial}{\partial y^{(k)i}}$ and $T_S = \dfrac{\partial S^i}{\partial y^{(k)j}} \dfrac{\partial}{\partial y^{(k)i}} \otimes dy^{(k)j}$ defines an endomorphism on the fibers of V_0T^kM.

A *Liouville type* section is a vertical section $S \in \Gamma(V_0T^kM)$ which T_S is the identity on the fibers of V_0T^kM; it has the local form
$S^i(x^i, y^{(1)i}, \ldots, y^{(k)i}) = (y^{(k)i} + t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))\dfrac{\partial}{\partial y^{(k)i}}$.

Proposition 2.2. *There is an one to one correspondence between the Liouville type sections in $\Gamma(V_0T^kM)$ and the affine sections in $T^kM \to T^{k-1}M$.*

The vector bundle canonically associated with the affine bundle $(T^kM, p_k, T^{k-1}M)$ is the vector bundle $q^*_{k-1}TM$, where $q_{k-1} : T^{k-1}M \to M$ is $q_{k-1} = p_1 \circ p_2 \circ \cdots \circ p_{k-1}$. The fibered manifold (T^kM, q_k, M) is systematically used in [14, 9] in the study of the geometrical objects of order k on M, in particular the Lagrangians of order k on M. The total space of the dual $q^*_{k-1}T^*M$ of the vector bundle $q^*_{k-1}TM$ is also the total space of the fibered manifold $(T^{k-1}M \times_M T^*M, r_k, M)$ and is used in [11, 12, 13] in the study of the dual geometrical objects of order k on M, in particular the Hamiltonians of order k on M. In the sequel we denote $q^*_{k-1}T^*M = T^{k*}M$ and we consider it as a vector bundle over $T^{k-1}M$.

The tensors defined on the fibers of the vertical vector bundle $V^k_{k-1}M \to T^kM$ of the affine bundle $(T^kM,\ p_k, T^{k-1}M)$ (even less the null section) are called *d-tensors of order* k on M.

3 Legendre transformations

In this section we study the Legendre transformations associated to Lagrangians and vectorial Hamiltonians of higher order (subsection 3.1) and to Lagrangians and vectorial Hamiltonians of higher order (subsection 3.2). The Legendre transformations studied in subsection 3.1 use essentially an affine section and it is not intrinsic associated with the Lagrangian and the vectorial Hamiltonian. It is used in [13]. The Legendre transformations studied in subsection 3.2 is intrinsic associated with the Lagrangian and the affine Hamiltonian.

3.1 Legendre transformations using affine sections

In this subsection we perform the constructions of Legendre and Legendre* transformations associated with a Lagrangian and a Hamiltonian respectively, using as an ingredient an affine section. This is in brief the way used in [11]-[13] to construct the Legendre transformations.

A *Lagrangian* of order k on M is a function $L : T^kM \to I\!R$, differentiable on $\widetilde{T^kM} = T^kM\backslash 0(M)$ (i.e. T^kM without the image $0(M)$ of the null section). The Lagrangian is *regular* if the vertical Hessian $\left(\dfrac{\partial^2 L}{\partial y^\alpha y^\beta}\right)$ of L is non-degenerate. In this case the vertical hessian defines a (pseudo)metric structure on the fibers of the vertical bundle $V\widetilde{T^kM}$.

A *vectorial Hamiltonian* of order k on M is a function $H : T^{k*}M \to I\!R$, differentiable on $\widetilde{T^{k*}M}$ (i.e. $T^{k*}M$ without the null section). The vectorial Hamiltonian is *regular* if the vertical Hessian $\left(\dfrac{\partial^2 H}{\partial p_i p_j}\right)$ of H is non-degenerate. In this case the vertical hessian defines a (pseudo)metric structure on the fibers of the vertical bundle $V\widetilde{T^{k*}M}$. The vectorial Hamiltonian defined here is called simply a Hamiltonian in [11]-[13]. We say that this Hamiltonian is *vectorial*, in order to distinguish it from the *affine Hamiltonian* defined latter in the paper.

If $L : T^kM \to I\!R$ is a Lagrangian, then the *Legendre transformation* is the fibered manifold mapping $\mathcal{L} : \widetilde{T^kM} \to \widetilde{T^{k*}M}$ (both on the base $T^{k-1}M$) defined in local coordinates on the fibers by $(y^{(k)i}) \xrightarrow{\mathcal{L}} (p_i = \dfrac{\partial L}{\partial y^{(k)i}}(x^i, y^{(1)i}, \ldots, y^{(k)i}))$. It is easy to see that if L is a regular Lagrangian, then $\mathcal{L}$ is a local diffeomorphism. Considering a regular Lagrangian locally, we can suppose that $\mathcal{L}$ is a global diffeomorphism.

The Legendre transformation defines an $\mathcal{L}$-morphism of the vertical vector bundles $V\widetilde{T^kM} \to V\widetilde{T^{k*}M}$ (called the *vertical Legendre morphism*) and expressed in local coordinates on fibers by $(y^{(k)i}, Y^j) \to (\dfrac{\partial L}{\partial y^{(k)i}}, Y^j \dfrac{\partial^2 L}{\partial y^{(k)j} y^{(k)k}})$.

Theorem 3.1. *Let $s : T^{k-1}M \to T^kM$ be an affine section and $L : T^kM \to I\!R$ be a regular Lagrangian.*

Then there is a Hamiltonian $H : T^{k*}M \to I\!R$ defined by L and s such that the vertical Legendre morphism is an isometry and the vertical hessian of H does not depend on the section s.

Proof. Let $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{s} (x^i, y^{(1)i}, \ldots, y^{(k-1)i},\ s^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$ be the local form of the section s. According to Proposition 2.2, the section s defines a Liouville section $S : T^{k-1}M \to VT^{k-1}M$ given in local coordinates by $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{S} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, y^{(k-1)i} - s^i(y^{(1)i}, \ldots, y^{(k-1)i}))$. Since L is non-degenerate it means that $\mathcal{L}$ is a diffeomorphism, thus consider $\mathcal{H} = \mathcal{L}^{-1} : T^{k*}M \to T^kM$ and denote by $\bar{S} = S \circ \mathcal{H}: T^{k*}M \to VT^{k-1}M$. Notice that $\mathcal{H}$ has the local form $(x^i, y^{(1)i}, \ldots,\ y^{(k-1)i},\ p_i) \xrightarrow{\mathcal{H}} (x^i,\ y^{(1)i}, \ldots,\ y^{(k-1)i},\ H^i(x^i, y^{(1)i}, \ldots,\ y^{(k-1)i},\ p_i))$, where $H^j(x^i, y^{(1)i}, \ldots,\ y^{(k-1)i},\ \dfrac{\partial L}{\partial y^{(k)i}}(x^i,\ y^{(1)i}, \ldots,\ y^{(k)i})) = y^j$ and $\dfrac{\partial L}{\partial y^{(k)j}}(x^i,\ y^{(1)i}, \ldots,\ y^{(k-1)i},\ H^i(x^i,\ y^{(1)i}, \ldots,\ y^{(k-1)i},\ p_i)) = p_j$. Differentiating the first formula, we obtain:

$$\frac{\partial^2 L}{\partial y^{(k)u} \partial y^{(k)w}}(x^i, y^{(1)i}, \ldots, y^{(k)i}) \cdot \tag{3.1}$$

$$\frac{\partial H^w}{\partial p_v}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial L}{\partial y^{(k)i}}(x^i, y^{(1)i}, \dots, y^{(k)i})){=}\delta_{uv}.$$

Substituting $y^{(k)j} = H^j(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i)$ we also have

$$\begin{aligned}(3.2)\qquad & \frac{\partial^2 L}{\partial y^{(k)u}\partial y^{(k)w}}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \dots, y^{(k)i}, p_i)) \cdot \\ & \frac{\partial H^w}{\partial p_v}(x^i, y^{(1)i}, \dots, y^{(k)i}){=}\delta_{uv}.\end{aligned}$$

Then $\bar{S}$ has the form

$$\begin{aligned}&(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \xrightarrow{\bar{S}} \\ &(x^i, y^{(1)i}, \dots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i), H^i(x^i, y^{(1)i}, \dots, \\ &y^{(k-1)i}, p_i) - s^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i)).\end{aligned}$$

We define $H : T^{k*}M \to I\!R$ using the formula

$$\begin{aligned}(3.3)\qquad H(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \quad = \quad & p_j(H^j(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) - \\ & s^j(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i)) - \\ & L(x^i, y^{(1)i}, \dots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i)).\end{aligned}$$

It is easy to see that H is globally defined on $T^{k*}M$. In order to prove that the vertical hessian of H is non-degenerate and also that the vertical bundle morphism is an isometry, it suffices to prove that

$$\left(\frac{\partial H^2}{\partial p_u \partial p_v}\left(x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial L}{\partial y^i}\left(x^i, y^{(1)i}, \dots, y^{(k)i}\right)\right)\right) = \left(\frac{\partial L^2}{\partial y^u \partial y^v}\left(x^i, y^{(1)i}, \dots, y^{(k)i}\right)\right)^{-1}.$$

This can be obtained by a straightforward computation, as follows. Using formula (3.1), we obtain $\frac{\partial H}{\partial p_j}(x^i, y^{(1)i}, \dots, y^{(k-1)i},\ p_i) = H^j(x^i, y^{(1)i}, \dots,$ $y^{(k-1)i}, p_i)$, then using the relations (3.2) and (3.1), the above formula follows. It is easy to see that the vertical hessian of the Hamiltonian does not depend on the section s. □

An inverse construction is performed in the sequel. Starting from a Hamiltonian, a Lagrangian on T^kM can be constructed.

Given a Hamiltonian $H : \widetilde{T^{k*}M} \to I\!R$ and a section s of T^kM, the *Legendre* transformation* is the fibered manifold morphism $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^kM}$ defined by the local formula $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \xrightarrow{\mathcal{H}}$ $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, p_j){+}s^i(x^j))$. If the Hamiltonian is regular, then the Legendre* transformation is a diffeomorphism.

The Legendre* transformation defines an $\mathcal{H}$-morphism of the vertical vector bundles $V\widetilde{T^{k*}M} \to V\widetilde{T^kM}$ (called the *vertical Legendre* morphism*) and expressed in

local coordinates by $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, P_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i},$
$\frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j) + s^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}),\, P^u \frac{\partial^2 H}{\partial p_i \partial p_u}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j)).$

Theorem 3.2. *Let $s : \widetilde{T^{k-1}M} \to \widetilde{T^k M}$ be an affine section and $H : T^{k*}M \to I\!R$ be a non-degenerate Hamiltonian.*

Then there is a Lagrangian $L : \widetilde{T^k M} \to I\!R$ of order k on M such that the vertical Legendre* morphism is an isometry and the vertical hessian of L does not depend on the section s.

Proof. The proof is analogous to the proof of Theorem 3.1. In fact we reverse the order of H and L in the construction of H in the formula (3.3). We denote by $\mathcal{L} = \mathcal{H}^{-1} : T^k M \to T^{k*}M$ the inverse of the Legendre* transformation. It has the local form

$$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\mathcal{L}}{\to} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^j, y^{(1)j}, \ldots, y^{(k)j})),$$

where

$$\begin{aligned} & L_i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, \frac{\partial H}{\partial p_j}(x^u, y^{(1)u}, \ldots, y^{(k-1)u}, p_u) + \\ & s^j(x^u, y^{(1)u}, \ldots, y^{(k-1)u})) = p_i \text{ and} \\ & \frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, L_j(x^u, y^{(1)u}, \ldots, y^{(k)u})) + \\ & s^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}) = y^i. \end{aligned}$$

One defines $H : \widetilde{T^{k*}M} \to I\!R$ using the formula

$$\begin{aligned} & L(x^i, y^{(1)i}, \ldots, y^{(k)i}) = \\ & L_i(x^j, y^{(1)j}, \ldots, y^{(k)j}) \left(y^i - s^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j})\right) - \\ & H(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^j, y^{(1)j}, \ldots, y^{(k)j})). \end{aligned} \tag{3.4}$$

The proof follows that of Theorem 3.1. □

3.2 Legendre transformations without using affine sections

In this subsection we define the affine Hamiltonian of order $k \geq 2$. This definition allows us to perform a construction of some new Legendre and Legendre* transformations associated with a Lagrangian and an affine Hamiltonian respectively, without using an affine section. The use of an affine section in the construction of the Legendre transformations, as in [11]-[13], makes the Legendre and Legendre* transformations non-canonical associated to Lagrangians and Hamiltonians, thus our construction improves this aspect.

Let us consider the affine bundle $T^k M \stackrel{\pi_k}{\to} T^{k-1}M$ and $u \in T^{k-1}M$. The fiber $T^k_u M = \pi_k^{-1}(u) \subset T^k M$ is a real affine space, modelled on the real vector space $T_{\pi(u)}M$. The *vectorial dual* of the affine space $T^k_u M$ is $T^k_u M^\dagger = Aff(T^k_u M, I\!R)$, where Aff denotes affine morphisms (for details see the Appendix). Denoting by

$T^kM^\dagger = \bigcup_{u\in T^{k-1}M} T_u^kM^\dagger$ and $\pi^\dagger : T^kM^\dagger \to T^{k-1}M$ the canonical projection, then it is clear that $(T^kM^\dagger, \pi^\dagger, T^{k-1}M)$ is a vector bundle. There is a canonical vector bundle morphism of vector bundles over $T^{k-1}M$, $\Pi : T^kM^\dagger \to T^{k*}M$.

Considering local coordinates (x^i) on M, $(x^i, y^{(1)i}, \ldots, y^{(k-1)i})$ on $T^{k-1}M$, and $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, T)$ on $T^kM^\dagger$, then the coordinates p_i and T' change according the rules $p_{i'} = \dfrac{\partial x^i}{\partial x^{i'}} p_i$ and $T' = T - \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\dfrac{\partial x^i}{\partial x^{i'}} p_i$ respectively. The vector bundle morphism Π is given in local coordinates by $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, T) \stackrel{\Pi}{\to} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$. An *affine Hamiltonian* of order k on M is a differentiable map $h : \widetilde{T^{k*}M} \to \widetilde{T^kM^\dagger}$, such that $\Pi \circ h = 1_{\widetilde{T^{k*}M}}$. Thus h has the local form $h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, -H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. The local functions H_0 change according the rules $H_0'(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\dfrac{\partial x^i}{\partial x^{i'}} p_i$. It is easy to see that $\dfrac{\partial H_0'}{\partial p_{i'}} = \dfrac{\partial x^{i'}}{\partial x^i}\dfrac{\partial H_0}{\partial p_i} + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})$. Thus there is a map $\mathcal{H} : T^{k*}M \to T^kM$, given in local coordinates by $\mathcal{H}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = \dfrac{\partial H_0}{\partial p_i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$. We say that $\mathcal{H}$ is the Legendre* mapping of the affine Hamiltonian h and h is *regular* if $\mathcal{H}$ is a local diffeomorphism. Since $\dfrac{\partial^2 H_0'}{\partial p_{i'}\partial p_{j'}} = \dfrac{\partial x^{i'}}{\partial x^i}\dfrac{\partial x^{j'}}{\partial x^j}\dfrac{\partial^2 H_0}{\partial p_i \partial p_j}$, it follows that $h^{ij} = \dfrac{\partial^2 H_0}{\partial p_i \partial p_j}$ is a symmetric 2-contravariant d-tensor, which is non-degenerate iff h is regular.

Theorem 3.3. *a) Let $L : T^kM \to I\!R$ be a regular Lagrangian of order k on M. Considering local coordinates, let $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \stackrel{\mathcal{L}^{-1}}{\to} H^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$ denote the local form of the inverse $\mathcal{L}^{-1}$ of the Legendre transformation. Then the local functions given by*

$$\begin{aligned} H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) &= p_j H^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) - \\ & L(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)) \end{aligned}$$

define a regular affine Hamiltonian of order k on M and the vertical Legendre morphism is an isometry .

b) Conversely, let $h : \widetilde{T^{k*}M} \to \widetilde{T^kM^\dagger}$ be a regular affine Hamiltonian of order k on M. Considering local coordinates, let $h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$ be the local form of h and $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \stackrel{\mathcal{H}^{-1}}{\to} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^i, y^{(1)i}, \ldots, y^{(k)i}))$ denote the local form of the inverse $\mathcal{H}^{-1}$ of the Legendre* transformation. Then the local functions given by

$$\begin{aligned} L(x^i, y^{(1)i}, \ldots, y^{(k)i}) &= y^{(k)j} L_j(x^i, y^{(1)i}, \ldots, y^{(k)i}) - \\ & H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^i, y^{(1)i}, \ldots, y^{(k)i}) \end{aligned}$$

define a global regular Lagrangian of order k on M and the vertical Legendre* morphism is an isometry .

c) The constructions a) and b) are inverse each to the other.

The proof follows using Propositions 1.2 and 1.2 in Appendix.

4 Induced Hamiltonians on submanifolds

Besides the theory of Lagrange and Finsler submanifolds, which is studied by many authors, (see the Bibliography), an attempt to study the Hamilton submanifolds is performed in [16, 10], using an arbitrary section of the natural projection of the cotangent bundles. In [24] it is shown that there is a distinguished section, which depends only on the Hamiltonian (see also [7]). It solves a problem from [16, 10], concerning the possibility to induce in an intrinsic way a Hamiltonian on a submanifold. Following a similar idea, we show that an analogous result holds in the higher order. Since the higher order tangent spaces are affine bundles, we have seen in the previous section that a distinction between vectorial and affine Hamiltonians is necessary. Thus we investigate the possibility to induce vectorial and affine Hamiltonians on submanifolds.

If $E \xrightarrow{\pi} M$ is an affine bundle then an *affine subbundle* of E is an affine bundle $E' \xrightarrow{\pi'} M'$ such that $E' \subset E$ and $M' \subset M$ are submanifolds, π' is the restriction of π and the affine structure on the fibers of E' is induced by the affine structure on the fibers of E.

Consider $M' \subset M$ a submanifold and denote by $i : M' \to M$ the inclusion. Consider some coordinates on M, along M', adapted to the submanifold M'. It means that the coordinates have the form $(x^i)_{i=\overline{1,m}} = (x^\alpha)_{\alpha=\overline{1,m'}} \cup (x^{\bar\alpha})_{\bar\alpha=\overline{m'+1,m}}$ and the points in M' are characterized by $x^{\bar\alpha} = 0$, $(\forall)\bar\alpha = \overline{m'+1,m}$. Using these coordinates on T^kM, the inclusion $i^k : T^kM' \to T^kM$ has the local form $(x^\alpha, y^{(1)\alpha}, \dots, y^{(k)\alpha}) \to (x^\alpha, x^{\bar\alpha} = 0, y^{(1)\alpha}, y^{(1)\bar\alpha} = 0, \dots, y^{(k)\alpha}, y^{(k)\bar\alpha} = 0)$.

There are some local coordinates (x^α) on M' and $(x^\alpha, y^{(1)\alpha}, \dots, y^{(k)\alpha})$ on T^kM' which extend to local coordinates $(x^i) = (x^\alpha, x^{\bar\alpha})$ on M and $(x^i, y^\alpha) = (x^\alpha, x^{\bar\alpha}, y^{(1)\alpha}, y^{(1)\bar\alpha}, \dots, y^{(k)\alpha}, y^{(k)\bar\alpha})$ on T^kM respectively, such that the points in M' and in T^kM' are characterized by the conditions $x^{\bar\alpha} = 0$ and $x^{\bar\alpha} = y^{(1)\bar\alpha} = \cdots = y^{(k)\bar\alpha} = 0$ respectively. ($i, j, k, \ldots = \overline{1,m}$, $m = \dim M$, $\alpha, \beta, \ldots = \overline{1,m'}$, $\bar\alpha, \bar\beta\bar v, \ldots \in \overline{m'+1,m}$, $m' = \dim M'$.

We consider also local coordinates $(x^\alpha, , y^{(1)\alpha}, \dots, y^{(k-1)\alpha}, p_a)$ on $T^{k*}M'$ and $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) = (x^\alpha,\ y^{(1)\alpha}, y^{(1)\bar\alpha}, \dots,\ y^{(k-1)\alpha}, y^{(k-1)\bar\alpha}, p_a, p_{\bar a})$ on $T^{k*}M$, which are adapted to the vector bundle structures and to the submanifolds structures.

4.1 Induced vectorial Hamiltonians on submanifolds using sections

In this subsection we follow [26].

A section $s : T^{k-1}M \to T^kM$ may not restrict, in general, to a section $s' : T^{k-1}M' \to T^kM'$. If the section $s : T^{k-1}M \to T^kM$ restricts to a section $s' : T^{k-1}M' \to T^kM'$, we say that s is *adapted* to the submanifold M. The local form of the adapted section s is $(x^i,\ y^{(1)i}, \dots,\ y^{(k-1)i}) \xrightarrow{s} (x^i, y^{(1)i}, \dots, y^{(k-1)i}, s^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}))$, where $s^{\bar a}(x^u,\ 0) = 0$.

Let us consider a Hamiltonian $H : T^{k*}M \to I\!R$. The local form of the Legendre* transformation $\mathcal{H}$ is

$$(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \quad \to \quad (x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \dots,$$

$$y^{(k)j}, p_j) + s^i(x^i, , y^{(1)i}, \ldots, y^{(k-1)i})),$$

and we denote $\frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j) = H^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j)$. The local forms of the inclusions $i : M' \to M$, $I : T^kM' \to T^kM$ and of the canonical projection $I^* : T^{k*}M \to T^{k*}M'$ are

$$(x^\alpha) \xrightarrow{i} (x^\alpha, 0), (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \xrightarrow{I} (x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k)\alpha}, 0),$$

and

$$(x^\alpha, x^{\bar{\alpha}}, y^{(1)\alpha}, y^{(1)\bar{\alpha}}, \ldots, y^{(k-1)\alpha}, y^{(k-1)\bar{\alpha}}, p_a, p_{\bar{\alpha}}) \xrightarrow{I^*} (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_a)$$

respectively.

If the Hamiltonian H is regular, the Legendre* transformation $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^kM}$ is a local diffeomorphism, which we suppose to be a global one. We denote by $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^i, y^{(1)i}, \ldots, y^{(k)i}))$ the local form of $\mathcal{L} = \mathcal{H}^{-1} : \widetilde{T^kM} \to \widetilde{T^{k*}M}$, the inverse of the Legendre* transformation.

We have that $W' = \mathcal{L} \circ I(\widetilde{T^kM'})$ is a submanifold of $\widetilde{T^{k*}M}$.

Proposition 4.1. *The restriction of I^* to W', $I^*_{|W'} : W' \to \widetilde{T^{k*}M'}$ is a diffeomorphism.*

Proof. We have: $\mathcal{L}$ is a diffeomorphism, I^* is a surjective submersion and I is an injective immersion. The local form of $I^* \circ \mathcal{L} \circ I$ is $(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \to (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, L_\alpha(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k)\alpha}, 0))$, thus it is a local diffeomorphism. In fact $I^* \circ \mathcal{L} \circ I$ is a diffeomorphism, since it sends the fibre $\widetilde{T^kM'}_x$ in the fibre $\widetilde{T^{k*}M'}_x$ for every $x \in T^{k-1}M'$ and $\mathcal{L}$ is a diffeomorphism when it is restricted to the fiber, thus $I^*_{|W'}$ is also a diffeomorphism. □

Taking into account of the local form of the Legendre* transformation and of the local coordinates, it follows that the points of the submanifold W' have as coordinates $(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar{\alpha}}(x^\alpha, y^{(1)\alpha}, \ldots,$
$y^{(k-1)\alpha}, p_\alpha))$ in $\widetilde{T^{k*}M}$, where

$$\begin{aligned} &\frac{\partial H}{\partial p_{\bar{\alpha}}}(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar{\alpha}}(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_\alpha)) + \\ &s^{\bar{\alpha}}(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}) = 0. \end{aligned} \tag{4.1}$$

Differentiating this equation with respect to p_α, we get:

$$\frac{\partial^2 H}{\partial p_\alpha \partial p_{\bar{\alpha}}} + \frac{\partial^2 H}{\partial p_{\bar{\beta}} \partial p_{\bar{\alpha}}} \cdot \frac{\partial Q_{\bar{\beta}}}{\partial p_\alpha} = 0.$$

Denoting by $h^{\alpha\beta} = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$, we suppose that the matrix
$\tilde{h} = (h^{\bar{\alpha}\bar{\beta}})_{\bar{\alpha}, \bar{\beta} = \overline{m'+1, m}}$ is non-degenerate; if this condition holds, we say that the Hamiltonian is *non-degenerate* along the affine subbundle E' (notice that this condition

automatically holds when the vertical hessian of the Hamiltonian defines a positive quadratic form). Considering the inverse $\tilde{h}^{-1} = (\tilde{h}_{\bar{\alpha}\bar{\beta}})_{\bar{\alpha},\bar{\beta}=\overline{m'+1,m}}$, it follows that

$$\frac{\partial Q_{\bar{\beta}}}{\partial p_\alpha} = -h^{\alpha\bar{\alpha}}\tilde{h}_{\bar{\alpha}\bar{\beta}}. \tag{4.2}$$

Denote $\bar{I} = I^{*-1}_{|W'} : \widetilde{T^{k*}M'} \to W' \subset \widetilde{T^{k*}M}$. Using the above constructions, we obtain the following result.

Theorem 4.1. *The map $\bar{I}$ is a section of I^* which depends only on H and s. If the section s is adapted, then the map $\bar{I}$ depend only on the Hamiltonian H.*

We define $\bar{H} = H \circ \bar{I} : \widetilde{T^{k*}M'} \to I\!R$ and we consider the vertical Hessian of $\bar{H}$:

$$\left(\frac{\partial^2 \bar{H}}{\partial p_\alpha \partial p_\beta}(x^\gamma, y^{(1)\gamma}, \dots, y^{(k-1)\gamma}, p_\gamma)\right)_{\alpha,\beta=\overline{1,m'}}$$

in every point of $\widetilde{T^{k*}M'}$.

Proposition 4.2. *a) If the Hamiltonian H is non-degenerate along the submanifold W', then H' is a regular Hamiltonian.*

b) If the Hamiltonian H has a positive definite metric along the submanifold W', then $\bar{H}$ is a regular Hamiltonian with a positive defined metric.

Proof. We use local coordinates. We have $\bar{H}(x^\alpha, y^{(1)\alpha}, \dots, y^{(k-1)\alpha}, p_\alpha)$ $= H(x^\alpha, 0, y^{(1)\alpha}, 0, \dots, y^{(k-1)\alpha}, 0, p_\alpha, Q_{\bar{\alpha}}(x^\beta, y^{(1)\beta}, \dots, y^{(k-1)\beta}, p_\beta))$. Using formula (4.1) it follows that:

$$\frac{\partial \bar{H}}{\partial p_\alpha}(x^\beta, y^{(1)\beta}, \dots, y^{(k-1)\beta}, p_\beta) =$$
$$\frac{\partial H}{\partial p_\alpha}(x^\beta, 0, y^{(1)\beta}, 0, \dots, y^{(k-1)\beta}, 0, p_\beta, Q_{\bar{\beta}}(x^\gamma, y^{(1)\gamma}, \dots, y^{(k-1)\gamma}, p_\gamma)).$$

Differentiating this formula with respect to p_β, then using formula (4.2), we get:

$$\frac{\partial^2 \bar{H}}{\partial p_\alpha \partial p_\beta} = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta} + \frac{\partial Q_{\bar{\alpha}}}{\partial p_\beta}\frac{\partial^2 H}{\partial p_{\bar{\alpha}} \partial p_\alpha} = h^{\alpha\beta} - h^{\bar{\alpha}\alpha}\tilde{h}_{\bar{\alpha}\bar{\beta}}h^{\beta\bar{\beta}}.$$

We use now the following Lemma of linear algebra.

Lemma 4.1. *Let A be a symmetric matrix of dimension p, B a symmetric and non-degenerated matrix of dimension q and C a $p \times q$ matrix such that the symmetric matrix $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}$ of dimension $p+q$ is non-degenerate. Denote $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}^{-1} = \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$, where X, Y and Z have the same dimensions as the matrices A, B and C respectively.*

Then the matrix $A - C \cdot B^{-1}C^t$ is invertible and its inverse is X.

Turning back to the proof of the Proposition 4.2, consider the matrix $h = (h^{ij}) = \begin{pmatrix} h^{\alpha\beta} & h^{\bar\alpha\beta} \\ h^{\alpha\bar\beta} & h^{\bar\alpha\bar\beta} \end{pmatrix}$. Using the Lemma 4.1, it follows that the matrix

$$\left(h^{\alpha\beta} - h^{\bar\alpha\alpha}\tilde{h}_{\bar\alpha\bar\beta}h^{\beta\bar\beta}\right)_{\alpha,\beta=\overline{1,m'}}$$

is invertible and its inverse is $(h_{\alpha\beta})$, where $\begin{pmatrix} h_{\alpha\beta} & h_{\bar\alpha\beta} \\ h_{\alpha\bar\beta} & h_{\bar\alpha\bar\beta} \end{pmatrix} =$
$\begin{pmatrix} h^{\alpha\beta} & h^{\bar\alpha\beta} \\ h^{\alpha\bar\beta} & h^{\bar\alpha\bar\beta} \end{pmatrix}^{-1}$. $\square$

4.2 Induced affine Hamiltonians on submanifolds

Let us consider an affine Hamiltonian $h : \widetilde{T^{k*}M} \to \widetilde{T^kM^\dagger}$, thus $\Pi \circ h = 1_{\widetilde{T^{k*}M}}$; h has the local form $h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. The local form of the Legendre* transformation $\mathcal{H}$ is

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial H_0}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j)),$$

and we denote $\frac{\partial H_0}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j) = H_0^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j)$. The local forms of the inclusions $i : M' \to M$, $I : T^kM' \to T^kM$ and of the canonical projection $I^* : T^{k*}M_{T^{k-1}M'} \to T^{k*}M'$ are $(x^\alpha) \xrightarrow{i} (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \xrightarrow{I} (x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k)\alpha}, 0)$,and

$$(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, 0, p_a, p_{\bar\alpha}) \xrightarrow{I^*} (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_a)$$

respectively, where $T^{k*}M_{T^{k-1}M'}$ denotes the restriction of the total space of the affine bundle $T^{k*}M \to T^{k-1}M$ to $T^{k-1}M' \subset T^{k-1}M$.

If the affine Hamiltonian h is regular, the Legendre* transformation $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^kM}$ is a local diffeomorphism, which we suppose to be a global one. We denote by $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^i, y^{(1)i}, \ldots, y^{(k)i}))$ the local form of $\mathcal{L} = \mathcal{H}^{-1} : \widetilde{T^kM} \to \widetilde{T^{k*}M}$, its inverse.

We have that $W' = \mathcal{L} \circ I(\widetilde{T^kM'})$ is a submanifold of $\widetilde{T^{k*}M}$ and the Proposition 4.1 is valid also in this situation, the proof being the same.

Taking into account of the local form of the Legendre* transformation and of the local coordinates, it follows that the points of the submanifold W' have as coordinates $(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar a}(x^\alpha, y^{(1)\alpha}, \ldots,$
$y^{(k-1)\alpha}, p_\alpha))$ in $\widetilde{T^{k*}M}$, where

$$\text{(4.3)} \quad \frac{\partial H_0}{\partial p_{\bar\alpha}}(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar a}(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_\alpha)) = 0.$$

Differentiating this equation with respect to p_α, we get:

$$\frac{\partial^2 H_0}{\partial p_\alpha \partial p_{\bar\alpha}} + \frac{\partial^2 H_0}{\partial p_{\bar\beta} \partial p_{\bar\alpha}} \cdot \frac{\partial Q_{\bar\beta}}{\partial p_\alpha} = 0.$$

Denoting by $h^{\alpha\beta} = \dfrac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$, we suppose that the matrix $\tilde{h} = (h^{\bar{\alpha}\bar{\beta}})_{\bar{\alpha},\bar{\beta}=\overline{m'+1,m}}$ is non-degenerate; if this condition holds, we say that the affine Hamiltonian is *non-degenerate* along the affine subbundle $T^k M'$ (notice that this condition automatically holds when the vertical hessian of the affine Hamiltonian defines a positive quadratic form). Considering the inverse $\tilde{h}^{-1} = (\tilde{h}_{\bar{\alpha}\bar{\beta}})_{\bar{\alpha},\bar{\beta}=\overline{m'+1,m}}$, it follows that

$$\frac{\partial Q_{\bar{\beta}}}{\partial p_\alpha} = -h^{\alpha\bar{\alpha}} \tilde{h}_{\bar{\alpha}\bar{\beta}}. \tag{4.4}$$

Denote $\bar{I} = I^{*-1}_{|W'} : \widetilde{T^{k*}M'} \to W' \subset \widetilde{T^{k*}M}$. Using the above constructions, we obtain the following result.

Theorem 4.2. *The map $\bar{I}$ is a section of I^* which depends only on h.*

Considering a domain U of a local chart of coordinates on M, it defines a domain U^{k*} of a local chart of coordinates on $T^{k*}M$. An affine Hamiltonian h is defined by some local functions $H_0 : U^{k*} \to I\!R$, which change according the rules $H_0'(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\dfrac{\partial x^i}{\partial x^{i'}} p_i$.

Considering now some local charts on M, adapted to the submanifold M' (as in previous subsection), let us consider the corresponding charts on $T^{k*}M$ and define the local functions $\bar{H}_0 = H_0 \circ \bar{I}$.

Proposition 4.3. *The local functions $\bar{H}_0$ define an affine Hamiltonian $\bar{h}$ of order k on the submanifold M'.*

Proof. Since $x^{\bar{\alpha}}(x^{\alpha'}, 0) = 0$, it follows that $\dfrac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha'}}(x^{\alpha'}, 0) = 0$.

We have $\bar{H}_0(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_\alpha) =$
$H_0(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, 0,\ p_\alpha,\ Q_{\bar{\alpha}}(x^\beta, y^{(1)\beta}, \ldots, y^{(k-1)\beta}, p_\beta))$ and
$\bar{H}_0'(x^{\alpha'}, y^{(1)\alpha'}, \ldots, y^{(k-1)\alpha'}, p_{\alpha'}) =$
$H_0'(x^{\alpha'}, 0, y^{(1)\alpha'}, 0, \ldots, y^{(k-1)\alpha'}, 0, p_{\alpha'}, Q_{\bar{\alpha}'}(x^{\beta'}, y^{(1)\beta'}, \ldots, y^{(k-1)\beta'}, p_{\beta'}))$, where $p_{\alpha'} = p_\alpha \dfrac{\partial x^\alpha}{\partial x^{\alpha'}}$ and $Q_{\bar{\alpha}'} = Q_{\bar{\alpha}} \dfrac{\partial x^{\bar{\alpha}}}{\partial x^{\bar{\alpha}'}} + p_\alpha \dfrac{\partial x^\alpha}{\partial x^{\bar{\alpha}'}}$. But $H_0'(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\dfrac{\partial x^i}{\partial x^{i'}} p_i$, thus $\bar{H}_0'(x^{\alpha'}, y^{(1)\alpha'}, \ldots, y^{(k-1)\alpha'}, p_{\alpha'}) = \bar{H}_0(x^\beta, y^{(1)\beta}, \ldots, y^{(k-1)\beta}, p_\beta) + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)\alpha'})\dfrac{\partial x^\alpha}{\partial x^{\alpha'}} p_\alpha$. It follows that the local functions $\bar{H}_0$ define an affine Hamiltonian on M'. □

We consider the vertical Hessian of $\bar{H}_0$:

$$\left(\frac{\partial^2 \bar{H}_0}{\partial p_\alpha \partial p_\beta}(x^\gamma, y^{(1)\gamma}, \ldots, y^{(k-1)\gamma}, p_\gamma) \right)_{\alpha,\beta=\overline{1,m'}}$$

in every point of $\widetilde{T^{k*}M'}$. Using the formulas (4.3) and (4.4), one can prove, in a similar way as in Proposition 4.2, the following result.

Proposition 4.4. *a) If the affine Hamiltonian h is non-degenerate along the submanifold W', then $\bar{h}$ is a regular Hamiltonian.*

b) If the vertical hessian of the affine Hamiltonian h defines a positive definite metric along the submanifold W', then $\bar{h}$ is a regular affine Hamiltonian with a positive defined metric.

Notice that if $s : T^{k-1}M \to T^kM$ is a section and $H : T^{k*}M \to I\!R$ is a vectorial Hamiltonian, then using Proposition 1.1, the local functions $H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = H(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + p_i s^i H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i})$ define an affine Hamiltonian of order k on M and we can use Proposition 4.4. If the section s is adapted to the submanifold $T^{k-1}M' \subset T^{k-1}M$, we obtain in this way Proposition 4.2 as a particular case.

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